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# ON RUZSA'S DISCRETE BRUNN-MINKOWSKI CONJECTURE

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ABSTRACT. We prove a conjecture by Ruzsa from 2006 on a discrete version of the Brunn-Minkowski inequality, stating that for any $A, B \subset \mathbb{Z}^k$ and $\epsilon > 0$ with $B$ not contained in $n_{k,\epsilon}$ parallel hyperplanes we have $|A+B|^{1/k} \geq |A|^{1/k} + (1-\epsilon)\,|B|^{1/k}$.

## 1. INTRODUCTION

The Brunn-Minkowski inequality is a foundational result for Convex Geometry and Asymptotic Geometric Analysis, with diverse connections to other areas including Information Theory, Statistical Mechanics and Algebraic Geometry (see the survey by Gardner [3]). In the continuous setting, for measurable $A, B \subset \mathbb{R}^k$ such that $A+B$ is also measurable, it states that $|A+B|^{1/k} \geq |A|^{1/k} + |B|^{1/k}$. It is natural, and well-motivated by questions in Additive Combinatorics (see [13]), to ask for an analogous inequality in the discrete setting of $A, B \subset \mathbb{Z}^k$. However, this task is surprisingly difficult to define, let alone accomplish, as discussed in Ruzsa's ICM survey [12].

Naturally, one should impose some non-degeneracy condition, otherwise the question reduces to subsets of $\mathbb{Z}$, for which it is well-known and easy to show that $|A+B| \geq |A| + |B| - 1$, with equality for (dilated) intervals. Gardner and Gronchi [4] determined the minimum value of $|A+B|$ in terms of $|A|$ and $|B|$ assuming that one or both of $A$ and $B$ are full-dimensional. While this may at first seem to be a natural discrete analogue, it turns out that under this weak non-degeneracy assumption the extremal examples are somewhat close to intervals, and the resulting bound on $|A+B|$ is much smaller than the 'Brunn-Minkowski' one $|A|^{1/k} + |B|^{1/k}$. Green and Tao [5] showed that one can obtain an asymptotic Brunn-Minkowski bound assuming that $A$ and $B$ are dense subsets of a reasonable box. Thus they obtained the correct bound, but under a very strong assumption; this motivates the question of proving this bound under the natural minimal assumption, formulated by Ruzsa [12, Conjecture 4.12].

**Conjecture 1.1.** *For all $k \in \mathbb{N}$ and $\epsilon$ there exists $n_{k,\epsilon}$ so that if $A, B \subset \mathbb{Z}^k$ with $B$ not covered by $n_{k,\epsilon}$ parallel hyperplanes then $|A+B|^{1/k} \geq |A|^{1/k} + (1-\epsilon)|B|^{1/k}$.*

Our main result establishes Conjecture 1.1 in the following sharp form that provides the optimal quantitative dependence between the parameters.

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**Theorem 1.2.** *For every $k \in \mathbb{N}$ and $t > 0$ there exists $c^{1.2}_{k,t}$ so that if $A, B \subset \mathbb{Z}^k$ with $t|B| \le |A|$ and $B$ is not covered by $n$ parallel hyperplanes then $|A+B|^{1/k} \ge |A|^{1/k} + \left(1 - c^{1.2}_{k,t} n^{-1}\right) |B|^{1/k}$.*

The $n^{-1}$ dependence is optimal up to the constant $c_{k,t}$ as demonstrated by the example $A = B = \{1, \ldots, n\}^k$.

To see that the dependence on $t$ is necessary, consider the example where $A$ is an interval in $\mathbb{Z} \subset \mathbb{Z}^k$ and $B$ is an interval together with $k+n$ scattered points[1], with $t|B| = |A| = m \gg t^{-1} \gg n \gg k$. Here we have $|A+B| \sim |B| + n|A| = (1+nt)|B|$ and $(|A|^{1/k} + (1-c/n)|B|^{1/k})^k \sim ((1-c/n)^k + kt^{1/k})|B|$, so the conclusion of Theorem 1.2 cannot hold if $c < t^{1/k}$.

1.1. **Brunn-Minkowski and the additive hull.** We will deduce Theorem 1.2 from more general results in which we bound $|A+B|$ for $A, B \subset \mathbb{Z}$, where $B$ satisfies a $k$-dimensional non-degeneracy condition, in the sense of the additive hull introduced in [6]. To describe this, we make the following definitions.

Given an axis-aligned box $C \subset \mathbb{R}^k$ and a linear map $\phi : \mathbb{Z}^k \to \mathbb{Z}$ we call $P = \phi(C \cap \mathbb{Z}^k)$ a $k$-dimensional generalised arithmetic progression ($k$-GAP). We say $P$ is $t$-proper if $\phi$ is injective on $tC \cap \mathbb{Z}^k$. For $B \subseteq \mathbb{Z}$ we let $\text{gap}^k_n(B)$ be a smallest set containing $B$ of the form $X + P$ where $|X| \le n$ and $P$ is a (1-)proper $k$-GAP. One can think of $\text{gap}^k_n(B)$ as a parameterised family of hulls that aim to capture the additive structure of $B$.

