Discrete Comput Geom (2011) 46: 48–71
DOI 10.1007/s00454-011-9325-0

Bounded Direction–Length Frameworks

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Received: 30 July 2009 / Revised: 13 January 2011 / Accepted: 13 January 2011 /
Published online: 27 January 2011

Abstract A direction–length framework is a pair (G, p) where $G = (V; D, L)$ is a 'mixed' graph whose edges are labelled as 'direction' or 'length' edges and p is a map from V to $\mathbb{R}^d$ for some d. The label of an edge uv represents a direction or length constraint between $p(u)$ and $p(v)$. Let G^+ be obtained from G by adding, for each length edge e of G, a direction edge with the same end vertices as e. We show that (G, p) is bounded if and only if (G^+, p) is infinitesimally rigid. This gives a characterization of when (G, p) is bounded in terms of the rank of the rigidity matrix of (G^+, p). We use this to characterize when a mixed graph is generically bounded in $\mathbb{R}^d$. As an application we deduce that if (G, p) is a globally rigid generic framework with at least two length edges and e is a length edge of G then $(G \setminus e, p)$ is bounded.

Keywords Direction-length frameworks · Boundedness · Rigidity · Global rigidity

1 Introduction

Graphs with geometrical constraints provide natural models for a variety of applications, including Computer-Aided Design, sensor networks and flexibility in molecules. Given a graph G and prescribed lengths for its edges, a basic problem is to determine whether G has a straight line realisation in Euclidean d-dimensional space with these given lengths. We refer to such a realisation as a framework. Given a

Research supported in part by ERC grant 239696 and EPSRC grant EP/G056730/1.

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framework, one may also ask whether it is unique, either globally or with respect to local movement (rigidity). The rigidity question has a strong mathematical pedigree, going back to a conjecture of Euler [4] that every 3-dimensional polyhedron is rigid, when viewed as a 'panel-and-hinge' framework. (Connelly [2] gave a counterexample to this conjecture in 1978.) Saxe [19] has shown that both the existence and global uniqueness problems are NP-hard. However, this hardness relies on algebraic relations between co-ordinates of vertices, and for practical purposes it is natural to study generic realisations. Laman [12] (see also [14]) gave a combinatorial characterization for when a graph is rigid in any generic 2-dimensional realisation. Jackson and Jordan [7] gave a combinatorial characterization for when a graph is globally rigid in two dimensions, i.e. when every generic realisation is a unique realisation. No combinatorial characterizations are known in higher dimensions, although it is possible to give conditions in terms of the ranks of certain matrices. Other natural geometrical constraints include directions and angles, which arise in parallel drawing and map making. Combinatorial characterizations of generic rigidity were given for direction constraints in d-dimensions by Whiteley [22], and for mixed direction and length constraints in 2-dimensions by Servatius and Whiteley [20]. No characterization is known when angle constraints are involved, even in 2-dimensions.

Matroid theory is a valuable tool in many of these geometric problems. We refer the reader to [22] for a comprehensive survey of this method. Generic rigidity can be analysed in terms of a matroid defined on the edge set of a complete graph with n vertices in which the spanning sets in the matroid correspond to the rigid graphs with n vertices. Laman's characterization of 2-dimensional generic rigidity is a description of the bases of the corresponding matroid: for n vertices, a set E of edges is a basis if and only if $|E| = 2n - 3$ and $|E'| \leq 2|V(E')| - 3$ for all non-empty subsets E' of E. An equivalent description, using a theorem of Nash-Williams [17], is that for every $e \in E$, the graph obtained from G by adding a new edge with the same end-vertices as e, is the edge-disjoint union of two spanning trees (see Sect. 3.3 in [22] for discussion of this connection).

Servatius and Whiteley [20] gave an analogous counting characterization for rigidity of generic 2-dimensional frameworks which have constraints involving lengths and directions. In the corresponding matroid for n vertices, a set $E = D \cup L$ of edges is a basis if and only if (i) $|E| = 2n - 2$, (ii) $|E'| \leq 2|V(E')| - 2$ for all non-empty subsets E' of E, and (iii) $|E'| \leq 2|V(E')| - 3$ for all pure non-empty subsets E' of E, where a set of edges is *pure* if it involves either only the length constraints L or only the direction constraints D. Equivalently, using Nash-Williams' theorem, E is the disjoint union of two spanning trees, $D \cup \{e\}$ is the disjoint union of two forests for every $e \in D$, and $L \cup \{e\}$ is the disjoint union of two forests for every $e \in L$.

In this paper, we consider when a framework has the property that there is an absolute bound for the diameter of any framework that satisfies the same length and direction constraints as the original framework. This is a question of independent interest and it also gives insight on the question of global uniqueness. We characterize boundedness as rigidity of an augmented framework, and show that this is determined by the rank of its rigidity matrix. This enables us to obtain a combinatorial characterization for boundedness of d-dimensional generic direction–length frameworks. There are several known $O(n^2)$ algorithms for testing generic rigidity of

2-dimensional frameworks (see [1] for one algorithm and references to others). We will indicate how these algorithms can be adapted to test for generic rigidity in augmented direction–length frameworks. This implies that the boundedness of d-dimensional generic direction–length frameworks can be decided in $O(n^2)$ time.

An outline of the paper is as follows. We give precise definitions of the terms used in the above discussion of rigidity in the next section. In Sect. 3, we characterize rigidity for a special class of direction–length frameworks, namely those in which every length constraint is accompanied by a direction constraint with the same end vertices. The following section introduces a new geometrical scenario involving balls with directional constraints. We characterize boundedness and generic boundedness for such configurations. The boundedness characterization of ball–direction frameworks is then used in Sect. 5 to characterize boundedness for direction–length frameworks. We apply this result in Sect. 6 to give a combinatorial characterization for bounded d-dimensional generic frameworks. In Sect. 7, we show how to find the bounded components of a direction–length generic framework. Section 8 concerns necessary conditions for global rigidity: we show that if (G, p) is a globally rigid generic framework with at least two length edges and e is a length edge of G then $(G \setminus e, p)$ is bounded. The final section contains a summary of our results.

2 Definitions

Our graphs will not have loops but may have parallel edges. A *mixed graph* $G = (V; D, L)$ consists of a graph G on a vertex set V in which the edge set E is partitioned into two parts D and L. We refer to edges in D as *direction edges* and edges in L as *length edges*.

A *d-dimensional direction–length framework* is a pair (G, p) where $G = (V; D, L)$ is a mixed graph and p is a map from V to $\mathbb{R}^d$ such that $p(u) \neq p(v)$ for all $uv \in L$. We say that (G, p) is a *direction–length realisation* of G in $\mathbb{R}^d$.

Two direction–length frameworks (G, p) and (G, q) are *equivalent* if $q(u) - q(v)$ is a scalar multiple of $p(u) - p(v)$ for all $uv \in D$ with $p(u) \neq p(v)$ and $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for all $uv \in L$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. They are *congruent* if there exists a $\lambda \in \{1, -1\}$ such that $p(u) - p(v) = \lambda(q(u) - q(v))$ for all $u, v \in V$, i.e. (G, q) can be obtained from (G, p) by a translation and a dilation by ± 1.

The direction–length framework (G, p) is *globally rigid* if every framework which is equivalent to (G, p) is congruent to (G, p).

We say (G, p) is *rigid* if there exists an $\varepsilon > 0$ such that if a framework (G, q) is equivalent to (G, p) and satisfies $\|p(v) - q(v)\| < \varepsilon$ for all $v \in V$ then (G, q) is congruent to (G, p). Equivalently, every continuous motion of the points $p(v)$, $v \in V$ respecting the length and direction constraints results in a framework which is congruent to (G, p).

The above definitions are illustrated in Fig. 1.

A direction–length framework (G, p) is *bounded* if there exists a real number K such that $\|q(u) - q(v)\| < K$ for all $u, v \in V$ whenever (G, q) is a framework equivalent to (G, p), see Fig. 2.

A direction–length framework (G, p) is *generic* if the set containing the coordinates of all of the vertices is algebraically independent over the rationals.

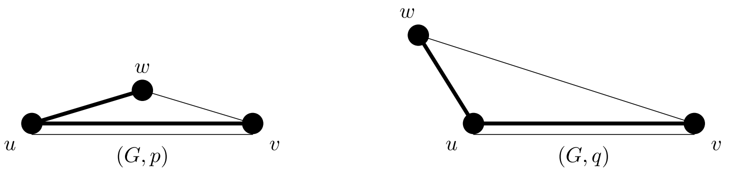


Fig. 1 Two equivalent but non-congruent 2-dimensional direction–length frameworks. We use *thick* or *thin lines* to indicate edges representing length or direction constraints, respectively. The frameworks are rigid but not globally rigid

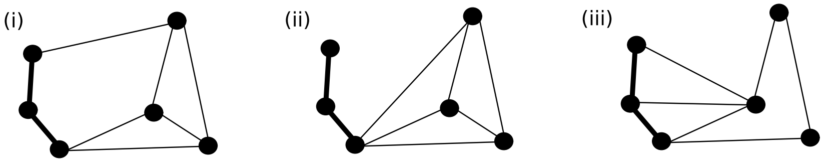
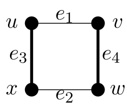


Fig. 2 2-dimensional direction–length frameworks, (i) bounded, (ii) & (iii) unbounded



$$\begin{array}{c|cccc|} & u & v & w & x \\ e_1 & [p(u)-p(v)]^\perp & [p(v)-p(u)]^\perp & 0 & 0 \\ e_2 & 0 & 0 & [p(w)-p(x)]^\perp & [p(x)-p(w)]^\perp \\ e_3 & 0 & [p(v)-p(w)] & [p(w)-p(v)] & 0 \\ e_4 & [p(u)-p(x)] & 0 & 0 & [p(x)-p(u)] \end{array}$$

Fig. 3 A 2-dimensional direction–length framework and its rigidity matrix

We say that a property P of frameworks is *generic* if whenever some generic realisation of a graph G has property P then all generic realisations of G have property P. If P is a generic property then we say that a graph G has property P if some generic realisation of G has property P (or equivalently, all generic realisations of G have property P).

Given a realisation p of G and a direction edge $e = uv$ we let B_e be a $(d-1) \times d$-matrix whose rows are a basis for the subspace of $\mathbb{R}^d$ orthogonal to $\langle p(u) - p(v)\rangle$ if $p(u) \neq p(v)$, and $B_e = 0$ otherwise. A *rigidity matrix* for (G, p) is a $((d-1)|D| + |L|) \times d|V|$ matrix $R(G, p)$ constructed as follows. We first choose an arbitrary reference orientation for the edges of D, and use the notation $e = uv$ to mean that e has been oriented from u to v. Each edge in D corresponds to $d-1$ consecutive rows of $R(G, p)$, each edge in L to one row of $R(G, p)$, and each vertex in V to d consecutive columns of $R(G, p)$. The submatrix of $R(G, p)$ with rows labelled by $e = uv \in D$ and columns labelled by $x \in V$ is B_e if $x = u$, is $-B_e$ if $x = v$, and is the $(d-1) \times d$ zero matrix otherwise. The submatrix of $B(G, p)$ with row labelled by $e = uv \in L$ and columns labelled by $x \in V$ is $p(u) - p(v)$ if $x = u$, is $p(v) - p(u)$ if $x = v$, and is zero otherwise, see Fig. 3.

