Global rigidity of 2-dimensional direction-length frameworks

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Abstract

A 2-dimensional direction-length framework is a collection of points in the plane which are linked by pairwise constraints that fix the direction or length of the line segments joining certain pairs of points. We represent it as a pair (G, p), where $G = (V; D, L)$ is a 'mixed' graph and $p : V \to \mathbb{R}^2$ is a point configuration for V. It is globally rigid if every direction-length framework (G, q) which satisfies the same constraints can be obtained from (G, p) by a translation or a rotation by $180°$. We show that the problem of characterising when a generic (G, p) is globally rigid can be reduced to the case when G belongs to a special family of 'direction irreducible' mixed graphs, and prove that every generic realisation of a direction irreducible mixed graph G is globally rigid if and only if G is 2-connected, direction balanced and redundantly rigid.

1 Introduction

A finite configuration of points in Euclidean space with local constraints may be informally described as globally rigid if the constraints determine the point set up to congruence. It is a fundamental open problem to give a nice characterisation of global rigidity in various settings. Our setting here is that of a d-dimensional *direction-length framework*, which is a pair (G, p), where $G = (V; D, L)$ is a 'mixed' graph and $p : V \to \mathbb{R}^d$ is a point configuration for V. We call the graph G *mixed* because it has two types of edges: we refer to edges in D as *direction edges* and edges in L as *length edges*. Two direction-length frameworks (G, p) and (G, q) are *equivalent* if $p(u) - p(v)$ is a scalar multiple of $q(u) - q(v)$ for all $uv \in D$ with $q(u) \neq q(v)$, and $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for all $uv \in L$. Two point configurations p and q for V are *congruent* if either $p(u) - p(v) = q(u) - q(v)$ for all $u, v \in V$, or $p(u) - p(v) = q(v) - q(u)$ for all $u, v \in V$. (Thus p and q are congruent if p can be obtained from q by a translation, possibly followed by a rotation by $180°$.) A direction-length framework (G, p) is *globally rigid* if p is congruent to q for every framework (G, q) which is equivalent to (G, p). It is *rigid* if there exists an $\varepsilon > 0$ such that if a framework (G, q) is equivalent to (G, p) and satisfies $\|p(v) - q(v)\| < \varepsilon$ for all $v \in V$ then p is congruent to q (equivalently every continuous motion of the vertices of (G, p) which satisfies the direction and length constraints given by the edges results in a framework (G, q) with p congruent to q). The framework (G, p) is *redundantly rigid* if $(G - e, p)$ is rigid for all $e \in D \cup L$.

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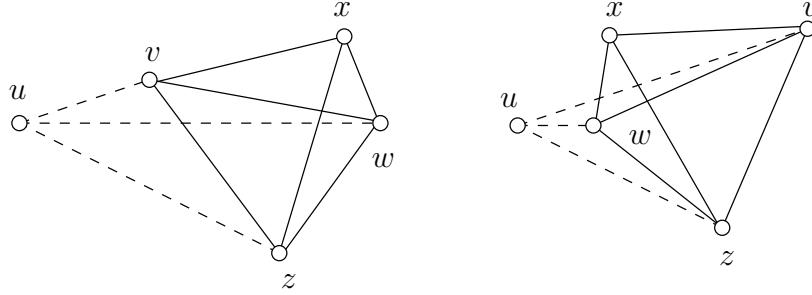
Figure 1: Two equivalent but non-congruent direction-length frameworks. We use solid or dashed lines to indicate length or direction constraints, respectively. The frameworks are rigid but not globally rigid.

We will only consider *generic* frameworks, meaning that the set containing the coordinates of all of the vertices is algebraically independent over the rationals: this eliminates many pathologies. It follows from [9, 11] that rigidity is a 'generic property' in the sense that if some generic realisation of G in $\mathbb{R}^d$ is rigid then all generic realisations of G in $\mathbb{R}^d$ are rigid. This implies that redundant rigidity is also a generic property and allows us to describe a mixed graph G as being *rigid* or *redundantly rigid* in $\mathbb{R}^d$ if some (or equivalently if every) generic realisation of G has these properties. It is not known whether global rigidity is a generic property (however this statement would follow from Conjecture 8.1 below in the 2-dimensional case).

Both rigidity and global rigidity are known to be generic properties for *pure frameworks*, i.e. frameworks which only contain length constraints or direction constraints. We will not give formal definitions for such frameworks, but note that they are similar to those for direction-length frameworks, except that the notion of congruence has to allow not only translations, but also rotations in the case of length-pure frameworks, and dilations in the case of direction-pure frameworks.