Theorem 1.2 will follow immediately from the following two results in which we consider separately two regimes for $|B|/|A|$, as when $|B|/|A|$ is small we obtain a stronger bound not depending on this ratio.

**Theorem 1.3.** *For every $k \in \mathbb{N}$ and $t > 0$ there exist $c^{1.3}_{k,t}, e^{1.3}_{k,t}$ so that if $A, B \subset \mathbb{Z}$ with $t|B| \le |A| \le t^{-1}|B|$ and $|\text{gap}^{k-1}_n(B)| \ge e^{1.3}_{k,t}|B|$ then $|A+B|^{1/k} \ge |A|^{1/k} + (1 - c^{1.3}_{k,t} n^{-1})|B|^{1/k}$.*

**Theorem 1.4.** *For every $k \in \mathbb{N}$ there exist $L^{1.4}_k$, $e^{1.4}_k$, $c^{1.4}_k$ so that the following holds for any $A, B \subset \mathbb{Z}$ where $L^{1.4}_k \le 2^j \le (|A|/|B|)^{1/k} \le 2^{j+1}$ with $j \in \mathbb{N}$. If $|\text{gap}^{k-1}_{\ell' n}(\ell' \cdot B)| \ge e^{1.4}_k \ell'^k |B|$ for any $\ell' = 2^{i'}$, $i' \in \{0, \ldots, j+1\}$ then $|A+B|^{1/k} \ge |A|^{1/k} + (1 - c^{1.4}_k n^{-1})|B|^{1/k}$.*

The non-degeneracy conditions in Theorems 1.3 and 1.4 might at first seem awkward to verify, but in fact they hold trivially when applying these theorems to prove Theorem 1.2. Indeed, consider $A, B \subseteq \mathbb{Z}^k$ such that $B$ is not contained in $n$ parallel hyperplanes. We note for any $\ell' \in \mathbb{N}$ that the $\ell'$-fold sumset $\ell' \cdot B$ is not contained in $\ell' n$ parallel hyperplanes. Now fix a linear map $\phi : \mathbb{Z}^k \to \mathbb{Z}$ with sufficiently disparate coefficients that is injective on a large box containing $A$, $A+B$ and $L^{1.4}_k \cdot B$. We note that $|\text{gap}^{k-1}_{\ell' n}(\ell' \cdot B)| > e^{1.4}_k \ell'^k |B|$ for all $\ell' \le L^{1.4}_k$. Thus we can apply Theorem 1.3 or 1.4 to $\phi(A)$ and $\phi(B)$ and deduce Theorem 1.2.

1.2. **Stability.** We will prove Theorems 1.3 and 1.4 by the Stability Method. This is a well-known technique in Extremal Combinatorics, but apparently not as widely exploited in Additive Combinatorics prior to [6].

[1] That is, a set $X \subset \mathbb{Z}^k$ with $|X| = k+n$, which is not covered by $n$ hyperplanes, for which distinct points $x, x' \in X$ have $|x - x'| \gg n$, and so that $|X+X| = \binom{|X|}{2}$

The idea will be to describe the approximate structure of sets $A, B$ as in our theorems for which $|A+B|$ is approximately minimised, from which it will then be straightforward to deduce the required bound on $|A+B|$ from a variant of the Green-Tao discrete Brunn-Minkowski result mentioned above.

When $A$ and $B$ are of roughly comparable sizes, the following result shows that they are dense subsets of translates of the same box.

**Theorem 1.5.** *For every $k \in \mathbb{N}$ and $t > 0$ there exist $n^{1.5}_{k,t}, e^{1.5}_{k,t}, \Delta^{1.5}_{k,t}, f^{1.5}_{k,t} > 0$ so that the following holds. Let $A, B \subset \mathbb{Z}$ with $t|B| \leq |A| \leq t^{-1}|B|$ such that $|\operatorname{gap}^{k-1}_{n^{1.5}_{k,t}}(B)| \geq e^{1.5}_{k,t}|B|$ and $|A+B|^{1/k} \leq |A|^{1/k} + (1+\Delta^{1.5}_{k,t})|B|^{1/k}$. Then there is a 10-proper $k$-GAP $P$ with $|P| \leq f^{1.5}_{k,t}|A|$ such that $A$ and $B$ are contained in translates of $P$.*

In the unbalanced case $|A| \gg |B|$ we obtain a similar structural result for $B$, but for $A$ our description is much weaker (due to applying Plünnecke's inequality), so we will not explicitly state the stability result that follows from the proof. For the case $A = B$, a sharp stability result in this direction was established in $\mathbb{Z}^k$ in [8], which was extended to $\mathbb{Z}$ in [6].