Let $Z(G, p)$ be the null space of $R(G, p)$. We refer to vectors in $Z(G, p)$ as *infinitesimal motions* of (G, p). The labelling of the columns of $R(G, p)$ allows us to consider each infinitesimal motion z as a map from V to $\mathbb{R}^d$ with the properties that $B_{uv}(z(u) - z(v)) = 0$ for all $e = uv \in D$ and $(p(u) - p(v)) \cdot (z(u) - z(v)) = 0$ for all

$uv \in L$. For $e = uv \in D$ the condition $B_e(z(u) - z(v)) = 0$ is equivalent to $z(u) - z(v)$ being parallel to $p(u) - p(v)$. It follows that $Z(G, p)$, and hence also rank $R(G, p)$, depend only on the framework (G, p): they are independent of the choice of the bases B_e, $e \in D$. Nevertheless, it will sometimes serve our purposes to refer to a particular rigidity matrix which we call the *standard rigidity matrix of* $R(G, p)$. This is defined as follows: for each $e = uv \in D$ and $p(u) - p(v) = (a_1, \dots, a_d)$, we take the rows of B_e as the vectors $b^1, \dots, b^{d-1}$, where b^i is equal to a_d in co-ordinate i, to $-a_i$ in co-ordinate d, and 0 in the other co-ordinates.

For any $a \in \mathbb{R}^d$ the translation given by $z(v) = a$ for all $v \in V$ is an infinitesimal motion, so $\dim Z(G, p) \geq d$ and rank $R(G, p) \leq d|V| - d$. We say that the framework (G, p) is *infinitesimally rigid* if rank $R(G, p) = d|V| - d$, and is *independent* if the rows of $R(G, p)$ are linearly independent. Infinitesimal rigidity and independence are both generic properties of graphs, as the rank of $R(G, p)$ is the same for all generic realisations of G. To see this, note that it is determined by the maximum size of a square submatrix of $R(G, p)$ which has non-zero determinant. If we take $R(G, p)$ to be the standard rigidity matrix for (G, p), then its entries are linear functions of the co-ordinates of the points $p(v)$. Thus the relevant determinants are non-zero polynomials in the co-ordinates, so are non-zero at all generic realisation whenever they are not identically zero. We denote the rank of the rigidity matrix of a generic realisation of G in $\mathbb{R}^d$ by $r_d(G)$. Then G is independent in $\mathbb{R}^d$ if $r_d(G) = (d-1)|D| + |L|$ and infinitesimally rigid if $r_d(G) = d|V| - d$.

A *matroid* $M = (E, I)$ consists of a ground set E and a collection I of subsets of E called *independent sets*, such that (i) $\emptyset \in I$, (ii) if $A \in I$ and $B \subseteq A$ then $B \in I$, and (iii) for any $E' \subseteq E$ there is a number $r(E')$, called the *rank* of E', such that any maximal independent subset of E' has size $r(E')$. We refer the reader to [18] for an introduction to the theory of matroids. Given a matrix R, one can define a matroid $M(R)$ in which the ground set E corresponds to rows of R and a subset of E is independent in $M(R)$ if and only if the corresponding rows of R are linearly independent. This definition depends on the field, which will always be taken as the real numbers in this paper. Conversely, given a matroid M, we say that R is a linear representation of M if $M(R) = M$. Another matroid that will be used in this paper is the *cycle matroid* $M(G)$ of a graph G: this has ground set $E = E(G)$ and a set $A \subseteq E$ is independent if it forms an acyclic subgraph of G. The standard rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p): the ground set $(d-1)D \cup L$ corresponds to rows of the rigidity matrix, and a subset is independent when the corresponding rows are linearly independent. Any two generic realisations of G have the same rigidity matroid, which we call the *rigidity matroid* of G.

3 A Special Class of Direction–Length Frameworks

Characterising when a direction–length framework is rigid seems to be a difficult problem. The only known result is the above-mentioned characterization by Servatius and Whiteley of rigid 2-dimensional generic direction–length frameworks. Infinitesimal rigidity is a sufficient condition for rigidity, as shown by Lemma 5.1 in [10] (the proof there is written for 2-dimensional frameworks but it can be easily modified

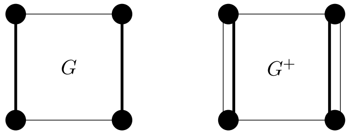


Fig. 4 A mixed graph G and its augmentation G^+

for the general case). Rigidity and infinitesimal rigidity are not equivalent in general, since a framework may have infinitesimal motions which are not induced by continuous motions. (They are equivalent, however, for generic frameworks, see Lemma 8.1 below.)

In this section, we consider the following special class of framework. Given a mixed graph G, we construct the *augmented graph* G^+ by adding a direction edge with the same end vertices as e, for each length edge e of G, see Fig. 4. The following theorem shows that rigidity and infinitesimal rigidity are equivalent for any given realisation (G^+, p) of G^+, using a relationship between the infinitesimal motions and equivalent realisations of (G^+, p).

Theorem 3.1 *Let $G = (V; D, L)$ be a mixed graph and (G, p) be a realisation of G.*

(a) *Suppose that z is an infinitesimal motion of (G^+, p). Then $z(u) = z(v)$ for all $uv \in L$. Also, if $q = p + z$ then (G^+, q) is equivalent to (G^+, p).*
(b) *Suppose (G^+, q) is equivalent to (G^+, p) and for all $v \in V$ we have $\|p(v) - q(v)\| < \delta := \min_{xy\in L} \|p(x) - p(y)\|$. Then $z = q - p$ is an infinitesimal motion of (G^+, p).*
(c) *(G^+, p) is rigid if and only if (G^+, p) is infinitesimally rigid.*

Proof Write $G^+ = (V; D^+, L^+)$, where $D^+ = D \cup L$ and $L^+ = L$.

(a) Suppose $z \in Z(G^+, p)$ and write $q = p + z$. Consider any edge $e = xy \in D^+$ with $p(x) \neq p(y)$. By definition of the rigidity matrix, we have $B_e(z(x) - z(y)) = 0$, and so $B_e(q(x) - q(y)) = B_e(p(x) - p(y))$. By construction, the rows of B_e form a basis for the subspace of $\mathbb{R}^d$ orthogonal to $\langle p(x) - p(y)\rangle$, so $q(x) - q(y) = \lambda(p(x) - p(y))$ for some $\lambda \in \mathbb{R}$.

Next consider any edge $uv \in L$. By definition of the rigidity matrix, we have $(p(u) - p(v)) \cdot (z(u) - z(v)) = 0$. Hence

$$\big(p(u) - p(v)\big) \cdot \big(p(u) - p(v)\big) = \big(p(u) - p(v)\big) \cdot \big(q(u) - q(v)\big). \tag{1}$$

The definition of G^+ implies that we also have $uv \in D^+$. As already shown, this implies that $q(u) - q(v) = \lambda(p(u) - p(v))$ for some $\lambda \in \mathbb{R}$. Substituting into (1) gives $\|p(u) - p(v)\|^2 = \lambda\|p(u) - p(v)\|^2$. Thus $\lambda = 1$ and $p(u) - p(v) = q(u) - q(v)$. This gives $z(u) = z(v)$ for all $uv \in L$. We also have $q(x) - q(y) \in \langle p(x) - p(y)\rangle$ for all $xy \in D^+$ with $p(x) \neq p(y)$, so (G^+, q) is equivalent to (G^+, p) by definition.

(b) Suppose (G^+, q) is equivalent to (G^+, p) and $\|p(v) - q(v)\| < \delta$ for all $v \in V$. Write $z = q - p$. Consider any edge $e = xy \in D^+$ with $p(x) \neq p(y)$. Then $q(x) - q(y) \in \langle p(x) - p(y)\rangle$, so $B_e(q(x) - q(y)) = B_e(p(x) - p(y)) = 0$. Therefore, $B_e(z(x) - z(y)) = 0$.

Next consider any edge $uv \in L$. Then $\|p(u) - p(v)\| = \|q(u) - q(v)\|$. The definition of G^+ implies that we also have $uv \in D^+$. Thus $q(u) - q(v) = \lambda(p(u) - p(v))$

for some $\lambda \in \mathbb{R}$. The last two equalities imply that $p(u)-p(v)=\pm(q(u)-q(v))$. We cannot have $p(u)-p(v)=-(q(u)-q(v))$, as then $2(p(u)-p(v))=p(u)-q(u)+q(v)-p(v)$, so the triangle inequality gives $2\|p(u)-p(v)\| \le \|p(u)-q(u)\| + \|p(v)-q(v)\| < 2\delta$, contradicting the choice of δ. It follows that $p(u)-p(v)=q(u)-q(v)$ and hence $z(u)-z(v)=0$. Recalling that $B_e(z(x)-z(y))=0$ for all $e=xy\in D^+$ with $p(x)\neq p(y)$ we see that $z\in Z(G^+,p)$.

(c) We may use part (b) of the theorem to deduce that (G^+,p) is not infinitesimally rigid if it is not rigid. (This would also follow from the above mentioned result that infinitesimal rigidity is a sufficient condition for rigidity in all direction–length frameworks.) Now suppose that (G^+,p) is not infinitesimally rigid. We need to show that (G^+,p) is not rigid. Choose $\varepsilon>0$. Since $\dim Z(G^+,p)>d$ we can choose $z\in Z(G^+,p)$ that is not a translation by a fixed vector. Multiplying by a real constant we can assume that $\|z\|<\varepsilon$. Let $q=p+z$. Then (G^+,q) is equivalent to (G^+,p) by part (a) of the theorem. But $\|p(v)-q(v)\|<\varepsilon$ for all $v\in V$, (G^+,q) is not congruent to (G^+,p) and ε is arbitrary, so (G^+,p) is not rigid. □

Note that part (a) of Theorem 3.1 implies that any infinitesimal motion of (G^+,p) is constant on the connected components of the graph $F=(V,L)$ obtained by taking just the length edges of G. We can use this observation to characterize rigidity of (G^+,p) in terms of the rank of the following reduced rigidity matrix.

Let $F_1,F_2,\dots,F_m$ be the components of $F=(V,L)$. Let $E\subseteq D$ be the set of direction edges $e=uv$ in G which join distinct components of F and satisfy $p(u)\neq p(v)$. As before, for each $e=uv\in E$ we choose a reference orientation for e and let B_e be a $(d-1)\times d$-matrix whose rows are a basis for the subspace of $\mathbb{R}^d$ orthogonal to $\langle p(u)-p(v)\rangle$. A *reduced rigidity matrix* for (G^+,p) is a $(d-1)|E|\times dm$ matrix $\widetilde{R}(G^+,p)$ constructed as follows. Each edge in E corresponds to $d-1$ consecutive rows and each component of F to d consecutive columns. The submatrix of $\widetilde{R}(G^+,p)$ with rows labelled by $e=uv\in E$ and columns labelled by F_i is B_e if $u\in V(F_i)$ and $v\notin V(F_i)$, $-B_e$ if $v\in V(F_i)$ and $u\notin V(F_i)$, and is the $(d-1)\times d$ zero matrix otherwise. See Fig. 5.