## 2. Lemmas and tools

We start by gathering various tools and lemmas needed for the proofs of our theorems.

2.1. **Coverings.** Our main tool will be the following version of Freiman's Theorem with separation properties, which is a special case of Theorem 2.2 and Theorem 1.4 from [6]. We require the following definitions for the statement. Given an abelian group $G$ and $A, B \subseteq G$ with $0 \in B = -B$, we say $A$ is $B$-separated if $a - a' \notin B$ for all distinct $a, a' \in A$. We say that a $k$-GAP $P = \phi(C \cap \mathbb{Z}^k)$ is $n$-full if each side of $C$ has length at least $n$.

**Theorem 2.1** ([6])**.** *Let $d, s, m \in \mathbb{N}$, $n_1, \dots, n_d \in \mathbb{N}$, and $S \subset \mathbb{Z}$. If $|S+S| \leq 1.9 \cdot 2^d|S|$ then for some $d' \leq d$ we have $S \subset X + P$ where $P$ is an $s$-proper $n_{d'}$-full $d'$-GAP with $P = -P$ and $X+X$ is $m^2 \cdot P/m$-separated, with $|X| = O_{d,n_{d'+1},\dots,n_d}(1)$ and $|P| = O_{d,s,m,n_{d'+1},\dots,n_d}(|S|)$. Moreover, if $|S+S| \leq (2^{d'}+1/2)|S|$ then $|X| = 1$.*

We also use the well-known Ruzsa Covering Lemma [11].

**Lemma 2.2.** *Suppose $A, B \subset \mathbb{Z}$ with $|A+B| \leq K|B|$. Then $A \subset X + B - B$ for some $X \subset \mathbb{Z}$ with $|X| \leq K$.*

2.2. **Iterated sumsets.** Here we collect various results on iterated sumsets. We start with a version of Plünnecke's inequality, in a strengthened form due to Petridis [10].

**Lemma 2.3.** *Let $A, B \subseteq \mathbb{Z}$ with $K = |A+B|/|A| = \min\{|A'+B|/|A'| : \emptyset \neq A' \subset A\}$. Then $|A + \ell \cdot B| \leq K^\ell |A|$ for all $\ell \in \mathbb{N}$.*

Next we require the following result, which is a special case of [6, Theorem 1.4], establishing the case $A = B$ of Theorem 1.2. A less quantitative version of this result was obtained by Freiman (also presented by Bilu [1]). We remark (although it is not needed here) that the asymptotically optimal constant $c^{2.4}_k$ is determined in [6].

**Theorem 2.4.** *For all $k \in \mathbb{N}$ there exist $c_k^{2.4}, e_k^{2.4}$ so that if $A \subset \mathbb{Z}$ with $|\operatorname{gap}_n^{k-1}(A)| \geq e_k^{2.4}|A|$ then $|A+A| \geq 2^k(1-c_k^{2.4}n^{-1})|A|$.*

By iteration we can deduce the following estimate for iterated sumsets.

**Lemma 2.5.** *For every $k$ there exist $c_k^{2.5}, e_k^{2.5} > 0$ so that if $B \subset \mathbb{Z}$ with $|\operatorname{gap}_{\ell' n}^{k-1}(\ell' \cdot B)| \geq e_k^{2.5}\ell'^k|B|$ for all $\ell' = 2^{i'}$ with $0 \leq i' \leq j$ then $|2^j \cdot B| \geq (1-c_k^{2.5}n^{-1})2^{jk}|B|$.*

*Proof of Lemma* 2.5. We let $e_k^{2.5} = 2^k e_k^{2.4}$ and $c_k^{2.5} = 2c_k^{2.4}$. The case $j=0$ is trivial and the case $j=1$ follows from Theorem 2.4. In general, for any $i<j$ we note that if $|2^i \cdot B| \leq 2^{(i+1)k}|B|$ then $|\operatorname{gap}_{2^i n}^{k-1}(2^i \cdot B)| \geq e_k^{2.4}|2^i \cdot B|$, so we can apply Theorem 2.4 to get

$$|2^{i+1} \cdot B| = |2^i \cdot B + 2^i \cdot B| \geq (1-c_k^{2.4}(2^i n)^{-1})2^k|2^i \cdot B|.$$

If this holds for all $i<j$ then the lemma follows from

$$\begin{aligned}|2^j \cdot B| &\geq |B| \prod_{i<j} 2^k(1-c_k^{2.4}(2^i n)^{-1}) \\ &\geq 2^{jk}|B|\Big(1-c_k^{2.4}n^{-1}\sum_{i<j}2^{-i}\Big) \\ &\geq (1-2c_k^{2.4}n^{-1})2^{jk}|B|.\end{aligned}$$

Otherwise, we can consider the largest $i_0 \leq j$ with $|2^{i_0} \cdot B| \geq 2^{(i_0+1)k}|B|$, note that $|2^{i_0+1} \cdot B| \geq |2^{i_0} \cdot B|$, and similarly deduce the lemma from $|2^j \cdot B| \geq |2^{i_0+1} \cdot B| \prod_{i=i_0+1}^{j-1} 2^k(1-c_k^{2.4}(2^i n)^{-1})$. □

Next we require the following result of Lev [9] (see Corollary 1 and its proof) that gives a sharp bound for iterated sumsets (although we state a simpler bound that suffices here).

**Theorem 2.6** ([9]). *Suppose $A \subseteq \{0,\dots,\ell\}$ with $0,\ell \in A$ and $\gcd(A)=1$. Let $A' = \{0,\dots,|A|-2\} \cup \{\ell\}$ and $h \in \mathbb{N}$. Then $|h \cdot A| \geq |h \cdot A'| \geq h\ell - \ell^2/(|A|-2)$.*

We deduce Lemma 2.7 that provides long arithmetic progressions in iterated sumsets.

**Lemma 2.7.** *Suppose $A \subseteq [-\ell,\ell]$ with $0,\ell \in A = -A$ and $|A| > \frac{2\ell+1}{m+1}$, where $m \in [\ell/2]$. Then $r[-2m\ell', 2m\ell'] \subseteq 20m \cdot A$, where $r := \gcd(A) \in [m]$ and $\ell' = \ell/r$.*

*Proof.* We first note that $\frac{2\ell+1}{m+1} < |A| \leq (2\ell+1)/r$, so $r \leq m$. We apply Theorem 2.6 to $A' := A/r \cap [0,\ell']$, obtaining $|10m \cdot A'| \geq 10m\ell' - \ell'^2/(\ell/m-2) \geq 8m\ell'$. Thus $10m \cdot A'$ has density at least $4/5$ in $\operatorname{co}(10m \cdot A') = [0, 10m\ell']$. Now fix any $z \in [-2m\ell', 2m\ell']$ and consider $S := \{(x,y) \in [-10m\ell', 10m\ell']^2 : x<y, x+y=z\}$. Then $|S| \geq 16m\ell'$ and $\{x,y\} \subseteq 10m \cdot (A' \cup -A')$ for all but at most $4m\ell'$ pairs $(x,y) \in S$. Thus we can write $z = x+y$ with $\{x,y\} \subseteq 10m \cdot (A' \cup -A')$, and so $rz \in 20m \cdot A$. □

Now we apply the previous lemma to the following one, that takes a covering of an iterated sumset $\ell \cdot B$ by boxes and shrinks these boxes to obtain a covering by $B$; crucially, the size of the boxes scales correctly with $\ell$, up to a constant only depending on the number of boxes.

**Lemma 2.8.** *Suppose $P$ is a $40m\ell$-proper $k$-GAP and $\ell \cdot B \subseteq X + \ell \cdot P$ with $|X| \le m$. Then $X + 20mm! \cdot P/m!$ contains a translate of $B$.*

*Proof.* By translating, we can assume $0 \in B$. By enlarging $P$, we can assume $P = -P$ and $P$ is $20m\ell$-proper. We can also assume $\ell > 20m$, otherwise we are done by $B \subseteq \ell \cdot B \subseteq X + \ell \cdot P \subseteq X + 20mm! \cdot P/m!$.

Fix any $b \in B$ and consider $\{0, b, \dots, \ell b\} \subseteq \ell \cdot B \subseteq X + \ell \cdot P$. As $|X| \le m$, we can find $Z \subseteq [0, \ell]$ with $|Z| \ge (\ell + 1)/m \ge 4$ such that $bZ$ is contained in a single translate of $\ell \cdot P$. We note that $Z - Z \subseteq [-\ell, \ell]$ is symmetric and $|Z - Z| \ge 2\lceil(\ell + 1)/m\rceil - 1 > (2\ell + 1)/(m + 1)$.

By Lemma 2.7 we find $r[-2m\ell', 2m\ell'] \subseteq 20m \cdot (Z - Z)$, where $r := \gcd(Z - Z) \in [m]$ and $\ell' = r^{-1} \max(Z - Z) \ge |Z| - 1 \ge \ell/2m$. As $b(Z - Z) \subseteq 2\ell \cdot P$ we deduce $r[-\ell, \ell]b \subseteq 20m[-\ell, \ell]b \subseteq 20m\ell \cdot P$.