Corollary 3.2 *Let (G,p) be a mixed framework and suppose that $F=(V,L)$ has m components. Then*

(a) *The null spaces of $\widetilde{R}(G^+,p)$ and $R(G^+,p)$ are isomorphic*;
(b) rank $\widetilde{R}(G^+,p)\le dm-d$;
(c) *(G^+,p) is rigid if and only if* rank $\widetilde{R}(G^+,p)=dm-d$.

Proof Let $\widetilde{Z}(G^+,p)$ be the null space of $\widetilde{R}(G^+,p)$. We may consider any $\widetilde{z}\in\widetilde{Z}(G^+,p)$ as a map from $\{F_1,F_2,\dots,F_m\}$ to $\mathbb{R}^d$. Then we can extend $\widetilde{z}$ to a map $z:$

$$\begin{array}{c|cc|} & H_1 & H_2 \\ e_1 & [p(u)-p(v)]^\perp & [p(v)-p(u)]^\perp \\ e_2 & [p(w)-p(x)]^\perp & [p(x)-p(w)]^\perp \end{array}$$

Fig. 5 A reduced rigidity matrix for the direction–length framework in Fig. 3, where H_1 and H_2 are the length components induced by $\{u,x\}$ and $\{v,w\}$, respectively

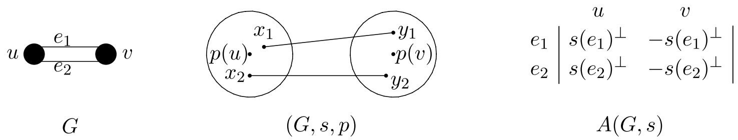


Fig. 6 A graph G, a 2-dimensional ball-direction realisation (G, s, p) of the weighted graph (G, s), and a ball-direction incidence matrix $A(G, s)$ for (G, s). We have $s : \{e_1, e_2\} \to \mathbb{R}^2 \setminus \{0\}$ and $x_i - y_i = \lambda_i s(e_i)$ for some $\lambda_i \in \mathbb{R}$ and all $1 \le i \le 2$

$V \to \mathbb{R}^d$ by setting $z(v) = \tilde{z}(F_i)$ for each $v \in V(F_i)$ and all $1 \le i \le m$. It is straightforward to check that $\tilde{z} \mapsto z$ is an isomorphism between $\tilde{Z}(G^+, p)$ and $Z(G^+, p)$ using Theorem 3.1(a). This proves (a) and implies that $dm - \operatorname{rank} \tilde{R}(G^+, p) = d|V| - \operatorname{rank} R(G^+, p)$. This gives (b) and implies that (G^+, p) is infinitesimally rigid if and only if $\operatorname{rank} \tilde{R}(G^+, p) = dm - d$. Part (c) now follows using Theorem 3.1(c). □

4 Ball–Direction Frameworks

In this section, we introduce a new type of framework and give a characterization for when it is bounded in terms of the rank of an associated matrix. Suppose $G = (V, E)$ is a graph and $s : E \to \mathbb{R}^d - \{0\}$ assigns a non-zero vector $s(e)$ in $\mathbb{R}^d$ to each edge e of G. Our goal is to place unit balls in $\mathbb{R}^d$ corresponding to the vertices of G so that for each edge $e = uv$ there is a line in the direction $s(e)$ that intersects the balls corresponding to u and v. We say that $p : V \to \mathbb{R}^d$ is a *ball–direction realisation* of (G, s) if for each edge $e = uv$ of G there exists $x, y \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ such that $\|x - p(u)\| \le 1$, $\|y - p(v)\| \le 1$ and $x - y = \lambda s(e)$. We refer to the triple (G, s, p) as a *d-dimensional ball–direction framework*. We say that the weighted graph (G, s) is *ball–direction bounded* if there exists $K \in \mathbb{R}$ such that $\|p(u) - p(v)\| < K$ for all $u, v \in V$ for all ball–direction realisations (G, s, p) of (G, s).

A *ball–direction incidence matrix* of (G, s) is a $(d-1)|E| \times d|V|$ matrix $A(G, s)$ defined as follows. For each $e \in E$, let A_e be a $(d-1) \times d$-matrix whose rows are a basis for the subspace of $\mathbb{R}^d$ orthogonal to $s(e)$. We choose an arbitrary reference orientation for the edges of E. Each edge in E corresponds to $d - 1$ consecutive rows and vertex of V to d consecutive columns. The submatrix of $A(G, s)$ with rows labelled by $e = uv \in D$ and columns labelled by $x \in V$ is A_e if $x = u$, is $-A_e$ if $x = v$, and is the $(d-1) \times d$ zero matrix otherwise.

Let $Z(G, s)$ be the null space of $A(G, s)$. The labelling on the columns of $A(G, s)$ allows us to consider each $z \in Z(G, s)$ as a map from V to $\mathbb{R}^d$. For any edge $e = uv$ and $z \in Z(G, s)$ we have $A_e(z(u) - z(v)) = 0$. Hence, if $z(u) \ne z(v)$, the line between $z(u)$ and $z(v)$ is in the direction $s(e)$. The null space $Z(G, s)$ contains the d-dimensional space of translations, so $\dim Z(G, s) \ge d$ and $\operatorname{rank} A(G, s) \le dn - d$.

Theorem 4.1 *Suppose $G = (V, E)$ is a graph on n vertices and $s : E \to \mathbb{R}^d - \{0\}$. Then (G, s) is ball–direction bounded if and only if* $\operatorname{rank} A(G, s) = dn - d$.

Proof Let $A = A(G, s)$. We may assume, without loss of generality, that the rows of A are vectors of unit length. Choose $v_0 \in V$.

First, suppose that rank $A < dn - d$. Since $\dim Z(G, s) \geq d + 1$, we may choose a non-zero $z \in Z(G, s)$ such that $z(v_0) = (0, \ldots, 0) \in \mathbb{R}^d$. Define $p : V \to \mathbb{R}^d$ by $p(v) = \lambda z(v)$ for each $v \in V$ and some fixed $\lambda \in \mathbb{R}$. Then (G, s, p) is a ball–direction realisation of (G, s). Since z is non-zero and λ can be arbitrarily large, (G, s) is unbounded.

Conversely, suppose that rank $A = dn - d$. Consider any ball–direction realisation p of (G, s). By translation we may suppose that $p(v_0) = (0, \ldots, 0)$ for some fixed $v_0 \in V$. Then by definition, for any edge $e = uv$ there are points $q(u) = p(u) + r(u)$ and $q(v) = p(v) + r(v)$ for which $\|r(u)\|, \|r(v)\| \leq 1$ and $q(u) - q(v) = \lambda s(e)$. Then $A_e(q(u) - q(v)) = 0$ and so $A_e(p(u) - p(v)) = A_e(r(u) - r(v))$. Since the rows of A_e are vectors of unit length, the triangle inequality gives $\|A_e(p(u) - p(v))\| \leq (d - 1)\|r(u) - r(v)\| \leq 2(d - 1)$. Now Ap is a vector in $\mathbb{R}^{(d-1)|E|}$ obtained by concatenating the vectors $A_e(p(u) - p(v))$ for each edge $e = uv$, so $\|Ap\| \leq 2(d - 1)|E|$.

Choose a set S of rows of A which form a basis for the row space of A. Let $\widetilde{A}$ be the $(dn - d) \times (dn - d)$-submatrix of A induced by the rows in S and columns indexed by $V - v_0$, and $\widetilde{p}$ the vector obtained from p by removing the d (zero) coordinates corresponding to v_0. For each congruence class modulo d, the sum of the columns of A with index in this congruence class is zero, and so rank $\widetilde{A} =$ rank $A = dn - d$. Hence $\widetilde{A}$ is invertible. Since $\|\widetilde{A}\widetilde{p}\| \leq \|Ap\| \leq 2(d - 1)|E|$, we have $\widetilde{p} \in \{\widetilde{A}^{-1}w : w \in \mathbb{R}^{dn-d}$ and $\|w\| \leq 2(d - 1)|E|\}$. Also $p(v_0) = (0, \ldots, 0)$, so p belongs to a bounded region of $\mathbb{R}^{dn}$. Thus $\|p(v_0) - p(v)\|$ is bounded for all $v \in V$ by a constant depending only on G and s, and hence (G, s) is bounded. □

A graph $G = (V, E)$ is said to be *ball–direction bounded in* $\mathbb{R}^d$ if (G, s) is ball–direction bounded for all generic $s : E \to \mathbb{R}^d$. We will use Theorem 4.1 to characterize ball–direction bounded graphs. We need the following concept. A *d-frame* is a graph $G = (V, E)$ together with a map $f : E \to \mathbb{R}^d$. The *incidence matrix* of the d-frame (G, f) is an $|E| \times d|V|$ matrix $I(G, f)$ defined as follows. We first choose an arbitrary reference orientation for the edges of E. Each edge in E corresponds to a row of $I(G, f)$ and each vertex of V to d consecutive columns. The submatrix of $I(G, p)$ with row labelled by $e = uv \in D$ and columns labelled by $x \in V$ is $f(e)$ if $x = u$, is $-f(e)$ if $x = v$, and is the d-dimensional zero matrix otherwise. It is known that when f is generic, $I(G, f)$ is a linear representation of the matroid union of d copies of the cycle matroid of G, see [23]. In particular, we have the following result. For $F \subseteq E$ and $X \subseteq V$, let $i_F(X)$ denote the number of edges of F between vertices in X.

Theorem 4.2 *Suppose $G = (V, E)$ is a graph and $f : E \to \mathbb{R}^d$ is generic. Then*

(a) *The rows of $I(G, f)$ are linearly independent if and only if $i_G(X) \leq d|X| - d$ for all $\emptyset \neq X \subseteq V$;*
(b) rank $I(G, f) = d|V| - d$ *if and only if G has d edge-disjoint spanning trees.*

We can use Theorems 4.1 and 4.2 to characterize ball–direction bounded graphs. For k a positive integer, we use kG to denote the graph obtained from G by replacing each edge by k parallel edges.

Corollary 4.3 *A graph $G=(V,E)$ is ball–direction bounded in $\mathbb{R}^d$ if and only if $(d-1)G$ has d edge-disjoint spanning trees.*

Proof We first suppose that $(d-1)G$ has d edge-disjoint spanning trees. Let $((d-1)G, f)$ be a generic d-frame. By Theorem 4.2, $\operatorname{rank} I((d-1)G, f) = d|V|-d$. For each $e \in E$, let $e_1, e_2, \ldots, e_{d-1}$ be the edges of $(d-1)G$ corresponding to e and let $s(e) \in \mathbb{R}^d$ be a non-zero vector which is orthogonal to $f(e_i)$ for all $1 \le i \le d-1$. Consider the ball–direction framework (G,s). We may take $A(G,s) = I((d-1)G, f)$ as a ball–direction incidence matrix for (G,s). Then $\operatorname{rank} A(G,s) = \operatorname{rank} I((d-1)G, f) = d|V|-d$. Since the rank of an incidence matrix of a generic ball–direction realisation of G in $\mathbb{R}^d$ will be at least $\operatorname{rank} A(G,s)$, Theorem 4.1 implies that G is ball–direction bounded in $\mathbb{R}^d$.

We may proceed similarly when $G=(V,E)$ is ball–direction bounded in $\mathbb{R}^d$. We choose a generic realisation (G,s) of G as a ball–direction framework and use it to construct a d-frame $((d-1)G, f)$ with $\operatorname{rank} I((d-1)G, f) = \operatorname{rank} A(G,s) = d|V|-d$. □

Remark Nash-Williams [16] and Tutte [21] independently characterized the graphs described in Corollary 4.3. A graph H contains d edge-disjoint spanning trees if and only if, for every partition $\{U_1, \ldots, U_t\}$ of $V(H)$, there are at least $d(t-1)$ edges of H with end vertices in two different sets. This property can be tested algorithmically in polynomial time.

5 Boundedness of Direction–Length Frameworks

In this section, we use the preceding result on boundedness of ball–direction frameworks to characterize boundedness in direction–length frameworks. Suppose that (G,p) is a d-dimensional direction–length framework and $G=(V;D,L)$. As in Sect. 3, we write $F_1, F_2, \ldots, F_m$ for the components of $F=(V,L)$ and let $E \subseteq D$ be the set of direction edges $e=uv$ in G which join distinct components of F and satisfy $p(u) \neq p(v)$. We also consider the graph $\widetilde{G}=(U,E)$ obtained from (V,E) by contracting $V(F_i)$ to a single vertex u_i for all $1 \le i \le m$. Now suppose that p is a realisation of G in $\mathbb{R}^d$. For each $e=uv \in E$, we let B_e be a $(d-1)\times d$-matrix whose rows are a basis for the subspace of $\mathbb{R}^d$ orthogonal to $\langle p(u)-p(v)\rangle$, as in Sect. 2. We also define $s: E \to \mathbb{R}^d - \{0\}$ by $s(e)=p(u)-p(v)$ for each $e=uv \in E$. The reduced rigidity matrix $\widetilde{R}(G^+, p)$ of (G,p) can be taken as a ball–direction incidence matrix $A(\widetilde{G}, s)$ for the weighted graph $(\widetilde{G}, s)$.

Theorem 5.1 *The following are equivalent*:

(a) *(G,p) is direction–length bounded*,
(b) *$(\widetilde{G}, s)$ is ball–direction bounded*,
(c) *(G^+, p) is rigid.*

Proof We show that (a) ⇒ (c) ⇒ (b) ⇒ (a).

(a) ⇒ (c): Suppose that (G^+,p) is not rigid. Then (G^+,p) is not infinitesimally rigid, so $\dim Z(G^+,p) \ge d+1$. Fix a vertex v_0. We can choose a non-zero infinitesimal motion z of G^+ such that $z(v_0) = 0$. Theorem 3.1(a) implies that $z(u) = z(v)$ for all $e = uv \in L$. Let (G^+, q) be the realisation of G^+ obtained by putting $q(v) = p(v) + \lambda z(v)$ for all $v \in V$, for some fixed $\lambda \in \mathbb{R}$. Then (G^+, q) is equivalent to (G^+, p) by Theorem 3.1(a). Now $q(v_0) = p(v_0)$, $z(v) \neq 0$ for some $v \in V$ and λ can be arbitrarily large, so (G^+, p) is unbounded. Hence (G, p) is unbounded.

(c) $\Rightarrow$ (b): Suppose that (G^+, p) is rigid. By Corollary 3.2, we have rank $\widetilde{R}(G^+, p) = dm - d$. Also, as remarked before the theorem, we may take $A(\widetilde{G}, s) = \widetilde{R}(G^+, p)$, so $(\widetilde{G}, s)$ is ball–direction bounded by Theorem 4.1.

(b) $\Rightarrow$ (a): Suppose that $(\widetilde{G}, s)$ is ball-direction bounded. Then rank $\widetilde{R}(G^+, p) =$ rank $A(\widetilde{G}, s) = dm - d$ so (G^+, p) is rigid and hence $L \neq \emptyset$. Consider any realisation (G, q) in $\mathbb{R}^d$ equivalent to (G, p). Choose $v_i \in V(F_i)$ for $1 \le i \le m$, let $\Delta = \sum_{e=xy \in L} \|p(x) - p(y)\|$, and define $r : U \to \mathbb{R}^d$ by $r(u_i) = q(v_i)/\Delta$. Then r is a ball–direction realisation of $(\widetilde{G}, s)$, since any edge $e = u_i u_j$ in E corresponds to an edge $u'_i u'_j$ in D with $u'_i \in V(F_i)$ and $u'_j \in V(F_j)$, and setting $x = q(u'_i)/\Delta$, $y = q(u'_j)/\Delta$ we have $\|x - r(u_i)\| \le 1$, $\|y - r(u_j)\| \le 1$ and $x - y = \lambda s(e)$. Since $(\widetilde{G}, s)$ is ball–direction bounded there is some $K \in \mathbb{R}$ depending only on G and s such that $\|r(u_i) - r(u_j)\| < K$ for all $1 \le i < j \le m$. Then $\|q(u) - q(v)\| \le (K + 2)\Delta$ for all $u, v \in V$, so (G, p) is bounded. □

Remarks

(a) Theorems 4.1 and 5.1 imply that we can determine whether or not a direction–length framework (G, p) is bounded by calculating the rank of the rigidity matrix $R(G^+, p)$. Equivalently, we could use the reduced rigidity matrix $\widetilde{R}(G^+, p)$.

(b) One can consider a relaxation of direction–length frameworks to *direction–cable* frameworks, where we have some direction constraints as before, and some cable constraints in which an edge places an upper bound on the distance between the end vertices of the edge. The arguments above show that (G, p) is bounded as a direction–length framework if and only if it is bounded as a direction–cable framework in which the length constraints are replaced by cable constraints.

6 Generic Boundedness

In this section, we characterize when a d-dimensional generic realisation of a mixed graph $G = (V; D, L)$ is bounded. It is not immediate that boundedness is a generic property, but we can see this from Theorems 3.1(c) and 5.1: (G, p) is bounded if and only if (G^+, p) is infinitesimally rigid, and infinitesimal rigidity is a generic property.

Recall that, by Corollary 3.2, (G, p) is bounded if and only if rank $\tilde{R}(G^+, p) = dm - d$, where $\tilde{R}(G^+, p)$ is a reduced rigidity matrix for (G^+, p), and m is the number of connected components in the length subgraph (V, L). When p is generic, we will give a combinatorial method for finding the rank of $\tilde{R}(G^+, p)$. This will enable us to characterize boundedness, and will also be used in the next section to describe the 'bounded components' of a mixed graph.

We start by motivating the formula for the rank and illustrating it with a simple example. Let $E \subseteq D$ be the set of the direction edges of G which join distinct components of (V, L), and let $H = (V, (d-1)E)$ be the graph obtained by taking $d-1$ copies of each edge in E. Since $\tilde{R}(G^+, p)$ has $d-1$ rows for each edge in E, we may associate each edge of H with a row in $\tilde{R}(G^+, p)$ and define a matroid on the edges of H in which independent sets correspond to linearly independent rows in $\tilde{R}(G^+, p)$. We expect an independent set $F \subseteq (d-1)E$ to satisfy the following two conditions.

1. F should be independent in the rigidity matroid for direction pure frameworks, for which a result of Whiteley [22, Theorem 8.2.2] gives the condition $i_F(X) \leq d|X| - d - 1$ for all $X \subseteq V$ with $|X| \geq 2$.
2. F should be independent when considered in the incidence matrix of a generic ball–direction framework $(\widetilde{G}, f)$ obtained by contracting each length component to a single point. Theorem 4.2 gives the condition $i_F(Y) \leq d|Y| - d$ for all $\emptyset \neq Y \subseteq V(\widetilde{G})$. Writing $t(X)$ for the number of components of (V, L) that $X \subseteq V$ intersects, we can write this condition as $i_F(X) \leq dt(X) - d$ for all $\emptyset \neq X \subseteq V$.

Figure 2 shows three 2-dimensional direction–length frameworks illustrating the role of these conditions. Each example has 4 (identical) length components and 6 direction edges (which is the minimum number of direction edges required to make them bounded by Corollary 4.3 and Theorem 5.1). Framework (i) satisfies both of the conditions above and is bounded. However, framework (ii) fails the first condition, and framework (iii) fails the second condition, and these frameworks are not bounded.

Now we consider these conditions in the following more general context. Suppose that d is a positive integer, $H = (V, E)$ is a graph and $\mathcal{P} = \{U_1, U_2, \ldots, U_m\}$ is a partition of V such that all edges in E join two distinct parts of $\mathcal{P}$. (We will later take $\mathcal{P}$ to be the components of (V, L) and $H = (V, (d-1)E)$ as defined above.) For $X \subseteq V$ let $t_{\mathcal{P}}(X)$ be the number of parts of $\mathcal{P}$ which intersect X, and define

$$f_{\mathcal{P}}(X) = \begin{cases} d|X| - d - 1 & \text{if } |X| \geq 2 \text{ and } t_{\mathcal{P}}(X) = |X|, \\ dt_{\mathcal{P}}(X) - d & \text{otherwise.} \end{cases}$$

We say that $F \subseteq E$ is *$\mathcal{P}$-independent* if $i_F(X) \leq f_{\mathcal{P}}(X)$ for all non-empty $X \subseteq V$. Note that the two conditions above separately define two distinct matroids on E, and the $\mathcal{P}$-independent sets are those subsets of E that are independent in both these matroids. In general, the intersection of two matroids may not be a matroid, but we will show in Theorem 6.2 that the $\mathcal{P}$-independent subsets of E do define a matroid. We achieve this by exhibiting a direct sum decomposition into 'critical' sets defined as follows. Fix any $\mathcal{P}$-independent $F \subseteq E$. We say that $X \subseteq V$ is *mixed $\mathcal{P}$-critical* if $i_F(X) = dt_{\mathcal{P}}(X) - d$, is *pure $\mathcal{P}$-critical* if $t_{\mathcal{P}}(X) = |X|$ and $i_F(X) = d|X| - d - 1$, and is *$\mathcal{P}$-critical* if it is either mixed $\mathcal{P}$-critical or pure $\mathcal{P}$-critical. Note that any mixed $\mathcal{P}$-critical set X with $|X| > 1$ satisfies $t_{\mathcal{P}}(X) < |X|$, since F is $\mathcal{P}$-independent. Note also that any X with $t_{\mathcal{P}}(X) = 1$ is mixed $\mathcal{P}$-critical, since all edges in E join two distinct parts of $\mathcal{P}$, so $i_F(X) = 0 = dt_{\mathcal{P}}(X) - d$.

We write $\mathcal{C}$ for the set of maximal $\mathcal{P}$-critical sets, $\mathcal{C}_P \subseteq \mathcal{C}$ for those sets in $\mathcal{C}$ that are pure $\mathcal{P}$-critical, and $\mathcal{C}_M = \mathcal{C} \setminus \mathcal{C}_P$ for those sets in $\mathcal{C}$ that are mixed $\mathcal{P}$-critical. We also write $\mathcal{B}$ for the set of maximal mixed $\mathcal{P}$-critical sets, which consists of all sets in $\mathcal{C}_M$ together with all sets $\{v\}$ such that v belongs to a pure $\mathcal{P}$-critical set but not to any larger mixed $\mathcal{P}$-critical set. Note that these definitions depend on F, but we suppress this in the notation. It will follow from Theorem 6.2 below that any maximal $\mathcal{P}$-independent set $F \subseteq E$ leads to the same sets $\mathcal{C}$, $\mathcal{C}_P$, $\mathcal{C}_M$ and $\mathcal{B}$.