To complete the proof, it suffices to show that $b \in 20m \cdot P/r$. To see this, recalling that $P$ is $20m\ell$-proper, we write $P = \phi(C \cap \mathbb{Z}^k)$ for some box $C \subset \mathbb{R}^k$ and linear map $\phi : \mathbb{Z}^k \to \mathbb{Z}$ injective on $m!\ell C \cap \mathbb{Z}^k$. Then $\phi^{-1}(r[-\ell, \ell]b)$ is an arithmetic progression in $20m\ell C$, so $\phi^{-1}(rb) \in 20mC$, giving $b \in 20m \cdot P/r$. □

2.3. **Brunn-Minkowski in boxes.** In Lemma 2.9 we adapt a Brunn-Minkowski result of Green and Tao [5] to the setting of dense sets in unbalanced boxes. For $t \in \mathbb{R}$ we write $(t)_+ = \max\{t, 0\}$.

**Lemma 2.9.** *Let $P$ be an $n$-full $(\ell + 1)$-proper $d$-GAP and let $Y \subset P$, $Z \subset \ell \cdot P$ be non-empty. Then*

$$|Y + Z|^{1/d} \ge (|Y| - dn^{-1}|P|)_+^{1/d} + |Z|^{1/d}.$$

*Proof.* Following Green and Tao [5], we introduce a cube summand: for non-empty sets $A, B \subset \mathbb{Z}^d$ we have (writing the measure space in the index)

$$\begin{aligned} \left|A + B + \{0,1\}^d\right|_{\mathbb{Z}^d}^{1/d} &= \left|A + B + \{0,1\}^d + [0,1]^d\right|_{\mathbb{R}^d}^{1/d} \\ &= \left|(A + [0,1]^d) + (B + [0,1]^d)\right|_{\mathbb{R}^d}^{1/d} \\ &\ge \left|A + [0,1]^d\right|_{\mathbb{R}^d}^{1/d} + \left|B + [0,1]^d\right|_{\mathbb{R}^d}^{1/d} = |A|_{\mathbb{Z}^d}^{1/d} + |B|_{\mathbb{Z}^d}^{1/d}. \end{aligned} \tag{2.1}$$

We may assume $P = \prod_{i=1}^d [0, n_i]$, where each $n_i \ge n$. Compress $Y$ and $Z$ onto the coordinate planes and note that this does not increase $|Y + Z|$ (see [5, Lemma 2.8]). Letting $Y' := \{y \in Y : y + \{0,1\}^d \subset Y\}$, we note that if $x, x + (1, 1, \dots, 1) \in Y$ then $x + \{0,1\}^d \subset Y$. As $Y$ is compressed, this implies that $Y \setminus Y'$ contains at most one point in every translate of $\mathbb{R}(1, \dots, 1)$. There are fewer than $\sum_i |P|/n_i \le dn^{-1}|P|$ such lines intersecting $P$, so $|Y'| \ge |Y| - dn^{-1}|P|$. We may assume $|Y| - dn^{-1}|P| > 0$, otherwise the lemma is immediate from $|Y + Z| \ge |Z|$. As $Y' + \{0,1\}^d \subset Y$, applying (2.1) to $Y'$ and $Z$ proves the lemma. □

The inequality in Lemma 2.10 will arise when applying the preceding lemmas to sets covered by several boxes. For the statement we let $\Sigma$ denote the counting measure on $\mathbb{Z}$, so that $\Sigma(f) = \sum_{x \in \mathbb{Z}} f(x)$.

**Lemma 2.10.** *Let $f, g, h\colon \mathbb{Z} \to \mathbb{R}_{\ge 0}$ so that $\Sigma(f), \Sigma(g) > 0$ and for all $x, y \in \mathbb{Z}$ with $f(x), g(y) > 0$ we have $h(x + y)^{1/d} \ge f(x)^{1/d} + g(y)^{1/d}$. Then $\Sigma(h)^{1/d} \ge \Sigma(f)^{1/d} + \Sigma(g)^{1/d}$.*

*Moreover, if $f$ or $g$ is supported on more than one integer then $\Sigma(h)^{1/d} \ge (1 + \Delta_{t,d}^{2.10})(\Sigma(f)^{1/d} + \Sigma(g)^{1/d})$, where $\Delta_{t,d}^{2.10} > 0$ only depends on $d$ and $t := \Sigma(f)/\Sigma(g)$.*

*Proof.* The first statement follows from the Brunn-Minkowski inequality as applied to $A_f + A_g \subseteq A_h$ where $A_f := \bigcup_{x\in\mathbb{Z}} xe_1 + \epsilon\left(0, f(x)^{1/d}\right)^d$, noting that $|A_f| = \epsilon^d\Sigma(f)$, etc. The second statement follows by stability (see e.g. [7] or [2]), noting that if $f$ is supported on more than one integer then $|\operatorname{co}(A_f)|/|A_f| \to \infty$ as $\epsilon \to 0$. □

## 3. Proofs

We are now ready to prove our main results. Considering Lemma 2.9 (Brunn-Minkowski in boxes), we see that Theorem 1.5 implies Theorem 1.3. Thus it remains to prove Theorem 1.5 and Theorem 1.4.