First, we need the following lemma describing the structure of maximal critical sets.

Lemma 6.1

(a) *If* $X, Y \in \mathcal{C}$ *with* $X \neq Y$ *then* $|X \cap Y| \leq 1$, *and if* $X, Y \in \mathcal{B}$ *then* X *and* Y *are disjoint*;
(b) $\mathcal{B}$ *is a partition of* V *and* $\mathcal{P}$ *is a refinement of* $\mathcal{B}$.

Proof We start by noting two super/sub-modular inequalities holding for any $X, Y \subseteq V$:

(i) $i_F(X) + i_F(Y) \leq i_F(X \cap Y) + i_F(X \cup Y)$,
(ii) $t_{\mathcal{P}}(X) + t_{\mathcal{P}}(Y) \geq t_{\mathcal{P}}(X \cap Y) + t_{\mathcal{P}}(X \cup Y)$.

We can verify (i) by considering the contribution of each edge of F to both sides of the inequality: if an edge is counted by at least one of $i_F(X), i_F(Y)$ then it is counted by $i_F(X \cup Y)$, and if it is counted by both then it is also counted by $i_F(X \cap Y)$. We can verify (ii) similarly by considering the contribution of each set in $\mathcal{P}$ to both sides of the inequality.

To prove statement (a) we consider cases as follows. Suppose first that $X, Y \in \mathcal{B}$ but $X \cap Y \neq \emptyset$. Using inequalities (i) and (ii), and the fact that F is $\mathcal{P}$-independent, we have

$$\begin{aligned} dt_{\mathcal{P}}(X) - d + dt_{\mathcal{P}}(Y) - d = i_F(X) + i_F(Y) &\leq i_F(X \cap Y) + i_F(X \cup Y) \\ &\leq dt_{\mathcal{P}}(X \cap Y) - d + dt_{\mathcal{P}}(X \cup Y) - d \\ &\leq dt_{\mathcal{P}}(X) - d + dt_{\mathcal{P}}(Y) - d. \end{aligned}$$

Hence equality must hold throughout. In particular, $i_F(X \cup Y) = dt_{\mathcal{P}}(X \cup Y) - d$, i.e. $X \cup Y$ is mixed $\mathcal{P}$-critical. This contradicts the maximality of X and Y. Therefore, X and Y are disjoint.

Next suppose that $X, Y \in \mathcal{C}$ but $|X \cap Y| \geq 2$. If X is pure $\mathcal{P}$-critical and Y is mixed $\mathcal{P}$-critical then in the above calculation we reduce each of $i_F(X)$ and $i_F(X \cap Y)$ by 1. Again equality holds throughout, so $X \cup Y$ is mixed $\mathcal{P}$-critical, which contradicts the maximality of X and Y. On the other hand, if both X, Y are pure $\mathcal{P}$-critical then there are two subcases to consider. One possibility is that $t_{\mathcal{P}}(X \cup Y) = |X \cup Y|$, when in the above calculation we reduce each of $i_F(X)$, $i_F(Y)$, $i_F(X \cap Y)$ and $i_F(X \cap Y)$ by 1. Then we deduce that $i_F(X \cup Y) = d|X \cup Y| - d - 1$, so $X \cup Y$ is pure $\mathcal{P}$-critical, contradicting maximality. The other possibility is that $t_{\mathcal{P}}(X \cup Y) < |X \cup Y|$, but this is impossible, as then in the above calculation we reduce the left-hand-side by 2 but the right-hand-side by at least $d + 1 > 2$. This covers all cases, so statement (a) holds.

Finally, we note that statement (b) follows from (a) and the fact that every set of $\mathcal{P}$ is mixed $\mathcal{P}$-critical. □

Now we show that the $\mathcal{P}$-independent subsets of E define a matroid. First, we need some definitions and notation. Suppose $\mathcal{X}$ is a family of non-empty subsets of V. For any $E' \subseteq E$ we write $c_{E'}(\mathcal{X})$ for the number of edges in E' that are 'crossing' with respect to $\mathcal{X}$, by which we mean that they are not contained within any set in $\mathcal{X}$. If $c_{E'}(\mathcal{X}) = 0$, i.e. every edge in E' is contained within some set of $\mathcal{X}$, then we say that $\mathcal{X}$ is a *cover* of E'.

Theorem 6.2 *Let $\mathcal{I}$ be the family of all sets $F \subseteq E$ such that (V, F) is $\mathcal{P}$-independent. Then*

(a) *$\mathcal{I}$ is the family of independent sets of a matroid $M(H, \mathcal{P})$ on E;*
(b) *For any $E' \subseteq E$, the rank of E' is*

$$\rho(E') = \min_{\mathcal{X}} \left\{ \sum_{X \in \mathcal{X}} f_{\mathcal{P}}(X) + c_{E'}(\mathcal{X}) \right\} \tag{2}$$

where the minimum is taken over all families $\mathcal{X}$ of non-empty subsets of V;
(c) *The minimum in (2) can be achieved by taking $\mathcal{X}$ equal to the maximal $\mathcal{P}$-critical sets with respect to some maximal $\mathcal{P}$-independent subset of E';*
(d) *Suppose every pair of adjacent vertices in V are joined by at least $d - 1$ edges of E. Let F be a maximal $\mathcal{P}$-independent subset of E and $\mathcal{C}$ the maximal $\mathcal{P}$-critical sets with respect to F. Then $\mathcal{C}$ is a cover of E, and so $M(H, \mathcal{P})$ has rank $\sum_{X \in \mathcal{C}} f_{\mathcal{P}}(X)$. Furthermore, $\mathcal{C}$ is independent of the choice of F.*

Proof Consider any $E' \subseteq E$ and let F' be a maximal subset of E' such that $F' \in \mathcal{I}$. Then $i_{F'}(X) \leq f_{\mathcal{P}}(X)$ for all non-empty $X \subseteq V$, and so $|F'| \leq \sum_{X \in \mathcal{X}} f_{\mathcal{P}}(X) + c_{E'}(\mathcal{X})$ for any family $\mathcal{X}$ of non-empty subsets of V. This establishes the upper bound in (b), so (b) will follow from (c).

Let $\mathcal{C}'$ be the maximal $\mathcal{P}$-critical sets with respect to F'. Then $i_{F'}(X) = f_{\mathcal{P}}(X)$ for all $X \in \mathcal{C}'$. Furthermore, $|X \cap Y| \leq 1$ for all $X, Y \in \mathcal{C}'$ by Lemma 6.1, so no edge in F' is induced by two different sets in $\mathcal{C}'$. Thus $|F'| = \sum_{X \in \mathcal{C}'} f_{\mathcal{P}}(X) + c_{F'}(\mathcal{C}')$. Next we claim that $c_{F'}(\mathcal{C}') = c_{E'}(\mathcal{C}')$. Consider any $e = uv \in E' \setminus F'$. The maximality of F' implies that $F' \cup \{e\} \notin \mathcal{I}$. Since $F' \in \mathcal{I}$, we deduce that there is a $\mathcal{P}$-critical set X containing $\{u, v\}$. Thus $c_{F'}(\mathcal{C}') = c_{E'}(\mathcal{C}')$, so $|F'| = \sum_{X \in \mathcal{C}'} f_{\mathcal{P}}(X) + c_{E'}(\mathcal{C}')$.

This proves (c), and (b) follows. We have also shown that all maximal $\mathcal{P}$-independent subsets F of E' have the same size, so (a) holds. It remains to prove (d). Suppose for a contradiction that $\mathcal{C}$ is not a cover of E, and choose $uv \in E$ that is not covered by $\mathcal{C}$. Since $c_F(\mathcal{C}) = c_E(\mathcal{C})$, all edges joining u and v belong to F. Since F is independent there cannot be more than $d - 1$ such edges, so there must be exactly $d - 1$ such edges, with u, v in different parts of $\mathcal{P}$. But then $\{u, v\}$ is pure $\mathcal{P}$-critical, so is contained in a set of $\mathcal{C}$. This contradiction shows that $\mathcal{C}$ is a cover of E, i.e. $c_E(\mathcal{C}) = 0$, and the rank formula follows.

Finally, we note that if $X \in \mathcal{C}$ then $H[X]$ has rank $i_F(X) = f_{\mathcal{P}}(X)$. Thus we have rank $M(H, \mathcal{P}) = \sum_{X \in \mathcal{C}} \operatorname{rank} M(H[X], \mathcal{P})$, i.e. a direct sum decomposition of the matroid. Now if F^* is any maximal $\mathcal{P}$-independent subset of E we have $i_{F^*}(X) = i_F(X) = f_{\mathcal{P}}(X)$ for all $X \in \mathcal{C}$, i.e. every maximal critical set with respect to F is critical with respect to F^*. We deduce that $\mathcal{C}$ does not depend on the choice of F. □

We can now characterize when a generic direction–length framework (G, p) is bounded. More generally, we can characterize the rank of the reduced rigidity matrix in terms of the matroid described by the previous theorem. The following notation will be used for the remainder of this section. Let $G = (V; D, L)$ be a mixed graph and $\mathcal{P}$ be the partition of V given by the vertex sets of the connected components of the length subgraph (V, L). Let E be the set of direction edges of G joining distinct sets in $\mathcal{P}$, and let $H = (V, (d-1)E)$ be the graph obtained by taking $d-1$ copies of each edge in E. Then we let F be a maximal independent set in $M(H, \mathcal{P})$ and $\mathcal{C}$ be the maximal $\mathcal{P}$-critical sets with respect to F. Note that $\mathcal{C}$ is a cover of $D \cup L$, as each class of $\mathcal{P}$ is mixed $\mathcal{P}$-critical, so is contained in a member of $\mathcal{C}$, and $\mathcal{C}$ covers E by Theorem 6.2(d). For $X \subseteq V$ let

$$g(X) = \begin{cases} d|X| - d - 1 & \text{if } |X| \geq 2 \text{ and } G[X] \text{ is direction pure,} \\ d|X| - d & \text{otherwise.} \end{cases}$$

The following theorem gives two closely related formulae for the ranks of the rigidity matrix and reduced rigidity matrix of G^+. Recall that these have isomorphic null spaces by Corollary 3.2(a), so the ranks are related by $\operatorname{rank} R(G^+, p) = d(|V| - |\mathcal{P}|) + \operatorname{rank} \widetilde{R}(G^+, p)$.

Theorem 6.3

(a) $r_d(G^+) := \operatorname{rank} R(G^+, p) = \sum_{X \in \mathcal{C}} g(X) = \min_{\mathcal{X}}\{\sum_{X \in \mathcal{X}} g(X)\}$, *where the minimum is taken over all covers* $\mathcal{X}$ *of* $D \cup L$. *Moreover,* $r_d(G^+[X]) = g(X)$ *for all* $X \in \mathcal{C}$. *In particular,* G *is bounded if and only if* $\sum_{X \in \mathcal{X}} g(X) \geq d|V| - d$ *for all covers* $\mathcal{X}$ *of* $D \cup L$.

(b) $\operatorname{rank} \widetilde{R}(G^+, p) = \operatorname{rank} M(H, \mathcal{P}) = |F| = \sum_{X \in \mathcal{C}} f_{\mathcal{P}}(X) = \min_{\mathcal{X}} \sum_{X \in \mathcal{X}} f_{\mathcal{P}}(X)$, *where the minimum is taken over all covers* $\mathcal{X}$ *of* E. *Moreover,* $i_F(X) = f_{\mathcal{P}}(X)$ *for all* $X \in \mathcal{C}$. *In particular,* G *is bounded if and only if* $\sum_{X \in \mathcal{X}} f_{\mathcal{P}}(X) \geq d|\mathcal{P}| - d$ *for all covers* $\mathcal{X}$ *of* E.

Proof We start by establishing the following identity which shows the equivalence of the formulae given in (a) and (b):

$$\sum_{X \in \mathcal{C}} g(X) = d\big(|V| - |\mathcal{P}|\big) + \sum_{X \in \mathcal{C}} f_{\mathcal{P}}(X). \tag{3}$$

To see this, observe that for each $X \in \mathcal{C}_P$ we have $f_{\mathcal{P}}(X) = dt_{\mathcal{P}}(X) - d - 1 = d|X| - d - 1 = g(X)$. Therefore,

$$\sum_{X \in \mathcal{C}} g(X) - f_{\mathcal{P}}(X) = \sum_{X \in \mathcal{C}_M} g(X) - f_{\mathcal{P}}(X) = \sum_{X \in \mathcal{C}_M} d\big(|X| - t_P(X)\big)$$
$$= d \sum_{X \in \mathcal{B}} |X| - t_P(X),$$

since when $X \in \mathcal{B} \setminus \mathcal{C}_M$ we have $|X| = 1$, so $|X| - t_{\mathcal{P}}(X) = 0$. But $\mathcal{B}$ is a partition of V and $\mathcal{P}$ is a refinement of $\mathcal{B}$, so $\sum_{X \in \mathcal{B}} |X| - t_{\mathcal{P}}(X) = |V| - |\mathcal{P}|$, giving (3).

In view of this equivalence, it suffices to prove the upper bound $r_d(G^+) \leq \sum_{X\in\mathcal{C}} g(X)$ in formula (a) and the lower bound rank $\widetilde{R}(G^+, p) \geq \sum_{X\in\mathcal{C}} f_{\mathcal{P}}(X)$ in formula (b). The upper bound is straightforward: we note that $r_d(G^+[X]) \leq g(X)$ for any $X \subseteq V$, so $r_d(G^+) \leq \sum_{X\in\mathcal{X}} r_d(G^+[X]) \leq \sum_{X\in\mathcal{X}} g(X)$ for any cover $\mathcal{X}$ of $D \cup L$. Our main task in the proof will be to establish the lower bound rank $\widetilde{R}(G^+, p) \geq |F|$.

Let $H' = (V, F)$, and let $H'' = (\mathcal{P}, F)$ be obtained from H' by contracting each part of $\mathcal{P}$ to a single vertex. Since F is independent in $M(H, \mathcal{P})$ we have $i_F(X) \leq dt_{\mathcal{P}}(X) - d$ for all $\emptyset \neq X \subseteq V$, and hence $i_{H''}(Y) \leq d|Y| - d$ for all $\emptyset \neq Y \subseteq U$. Let (H'', q) be a generic d-frame for H'' and let $I(H'', q)$ be its incidence matrix. Then Theorem 4.2(a) implies that the rows of $I(H'', q)$ are linearly independent.

Consider the generic d-frame (H', q) and let $I(H', q)$ be its incidence matrix. Since sets of d consecutive columns of $I(H', q)$ are labelled by the vertices in V, we may consider the vectors in the null space Z' of $I(H', q)$ as maps from V to $\mathbb{R}^d$. We need the following claim.

Claim 1 *The null space Z' of $I(H', q)$ contains a vector z such that $z(u) \neq z(v)$ for all $u, v \in V$.*

Proof Choose $u_0, v_0 \in V$ and let H^* be obtained by adding a new edge $e_0 = u_0v_0$ to H'. Let $(H^*, \tilde{q})$ be a generic d-frame such that $\tilde{q}(e) = q(e)$ for all $e \in F$ and let $I(H^*, \tilde{q})$ be its incidence matrix. Note that $I(H', q)$ is the submatrix of $I(H^*, \tilde{q})$ with rows indexed by F. Since F is independent in $M(H, \mathcal{P})$ we have $i_F(X) \leq f_{\mathcal{P}}(X) \leq d|X| - d - 1$ for all $X \subseteq V$ with $|X| \geq 2$. Thus $i_{F\cup\{e_0\}}(X) \leq d|X| - d$. Theorem 4.2(a) now implies that the rows of $I(H^*, \tilde{q})$ are linearly independent. Thus rank $I(H^*, \tilde{q}) =$ rank $I(H', q) + 1$. Writing Z^* for the null space of $I(H^*, \tilde{q})$ we see that $\dim Z^* < \dim Z'$, so we can choose $z_{e_0} \in Z' \setminus Z^*$. By definition of $I(H^*, \tilde{q})$, we have $z_{e_0}(u_0) \neq z_{e_0}(v_0)$. We may now construct the required vector z in Z' by taking a suitable linear combination of the vectors z_{e_0}, for all possible new edges $e_0 = u_0v_0$, $u_0, v_0 \in V$. □

Returning to the proof of the theorem, we consider the direction–length framework (G, z), where z is given by the previous claim. Since $z \in Z'$ we have $[z(u) - z(v)] \cdot q(e) = 0$ for each $e = uv \in F$. We may use this to construct a reduced rigidity matrix $\tilde{R}(G^+, z)$ for (G^+, z) which contains $I(H'', q)$ as a row induced submatrix as follows. For each $e = uv \in E$, we associate the $d - 1$ consecutive rows of $\tilde{R}(G^+, z)$ corresponding to e with the $d - 1$ copies $e^1, e^2, \ldots, e^{d-1}$ of e in the graph H. We then choose a basis B_e for $[z(u) - z(v)]^\perp$ in such a way that $q(e^i) \in B_e$ whenever $e^i \in F$. This is possible since $q(e^i)$ is orthogonal to $z(u) - z(v)$ if $e^i \in F$, and the vectors $\{q(e^i) : e^i \in F\}$ are linearly independent because q is generic. Then $I(H'', q)$ is the submatrix of $\tilde{R}(G^+, z)$ with rows indexed by F. But the rows of $I(H'', q)$ are linearly independent, so rank $\tilde{R}(G^+, z) \geq$ rank $I(H'', q) = |F|$. Since p is generic, we then have rank $\tilde{R}(G^+, p) \geq$ rank $\tilde{R}(G^+, z) \geq |F|$. This completes the proof of the lower bound, so we have established the formulae rank $R(G^+, p) = \sum_{X\in\mathcal{C}} g(X)$ and rank $\widetilde{R}(G^+, p) = \sum_{X\in\mathcal{C}} f_{\mathcal{P}}(X)$.

Now note that

$$r_d\big(G^+\big) \le \sum_{X\in\mathcal{C}} r_d\big(G^+[X]\big) \le \sum_{X\in\mathcal{C}} g(X) = r_d\big(G^+\big).$$

Therefore, equality holds throughout, so $r_d(G^+[X]) = g(X)$ for all $X \in \mathcal{C}$. A similar argument shows that $i_F(X) = f_{\mathcal{P}}(X)$ for all $X \in \mathcal{C}$. Finally, the characterizations of boundedness follow from the rank formulae and Corollary 3.2(c). □

Remarks

(a) Theorem 6.3 implies that, if a mixed graph is generically bounded in $\mathbb{R}^d$, then it is also generically bounded in $\mathbb{R}^{d+1}$. This follows from the fact that the corresponding functions g_d and g_{d+1} satisfy $dg_{d+1}(X) \ge (d+1)g_d(X)$ for all vertex sets X.

(b) Theorem 6.3 can also be used to deduce a result of Whiteley on d-dimensional direction-pure frameworks. Such a framework (G, p) is *direction rigid* if every equivalent framework can be obtained from (G, p) by a translation or dilation of $\mathbb{R}^d$. It is not difficult to see that (G, p) is direction rigid if and only if its rigidity matrix $R(G, p)$ has rank $d|V| - d - 1$. Whiteley [22] showed that a graph $G = (V, D)$ is generically direction rigid in $\mathbb{R}^d$ if and only if $\sum_{X\in\mathcal{X}}(d|X| - d - 1) \ge d|V| - d - 1$ for all covers $\mathcal{X}$ of D with $|X| \ge 2$ for all $X \in \mathcal{X}$. This follows from Theorem 6.3(a) using the fact that $G^+ = G$.

7 Bounded Components

In this section, we consider a local version of boundedness. Let $G = (V; D, L)$ be a mixed graph and (G, p) be a realisation of G in $\mathbb{R}^d$. We say that vertices $u, v \in V$ are *weakly linked* in (G, p) if there exists a $K \in \mathbb{R}$ such that $\|q(u) - q(v)\| < K$ for all realisations (G, q) of G in $\mathbb{R}^d$ which are equivalent to (G, p). This is an equivalence relation: we call the equivalence classes *bounded components* of (G, p) and refer to the partition of V into equivalence classes as the *bounded component partition* of G. We will see that the property of being weakly linked is generic, so all generic realisations of G have the same bounded components.

A *rigid component* of G in $\mathbb{R}^d$ is a maximal subgraph of G which is (generically) rigid in $\mathbb{R}^d$. It is easy to see that the rigid components of G are a family of induced subgraphs of G whose vertex sets partition V. We refer to this partition as the *rigid component partition* of G. We will show that the bounded component partition of G is identical to the rigid component partition of G^+.

As in the previous section, we suppose that $G = (V; D, L)$ is a mixed graph and $\mathcal{P}$ is the partition of V given by the vertex sets of the connected components of (V, L). We write E for the set of direction edges of G joining distinct sets in $\mathcal{P}$, and let $H = (V, (d-1)E)$ be the graph obtained by taking $d-1$ copies of each edge in E. Then we let F be a maximal independent set in $M(H, \mathcal{P})$, write $\mathcal{C}$ for the maximal $\mathcal{P}$-critical sets, $\mathcal{C}_P \subseteq \mathcal{C}$ for those sets in $\mathcal{C}$ that are pure $\mathcal{P}$-critical, $\mathcal{C}_M = \mathcal{C} \setminus \mathcal{C}_P$ for those sets in $\mathcal{C}$ that are mixed $\mathcal{P}$-critical, and $\mathcal{B}$ for the set of maximal mixed $\mathcal{P}$-critical sets. Recall that $\mathcal{B}$ is a partition of V and that $\mathcal{P}$ is a refinement of $\mathcal{B}$.

The following theorem shows that it is exactly the partition we need to describe the bounded components of G.