*Proof of Theorem* 1.5. Let $k, d \in \mathbb{N}$, $t \in (0,1)$ and fix constants satisfying

$$k/t \ll d \ll n_d \ll \cdots \ll n_1 \ll r \ll n.$$

Let $A, B \subset \mathbb{Z}$ with $t|B| \le |A| \le t^{-1}|B|$ such that $|\operatorname{gap}_n^{k-1}(B)| \ge n|B|$ and $|A+B|^{1/k} \le |A|^{1/k} + (1+n^{-1})|B|^{1/k}$. Write $D = |\operatorname{co}(A\cup B)|$. After translation, we can assume that the distance between $A$ and $B$ is at least $rD$. Let $S = A\cup B$. By Plünnecke's inequality, we find $|A+A|, |B+B| = O_{k,t}(|A|)$, so $|S+S| = O_{k,t}(|S|)$. We fix $d = O_{k,t}(1)$ so that $|S+S|/|S| \le 1.9\cdot 2^d$. Then we apply Theorem 2.1 to $S$ with $s = m = 10$ and $n_i$ as above to obtain $S \subset X + P$, where $P$ is a 10-proper $n_{d'}$-full $d'$-GAP with $P = -P$ for some $d' \le d$ and $X + X$ is $2\cdot P$-separated, such that $|X| + |P|/|S| \ll n_{d'}$. By the non-degeneracy condition $|\operatorname{gap}_n^{k-1}(B)| \ge n|B|$ we see that $d' \ge k$.

Next we claim that any translate of $P$ intersects only one of $A$ and $B$. To see this, we write $P = \phi(C\cap\mathbb{Z}^{d'})$ for some box $C = \prod_{i=1}^{d'}[0, a_i]$, where each $a_i \ge n_{d'}$. Due to the distance between $A$ and $B$, we can find $i \in [d']$ such that $x_i := \phi(b_ie_i)$ with $b_i := \lfloor a_i/n_{d'}\rfloor$ satisfies $|x_i| > 2D$. Writing $P = [0, a_i]\phi(e_i) + P'$, we see that at most $b_i$ translates $t\phi(e_i) + P'$ intersect $A$ or $B$. However, this gives $|A| + |B| \le |X||P|b_i/a_i$, which contradicts $|X| + |P|/|S| \ll n_{d'}$. Thus the claim holds, so we can find disjoint $Y, Z \subset X$ with $A \subset Y + P$ and $B \subset Z + P$.

For each $y \in Y$ and $z \in Z$ we write $A_y := A \cap y + P$, $B_z := B \cap z + P$ and $(A+B)_{y+z} := (A+B) \cap (y+z+P+P)$. As $X + X$ is $2\cdot P$-separated, the sets $A_y + B_z$ are pairwise disjoint. By Lemma 2.9, we have $|A_y + B_z|^{1/d'} \ge |A_y|^{1/d'} + (|B_z| - d'n_{d'}^{-1}|P|)^{1/d'}$. Thus we can apply Lemma 2.10 to the functions $f(y) = |A_y|$, $g(z) = |B_z| - d'n_{d'}^{-1}|P|)$, $h(x) = |(A+B)_x|$ (extended to the domain $\mathbb{Z}$ with zeroes where undefined), obtaining

$$|A+B|^{1/d'} \ge \Sigma(h)^{1/d'} \ge \Sigma(f)^{1/d'} + \Sigma(g)^{1/d'} \ge Q := |A|^{1/d'} + \left((1-n_{d'}^{-0.9})|B|\right)^{1/d'},$$

where we used $|X| + |P|/|S| \ll n_{d'}$ to estimate $\Sigma(g) \ge |B| - |X|d'n_{d'}^{-1}|P| \ge (1 - n_{d'}^{-0.9})|B|$.

We claim that this implies $d' = k$. Indeed, suppose $d' > k$, let $\theta = Q^{-d'}|A|$, and note that $\min\{\theta, 1-\theta\} > t/2$ as $t|B| \le |A| \le t^{-1}|B|$. Let $S := (1-t/2)^{1/k-1/d'}$ and consider $\theta^{1/k} + (1-\theta)^{1/k} \le S(\theta^{1/d'} + (1-t)^{1/d'}) = S$. We find that

$$|A+B|^{1/k} \ge Q^{d'/k} \ge S^{-1}Q^{d'/k}(\theta^{1/k} + (1-\theta)^{1/k}) = S^{-1}(|A|^{1/k} + \left((1-n_{d'}^{-0.9})|B|\right)^{1/k}).$$

However, this contradicts $|A+B|^{1/k} \le |A|^{1/k} + (1+n^{-1})|B|^{1/k}$, so we conclude $d' = k$.

Moreover, considering the stability statement in Lemma 2.10, we deduce that $|Y| = |Z| = 1$, i.e. $A$ and $B$ are each contained in one translate of $P$. □

*Proof of Theorem* 1.4. Fix constants satisfying $n \geq c \geq L \gg k$. Let $A, B \subset \mathbb{Z}$ where $L \leq \ell \leq (|A|/|B|)^{1/k} \leq 2\ell$ with $\ell = 2^j$, $j \in \mathbb{N}$. Suppose $|\operatorname{gap}^{k-1}_{\ell' n}(\ell' \cdot B)| \geq n\ell'^k|B|$ for any $\ell' = 2^{i'}$, $i' \in \{0, \dots, j+1\}$.

Fix $\emptyset \neq A' \subset A$ which minimizes $|A'+B|/|A'|$. It suffices to show the result for $A'$, i.e. that $|A'+B|^{1/k} \geq |A'|^{1/k} + (1-cn^{-1})|B|^{1/k}$, as this will imply $(|A+B|/|A|)^{1/k} \geq (|A'+B|/|A'|)^{1/k} \geq 1 + (1-cn^{-1})(|B|/|A|)^{1/k}$.

We can assume $|A'+B|/|A'| \leq |A+B|/|A| \leq (1+(|B|/|A|)^{1/k})^k$. By Lemma 2.3 (Petridis' form of Plünnecke's inequality) we deduce

$$|\ell \cdot B| \leq |A' + \ell \cdot B| \leq |A'|(1+(|B|/|A|)^{1/k})^{k\ell} \leq |A|(1+1/\ell)^{k\ell} \leq e^k|A| \leq (2e\ell)^k|B|.$$

Considering $\frac{2|\ell \cdot B|}{\ell^k|B|} = \prod_{i=0}^{j-1} \frac{|2^{i+1} \cdot B|}{2^k|2^i \cdot B|}$, we see that we can choose $\ell' = 2^{i'}$ such that $\frac{|2\ell' \cdot B|}{2^k|\ell' \cdot B|} \leq (2(2e)^k)^{1/j} < 1 + 2^{-2k}$, as $j \geq \log_2 L \gg k$. By Freiman's Theorem, we deduce $\ell' \cdot B \subseteq P'$ for some 80-proper $d$-GAP $P'$ with $d \leq k$ and $|P'| \leq O_k(|\ell' \cdot B|) \leq O_k(\ell'^k|B|)$, where the final inequality holds as otherwise we would contradict $|\ell \cdot B| \leq O_k(\ell^k|B|)$ by repeated application of Lemma 2.5 (Freiman-Bilu for iterated sumsets).

By the non-degeneracy condition $|\operatorname{gap}^{k-1}_{\ell' n}(\ell' \cdot B)| \geq n\ell'^k|B| \gg |P|$ we have $d = k$ and $P'$ is $\ell' n$-full. As $P'$ is 80-proper we have $\ell' \cdot B \subseteq P' \subseteq \ell' \cdot P''$ for some $40\ell'$-proper $n$-full $k$-GAP $P'' \subseteq P'$ with $\ell'^k|P''| = O_k(|P'|)$. Applying Lemma 2.8 we find $B \subseteq P$ for some translate $P$ of $20 \cdot P''$, where $|P| = O_k(|P''|) = O_k(|B|)$.

We let $Q = \ell \cdot P$, noting that $\ell \cdot B \subseteq Q$ and $|Q| \leq O_k(|\ell \cdot B|)$. Recalling $|A' + \ell \cdot B| = O_k(|\ell \cdot B|)$, by the Ruzsa Covering Lemma we have $A' \subseteq X + Q - Q$ with $|X| = O_k(1)$. Next we apply a merging process, where we start with $X' = X$ and $Q' = Q - Q$, and if we find any $x, y \in X'$ such that $x + Q' + P$ intersects $y + Q' + P$ then we replace $X'$ by $X' \setminus \{y\}$ and $Q'$ by $6 \cdot Q'$, noting that $y + Q' + P \subseteq (x + Q' - Q' + P - P) + Q' + P \subseteq x + 6 \cdot Q'$. Thus we terminate with $A' \subseteq X' + Q'$ for some $X' \subseteq X$ with $|Q'| \leq O_k(|Q|) \leq O_k(|\ell \cdot B|)$.