Theorem 7.1 *The bounded component partition of G and the rigid component partition of G^+ are both equal to $\mathcal{B}$.*

Proof First, we show that the bounded component partition of G is identical to the rigid component partition of G^+. Let (G,p) be a generic realisation of G in $\mathbb{R}^d$. Consider any rigid component $G^+[X]$ of G^+, where $X \subseteq V$. Then $(G[X], p|_X)$ is a generic realisation of $G[X]$. Since $(G^+[X], p|_X)$ is rigid, $(G[X], p|_X)$ is bounded by Theorem 5.1. Since all generic realisations have the same rigid components, we deduce that $G[X]$ is bounded.

Conversely, consider any $u, v \in V$ belonging to distinct rigid components, and let G_{uv} be obtained from G by adding a new length edge $e = uv$. We claim that $r_d(G^+_{uv}) > r_d(G^+)$. Suppose on the contrary that $r_d(G^+_{uv}) = r_d(G^+)$. By Theorem 6.3, there exists a cover $\mathcal{Z}$ of $D \cup (L \cup \{e\})$ such that $r_d(G^+_{uv}) = \sum_{Z\in\mathcal{Z}} g(Z)$. We now have

$$r_d\big(G^+\big) \le \sum_{Z\in\mathcal{Z}} r_d\big(G^+[Z]\big) \le \sum_{Z\in\mathcal{Z}} g(Z) = r_d\big(G^+_{uv}\big) = r_d\big(G^+\big).$$

Equality must hold throughout. In particular, if Z is the set in $\mathcal{Z}$ which covers e, then $r_d(G^+[Z]) = d|Z| - d$ and hence $G^+[Z]$ is rigid. This contradicts the fact that u and v belong to different rigid components of G^+. Hence $r_d(G^+_{uv}) = r_d(G^+) + 1$.

Let $R(G^+_{uv}, p)$ be a rigidity matrix for a generic realisation (G^+_{uv}, p) of G^+_{uv} in $\mathbb{R}^d$ and $R(G^+, p)$ the submatrix consisting of the rows indexed by $D \cup L$. Then $\operatorname{rank} R(G^+_{uv}, p) = \operatorname{rank} R(G^+, p) + 1$, so we can choose $z \in Z(G^+, p) \setminus Z(G^+_{uv}, p)$, i.e. an infinitesimal motion z of (G^+, p) with $z(u) \ne z(v)$. By Theorem 3.1(a), $(G^+, p + \lambda z)$ is equivalent to (G^+, p) for all $\lambda \in \mathbb{R}$. Since $z(u) \ne z(v)$ this implies that u and v do not belong to the same bounded component of (G^+, p), let alone (G, p).

This shows that the partitions are indeed identical. It remains to show that they are equal to $\mathcal{B}$. First, we show that if $X \in \mathcal{B}$ then $G^+[X]$ is rigid. This is clear if $|X| = 1$. Otherwise we have $X \in \mathcal{C}_M$, so by Theorem 6.3 $r_d(G^+[X]) = g(X) = d|X| - d$, and $G^+[X]$ is rigid. For the converse, we can suppose that $|\mathcal{B}| \ge 2$, otherwise $\mathcal{B} = \{V\}$ and G^+ is rigid. Consider any $u, v \in V$ belonging to distinct sets of $\mathcal{B}$. We claim that u and v do not belong to the same bounded component of G. In view of the argument above, it suffices to show that $r_d(G^+_{uv}) > r_d(G^+)$, where G_{uv} is obtained from G by adding a new length edge $e = uv$.

Suppose on the contrary that $r_d(G^+_{uv}) = r_d(G^+)$. Since u and v belong to distinct critical sets in $\mathcal{B}$, they belong to distinct parts $U_1, U_2 \in P$. Let $\mathcal{P}'$ be obtained from $\mathcal{P}$ by replacing U_1 and U_2 by $U_1 \cup U_2$ and H' be obtained from H by deleting all edges which join U_1 and U_2. Let F' be a maximum independent set in $M(H', \mathcal{P}')$ and $\mathcal{C}'$ be the maximal $\mathcal{P}'$-critical sets with respect to F'. By Theorem 6.3, we have $r_d(G^+) = d(|V| - |\mathcal{P}|) + |F|$ and $r_d(G^+_{uv}) = d(|V| - |\mathcal{P}'|) + |F'|$. Since $|\mathcal{P}| = |\mathcal{P}'| + 1$, our assumption that $r_d(G^+_{uv}) = r_d(G^+)$ gives $|F| = |F'| + d$. We also have $|F| = \sum_{X\in\mathcal{C}} f_{\mathcal{P}}(X)$ and $|F'| = \sum_{X\in\mathcal{C}'} f_{\mathcal{P}'}(X)$ by Theorem 6.2(d). Since $U_1 \cup U_2 \in \mathcal{P}'$,

$U_1 \cup U_2$ is mixed $\mathcal{P}'$-critical and so $U_1 \cup U_2 \subseteq X_0$ for some $X_0 \in \mathcal{C}'$. Note that $f_{\mathcal{P}}(X_0) \le f_{\mathcal{P}'}(X_0) + d$. Also, for all $X \in \mathcal{C}'$ distinct from X_0 we have $|X \cap X_0| \le 1$ by Lemma 6.1, so $f_{\mathcal{P}}(X) = f_{\mathcal{P}'}(X)$. Since $\mathcal{C}'$ covers E and F is $\mathcal{P}$-independent we have $|F| \le \sum_{X\in\mathcal{C}'} f_{\mathcal{P}}(X) \le d + \sum_{X\in\mathcal{C}'} f_{\mathcal{P}'}(X) = d + |F'|$. Equality holds, so we must have $i_F(X) = f_{\mathcal{P}}(X)$ for all $X \in \mathcal{C}'$. In particular, this implies that X_0 is mixed critical. However, this contradicts the fact that u and v belong to distinct critical sets in $\mathcal{B}$, so we are done. □

Remark Theorems 6.2, 6.3 and 7.1 give rise to an $O(|V|^2)$ algorithm for testing if the mixed graph G is bounded in $\mathbb{R}^d$, and more generally finding its bounded components. It suffices to construct a maximum independent set F in $M(H,\mathcal{P})$ and find the maximal mixed critical subsets in (V, F). A maximum independent set in a matroid can be constructed greedily starting with the empty set and adding or rejecting elements one by one, so we need only determine whether the addition of an edge $e \in (d-1)E$ to an independent set $J \subseteq (d-1)E$ satisfies $i_{J\cup\{e\}}(X) \le f_{\mathcal{P}}(X)$ for all $\emptyset \ne X \subseteq V$. This can be done in two stages by checking whether $i_{J+e}(X) \le d|X| - d - 1$ for all $X \subseteq V$ with $|X| \ge 2$, and whether, in the graph $(\mathcal{P}, (d-1)E)$ obtained from H by contracting the parts of $\mathcal{P}$ to single vertices, we have $i_{J+e}(Q) \le d|Q| - d$ for all $Q \subseteq \mathcal{P}$ with $|Q| \ge 1$. Both checks can be performed in $O(|V|)$ time, using for example the 'orientation algorithm' given in [1], or the 'pebble game algorithm' given in [13]. Each maximal mixed critical subset X in (V, F) induces a rigid component in G^+ and hence a bounded component in G. We have $X = \bigcup_{Y\in Q} Y$ where Q is a maximal subset of $\mathcal{P}$ for which $i_F(Q) = d|Q| - d$ in $(\mathcal{P}, F)$. Such a subset is referred to as a *d-brick* of $(\mathcal{P}, F)$ in [9]. Contracting the parts of $\mathcal{P}$ to single vertices transforms the bounded component partition of G into the d-brick partition of $(\mathcal{P}, F)$ studied in [9]. This can be constructed in $O(|V|^2)$ time as in [1].

8 Global Rigidity

In this section, we consider when a generic d-dimensional direction–length framework is globally rigid. Hendrickson [6] gave two necessary conditions for a d-dimensional generic length-pure framework (G, p) to be *globally length-rigid*, which means that all equivalent realisations (G, q) are length-congruent to (G, p) (i.e. satisfy $\|q(u) - q(v)\| = \|p(u) - p(v)\|$ for all $u, v \in V$). One is that the underlying graph G must be either complete or d-connected. The other is that G must be redundantly rigid in $\mathbb{R}^d$, i.e. $G \setminus e$ is rigid for any edge e of G. For general d these conditions are not sufficient for global rigidity, as shown by Connelly [3]. They do suffice for $d = 2$, as shown by Jackson and Jordan [7], who proved that a 2-dimensional generic length-pure framework (G, p) is globally length-rigid if and only if either G is a complete graph on at most 3 vertices, or G is 3-connected and redundantly rigid. In higher dimensions, no combinatorial characterization is known, although there is an algebraic condition that was shown to be sufficient by Connelly [3] and necessary by Gortler, Healy and Thurston [5]. This algebraic condition implies that global length-rigidity is a generic property.

Suppose (G, p) is a d-dimensional generic direction–length framework. It is certainly necessary for G to be connected if (G, p) is to be rigid, let alone globally rigid. Also, 2-connectivity is necessary for global rigidity, as if x is a cutvertex of G then we can obtain a realisation (G, p') that is equivalent but not congruent to (G, p) by inverting one component of $G \setminus x$ about the point $p(x)$, without changing the rest of the realisation. On the other hand, if $G = (V; D, L)$ is 2-connected and $D = L$ then (G, p) is globally rigid if and only if G is 2-connected, see [8, Theorem 7.2], so 3-connectivity is no longer necessary for global rigidity in $\mathbb{R}^d$. However, an analogue of Hendrickson's connectivity condition may be obtained by considering more restricted cuts: if (G, p) is globally rigid then there can be no cutset $X \subseteq V$ of size at most d such that there is a component C of $G \setminus X$ that contains only length edges, as then we could obtain a realisation (G, p') which is equivalent but not congruent to (G, p) by reflecting C in a hyperplane containing the points $p(x)$, $x \in X$. The main result of [8] is that this connectivity condition is both necessary and sufficient for the global rigidity of redundantly rigid 2-dimensional generic direction–length frameworks when $|D \cup L| = 2|V| - 2$, i.e. $D \cup L$ is a circuit in the corresponding rigidity matroid.

We next consider an analogue of Hendrickson's redundant rigidity condition. First, we need some definitions. Suppose that $G = (V, D)$ is a graph with direction constraints on its edges, but no length constraints. Let (G, p) and (G, q) be realisations of G in $\mathbb{R}^d$. We say that (G, p) and (G, q) are *direction equivalent* if $q(u) - q(v)$ is a scalar multiple of $p(u) - p(v)$ for all $uv \in D$ with $p(u) \neq p(v)$ and *direction congruent* if (G, q) can be obtained from (G, p) by a translation and/or a dilation. The definitions of rigidity and global rigidity of direction frameworks are as for direction–length frameworks but using direction equivalence and congruence. We say that (G, p) is *globally direction rigid* if every framework which is direction equivalent to (G, p) is direction congruent to (G, p). We say (G, p) is *direction rigid* if there exists an $\varepsilon > 0$ such that if a framework (G, q) is direction equivalent to (G, p) and satisfies $\|p(v) - q(v)\| < \varepsilon$ for all $v \in V$ then (G, q) is direction congruent to (G, p).