By the non-degeneracy condition $|\operatorname{gap}^{k-1}_{\ell n}(\ell \cdot B)| \geq n\ell^k|B| \gg |Q'|$ we can assume that $Q'$ is 2-proper (otherwise $Q'$ would be contained in a $(k-1)$-GAP of size $O_k(|Q'|)$; see e.g. [6, Lemma 5.4]). Recalling that $P$ is $n$-full, applying Lemma 2.10 as in the proof of Theorem 1.5 we deduce

$$|A' + B|^{1/k} \geq |A'|^{1/k} + (|B| - kn^{-1}|P|)^{1/k}.$$

As $|P| = O_k(|P''|) = O_k(|B|)$ this concludes the proof. □

## 4. Concluding remarks

Given Theorem 1.2 it would be interesting to determine the optimal constants $c^{1.2}_{k,t}$. In this direction the following asymptotically optimal result was derived in [6].

**Theorem 4.1.** *For all $k \in \mathbb{N}$, there exists $e_k$, and for all $n \in \mathbb{N}$ there exist $m_{n,k}$ so that if $A \subset \mathbb{Z}$ satisfies $|\operatorname{gap}^{k-2}_{m_{n,k}}(A)|, |\operatorname{gap}^{k-1}_n(A)| \geq e_k|A|$ then $|A+A| \geq 2^k(1 - (1+o(1))\frac{k}{4}n^{-1})|A|$, where $o(1) \to 0$ as $n \to \infty$.*

This result establishes, up to the $1+o(1)$ term, the optimal constant as shown by considering a discrete cone (see [6, Example 3.19]). However, in the context of this paper it is natural to ask for the optimal constant without the additional condition

$|\operatorname{gap}^{k-2}_{m_{n,k}}(A)| \geq e_k|A|$, the discrete simplex $S_n := \{x \in [0,n]^k : \sum_i x_i \leq n\}$ gives a worse constant. We leave you with Question 4.2.

**Question 4.2.** What is the smallest constant $c_{k,t}$ for which Theorem 1.2 holds? In particular, among all sets in $\mathbb{Z}^k$ not contained in $n$ hyperplanes does the discrete simplex $S_n$ have minimal doubling?

## References


[1] Yuri Bilu, *Structure of sets with small sumset* (English, with English and French summaries), Astérisque **258** (1999), xi, 77–108. Structure theory of set addition. MR1701189

[2] Alessio Figalli and David Jerison, *Quantitative stability for the Brunn-Minkowski inequality*, Adv. Math. **314** (2017), 1–47, DOI 10.1016/j.aim.2016.12.018. MR3658711

[3] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 3, 355–405, DOI 10.1090/S0273-0979-02-00941-2. MR1898210

[4] R. J. Gardner and P. Gronchi, *A Brunn-Minkowski inequality for the integer lattice*, Trans. Amer. Math. Soc. **353** (2001), no. 10, 3995–4024, DOI 10.1090/S0002-9947-01-02763-5. MR1837217

[5] B. Green and T. Tao, *Compressions, convex geometry and the Freiman-Bilu theorem*, Q. J. Math. **57** (2006), no. 4, 495–504, DOI 10.1093/qmath/hal009. MR2277597

[6] Peter van Hintum and Peter Keevash, *Locality in sumsets*, Preprint, arXiv:2304.01189, 2023.

[7] Peter van Hintum and Peter Keevash, *The sharp doubling threshold for approximate convexity*, Bull. Lond. Math. Soc. **56** (2024), no. 10, 3229–3239, DOI 10.1112/blms.13129. MR4816757

[8] Peter van Hintum, Hunter Spink, and Marius Tiba, *Sets in $\mathbb{Z}^k$ with doubling $2^k + \delta$ are near convex progressions*, Adv. Math. **413** (2023), Paper No. 108830, 72, DOI 10.1016/j.aim.2022.108830. MR4530619

[9] Vsevolod F. Lev, *Structure theorem for multiple addition and the Frobenius problem*, J. Number Theory **58** (1996), no. 1, 79–88, DOI 10.1006/jnth.1996.0065. MR1387726

[10] Giorgis Petridis, *New proofs of Plünnecke-type estimates for product sets in groups*, Combinatorica **32** (2012), no. 6, 721–733, DOI 10.1007/s00493-012-2818-5. MR3063158

[11] Imre Z. Ruzsa, *An analog of Freiman's theorem in groups* (English, with English and French summaries), Astérisque **258** (1999), xv, 323–326. Structure theory of set addition. MR1701207

[12] Imre Z. Ruzsa, *Additive combinatorics and geometry of numbers*, International Congress of Mathematicians. Vol. III, Eur. Math. Soc., Zürich, 2006, pp. 911–930. MR2275712

[13] Terence Tao and Van Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics, vol. 105, Cambridge University Press, Cambridge, 2006, DOI 10.1017/CBO9780511755149. MR2289012



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