The linearity of direction constraints makes the problem of characterizing (globally) rigid direction frameworks much easier than for length frameworks. Indeed, Whiteley [22] showed that rigidity, global rigidity and infinitesimal rigidity are equivalent for direction frameworks, and hence are determined by the rank of the rigidity matrix. He used this to characterize graphs which are (globally) direction rigid in $\mathbb{R}^d$.

It is possible to construct globally rigid generic realisations of a mixed graph $G = (V; D, L)$ which are not redundantly rigid. We first choose a graph $H = (V, D)$ which is (globally) direction rigid in $\mathbb{R}^d$ and let G be the mixed graph obtained by adding a length edge e to H. Let (G, p) be a generic realisation of G in $\mathbb{R}^d$ and let (G, q) be an equivalent realisation. Since (H, p) is globally direction rigid, (H, q) can be obtained from (H, p) by a translation and/or a dilation. Thus (G, q) can be obtained from (G, p) by a translation and/or a dilation. Since G contains a length edge, the only dilations of (G, p) which produce an equivalent direction–length framework are dilations by ± 1. Hence (G, q) is congruent to (G, p). Thus (G, p) is globally rigid. On the other hand, $(G \setminus e, p) = (H, p)$ is not rigid (as a direction–length framework), since it is direction pure, and so admits arbitrary dilations.

One reason that Hendrickson's redundant rigidity condition fails for a generic direction–length framework (G, p) is that his proof relies on a compactness argument which is not valid if $(G \setminus e, p)$ is unbounded. We shall use our characterization of bounded mixed graphs to show that the examples given in the preceding paragraph are the only examples of globally rigid generic direction–length frameworks (G, p) for which $(G \setminus e, p)$ is not rigid.

Our arguments will require the equivalence of rigidity and infinitesimal rigidity for generic direction–length frameworks. In two dimensions this follows from Lemmas 5.1 and 5.3 in [10]. The general case can be proved by very similar arguments. For completeness we give the proof here, although we will be brief on those points that are similar to the proofs given in [10].

We need to use a d-dimensional version of the rigidity map from [10]. For $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, let $l(x) = \|x\|^2$, and when $x_d \neq 0$ let $t(x) = (x_1/x_d, x_2/x_d, \dots, x_{d-1}/x_d)$. Let $G = (V; D, L)$ be a graph with $D \cup L = \{e_1, e_2, \dots, e_m\}$. Choose a reference orientation for the edges of $D \cup L$. Given a realisation (G, p) of G, we say that an edge $e = uv \in D$ is *vertical* in (G, p) if the last co-ordinate of $p(u) - p(v)$ is zero. Let T be the set of all points $p \in \mathbb{R}^{d|V|}$ such that (G, p) has no vertical direction edges. For each $p \in T$ and $e_i = uv \in D \cup L$ let $f_i(p) = t(p(u) - p(v))$ if $e_i \in D$, and $f_i(p) = l(p(u) - p(v))$ if $e_i \in L$. The *rigidity map* $f_G : T \to \mathbb{R}^{(d-1)|D|+|L|}$ is defined by putting $f_G(p) = (f_1(p), f_2(p), \dots, f_m(p))$. For each $p \in T$, we may use the Jacobian $df_G|_p$ as a rigidity matrix for (G, p). To see this, consider an edge $e_i = uv$ and write $p(u) = (a_1, \dots, a_d)$, $p(v) = (b_1, \dots, b_d)$. If $e_i \in L$ then $f_i(p) = \sum_{j=1}^d (a_j - b_j)^2$, so for $1 \le j \le d$ we have $\frac{\partial f_i(p)}{\partial a_j} = 2(a_j - b_j)$ and $\frac{\partial f_i(p)}{\partial b_j} = -2(a_j - b_j)$. (Derivatives with respect to variables not appearing in $p(u)$ or $p(v)$ are of course zero.) Thus the row in the Jacobian corresponding to e_i is obtained by multiplying that given in our earlier definition by 2, which does not affect the rank. Next suppose that $e_i \in D$. Then $f_i(p) \in \mathbb{R}^{d-1}$ has jth co-ordinate $f_i(p)_j = \frac{a_j - b_j}{a_d - b_d}$. We have $\frac{\partial f_i(p)_j}{\partial a_j} = \frac{1}{a_d - b_d}$ and $\frac{\partial f_i(p)_j}{\partial a_d} = -\frac{a_j - b_j}{(a_d - b_d)^2}$. Multiplying these rows through by $(a_d - b_d)^2$, we obtain the same basis of the space orthogonal to $p(u) - p(v)$ that we described at the end of Sect. 2.

Next we recall some basic concepts and facts of differential topology. We refer the reader to [15] for an introduction to this subject. Suppose U is an open subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ is a smooth map. Let k be the maximum rank of the Jacobian $df|_x$ over all $x \in U$. A point $x \in U$ is a *regular point* if rank $df|_x = k$, otherwise x is a *critical point*. A point $y \in f(U)$ is a *critical value* if $y = f(x)$ for some critical point x, otherwise y is a *regular value*. If y is a regular value then $f^{-1}(y)$ is an $(m - k)$-dimensional manifold (see [15, p. 11, Lemma 1]).

Lemma 8.1 *Let (G, p) be a generic direction–length framework in $\mathbb{R}^d$. Then (G, p) is rigid if and only if (G, p) is infinitesimally rigid.*

Proof We fix some vertex $v_0 \in V$ and restrict attention to realisations (G, q) in which v_0 is mapped to the origin in $\mathbb{R}^d$. We identify (G, q) with the vector $\hat{q} \in \mathbb{R}^{dn-d}$ of co-ordinates for the points $q(v)$, $v \neq v_0$. We can change our co-ordinate system so that $p(v_0) = 0$ and $\hat{p}$ is generic. Let $\hat{T} = \{\hat{q} : q \in T\}$, where T is as above, and define $f : \hat{T} \to \mathbb{R}^m$ by $f(\hat{q}) = f_G(q)$. Since $\hat{p}$ is generic, $\hat{p}$ is a regular point of f. Let $k = \operatorname{rank} df|_{\hat{p}}$. By continuity, there is an open neighbourhood $W \subseteq \hat{T}$ of $\hat{p}$ such that rank $df|_w = k$ for all $w \in W$. Let $g = f|_W$. Then $g(\hat{p})$ is a regular value of g, so $M = g^{-1}(g(\hat{p}))$ is a $(dn - d - k)$-dimensional manifold. Now k is the rank of the rigidity matrix at p, so $k = dn - d$ if and only if (G, p) is infinitesimally rigid. Also, M has non-zero dimension if and only if there is a sequence $\hat{p}_i \in M \setminus \{\hat{p}\}$ converging to p. Such a sequence $\hat{p}_i$ exists if and only if (G, p) is not rigid: (G, p_i) is not a translation of (G, p) as $p_i(v_0) = p(v_0) = 0$ and $p_i \neq -p$ for $\hat{p}_i$ sufficiently close to $\hat{p}$. The result follows. □

Now we can apply our characterization of bounded mixed graphs to derive a necessary condition for global rigidity.

Lemma 8.2 *Suppose that (G, p) is a generic globally rigid direction–length framework in $\mathbb{R}^d$ with at least two length edges and e is a length edge of G. Then $G \setminus e$ is bounded in $\mathbb{R}^d$.*

Proof Write $e = uv$. Suppose for a contradiction that $H = G \setminus e$ is not bounded. Then (H^+, p) is not rigid by Theorem 5.1. Therefore (H^+, p) is not infinitesimally rigid, so there is an infinitesimal motion z of (H^+, p) with $z \neq 0$ and $z(u) = 0$. Also (G, p) is rigid (since it is globally rigid) and so infinitesimally rigid (by Lemma 8.1, since p is generic). Thus z cannot be an infinitesimal motion of (G, p). Since e provides the only constraint in G that is not in H^+ this constraint is violated by z, i.e. $z(v) \cdot (p(u) - p(v)) \neq 0$. This implies that we can choose $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$\left\| p(u) - \big(p(v) + \lambda z(v)\big) \right\| = \left\| p(u) - p(v) \right\|.$$

Let $q = p + \lambda z$. Then (H^+, q) is equivalent to (H^+, p) by Lemma 3.1(a). Now $p(u) = q(u)$ but $p(v) \neq q(v)$, so (G, q) is not a translation of (G, p). Since (G, p) is globally rigid, (G, q) must be a dilation of (G, p) by -1 through the point $p(u)$. Then for every $w \in V$ we have $q(w) = 2p(u) - p(w)$, so $\lambda z(w) = 2(p(u) - p(w))$. Let $f = xy$ be a length edge of H. By Lemma 3.1(a), we have $z(x) = z(y)$, and so $p(x) = p(y)$. This contradicts the hypothesis that p is generic. □

We use Lemma 8.2 in [11] to show that Hendrickson's redundant rigidity condition holds for length edges in generic globally rigid direction–length frameworks.

Theorem 8.3 [11] *Suppose (G, p) is a d-dimensional generic globally rigid direction–length framework with at least two length edges and e is a length edge of G. Then $G \setminus e$ is rigid in $\mathbb{R}^d$.*

It is also natural to consider the result of deleting a direction edge, rather than a length edge, from a d-dimensional generic globally rigid direction–length framework. This can reduce the rank of the rigidity matrix by up to $d - 1$, so we expect a more complicated behaviour for $d \geq 3$. However, in the special case $d = 2$, we believe that the following weakening of Hendrickson's redundant rigidity condition holds.

Conjecture 8.4 *Suppose (G, p) is a 2-dimensional generic globally rigid direction–length framework with at least two length edges and e is a direction edge of G. Then $G \setminus e$ is either rigid or unbounded in $\mathbb{R}^2$.*

9 Concluding Remarks

We showed in Theorem 5.1 and Corollary 3.2(c) that a given direction–length framework (G, p) is bounded if and only if its augmented framework (G^+, p) is infinitesimally rigid, and that this can be determined by calculating the rank of the (reduced) rigidity matrix of (G^+, p). This is in sharp contrast to the problem of determining whether (G, p) is rigid which appears to be difficult. As an intermediate result, Theorem 4.1, we characterized boundedness for a new type of framework consisting of balls linked by direction constraints. We applied our results to obtain a combinatorial characterization of boundedness for d-dimensional generic direction–length frameworks, Theorem 6.3. This is again in contrast to the problem of determining generic rigidity which is solved only when $d = 2$. We then showed in Theorem 7.1 that, for any mixed graph G and fixed $d \geq 2$, the vertex set of G is partitioned by the d-dimensional bounded components of G, i.e. maximal subgraphs which are generically bounded in $\mathbb{R}^d$, and, by the remark at the end of Sect. 7, that this partition can be constructed in polynomial time. Finally, we applied our characterization of generic boundedness to obtain necessary conditions for the global rigidity of generic direction–length frameworks, Lemma 8.2 and Theorem 8.3. We believe that our characterization of generic boundedness will prove to be a useful tool in the problem of characterizing global rigidity for these frameworks.

Acknowledgement We thank John Bray, Mark Jerrum and Tibor Jordán for helpful discussions. We also thank an anonymous referee whose suggestions improved the presentation of this paper.

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