The problems of characterizing rigidity and global rigidity for 2-dimensional generic length pure frameworks were solved by Laman [13] and Jackson and Jordán [6], respectively. In particular, [6] proved that a 2-dimensional generic length pure framework (G, p) is globally length-rigid if and only if either G is a complete graph on at most 3 vertices, or G is 3-connected and redundantly length-rigid. The problems of characterizing rigidity and global rigidity for d-dimensional generic length pure frameworks are open for $d \geq 3$. In contrast, Whiteley [20] showed that rigidity and global rigidity are *equivalent* generic properties for direction frameworks and characterised the d-dimensional generic frameworks which have these properties for all d. Since direction-length frameworks are more general than length pure frameworks, we will restrict our attention to direction-length frameworks of dimension two.

The rigidity of generic 2-dimensional direction-length frameworks was characterised by Servatius and Whitely in [18]. In particular they showed that every minimally rigid mixed graph has $2|V| - 2$ edges, and that every generic realisation of a rigid mixed graph with exactly one length edge is globally rigid. Further results on global rigidity were obtained by Jackson and Jordán in [8] who showed that two necessary conditions for a generic 2-dimensional direction-length framework (G, p) to be globally rigid are that G is 2-connected and *direction-balanced* i.e. whenever H_1, H_2 are subgraphs of G with $G = H_1 \cup H_2$, $V(H_1) \cap V(H_2) = \{u, v\}$ and $V(H_1) \setminus V(H_2) \neq \emptyset \neq V(H_2) \setminus V(H_1)$, both H_1 and H_2 must contain a direction edge of G distinct from uv. They also showed that these conditions are sufficient when G is redundantly

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rigid and has $2|V| - 1$ edges.

We will see in Section 2.1 that we may define a matroid $M(G)$ on the edge set of a mixed graph G in such a way that G is rigid if and only if $M(G)$ has rank $2|V| - 2$. The above sufficient conditions for global rigidity, that G is redundantly rigid and has $2|V| - 1$ edges, are equivalent to the edge set of G being a rigid circuit of $M(G)$. A mixed graph G is redundantly rigid if and only if it is rigid and every edge of G is contained in a circuit of $M(G)$. We say that G is M-connected if it satisfies the stronger condition that every pair of edges of G is contained in a circuit of $M(G)$. (This is equivalent to $M(G)$ being a connected matroid.) Clinch [2] has recently shown that the above mentioned necessary conditions for generic global rigidity are also sufficient when the underlying mixed graph is M-connected and rigid.

Theorem 1.1. *Suppose (G, p) is a generic realisation of an M-connected rigid mixed graph G. Then (G, p) is globally rigid if and only if G is 2-connected and direction-balanced.*

Unfortunately, Clinch's result does not give a complete characterisation of generic global rigidity because M-connectivity is not a necessary condition for the global rigidity of generic rigid frameworks. This follows from the above mentioned fact that every generic realisation of a (minimally) rigid mixed graph with exactly one length edge is globally rigid, or from the fact that global rigidity is preserved if we join a new vertex to an existing globally rigid framework by two direction constraints. (The underlying graphs in both constructions are not even redundantly rigid.) We can generalise the second construction as follows.

Suppose (G, p) is a generic realisation of a mixed graph G which has a proper induced subgraph H such that the graph obtained from G by contracting H to a single vertex (and deleting all edges contained in H) has only direction edges and is the union of two edge-disjoint spanning trees. We will see in Section 4 that $G - e$ is not rigid for all direction edges e which do not belong to H (hence G is not redundantly rigid), and that (G, p) is globally rigid if and only if $(H, p|_H)$ is globally rigid.

These observations lead us to consider a more general reduction operation for a mixed graphs G. We say that G admits a *direction reduction* to a subgraph H if either:

(a) $H = G - e$ for some edge $e \in D$ which belongs to a direction-pure circuit in the rigidity matroid of G, or

(b) H is a proper induced subgraph of G, and the graph obtained by contracting H to a single vertex (and deleting all edges contained in H) is direction pure and is the union of two edge-disjoint spanning trees.

If G has no direction reduction, then we say that G is *direction irreducible*. (We will describe an efficient algorithm in Section 7 which either finds a direction reduction of a given mixed graph or concludes that it is direction irreducible.) An example of a direction reduction is given in Figure 2.

Our first result reduces the problem of characterizing the globally rigidity of a generic framework (G, p) to the case when G is direction irreducible.

Theorem 1.2. *Suppose (G, p) is a generic direction-length framework and G admits a direction reduction to a subgraph H. Then (G, p) is globally rigid if and only if $(H, p|_H)$ is globally rigid.*

We will obtain structural information about the family of direction irreducible mixed graphs which are not redundantly rigid and use it to prove our main result which characterises (direction irreducible) mixed graphs for which every generic realisation is globally rigid.
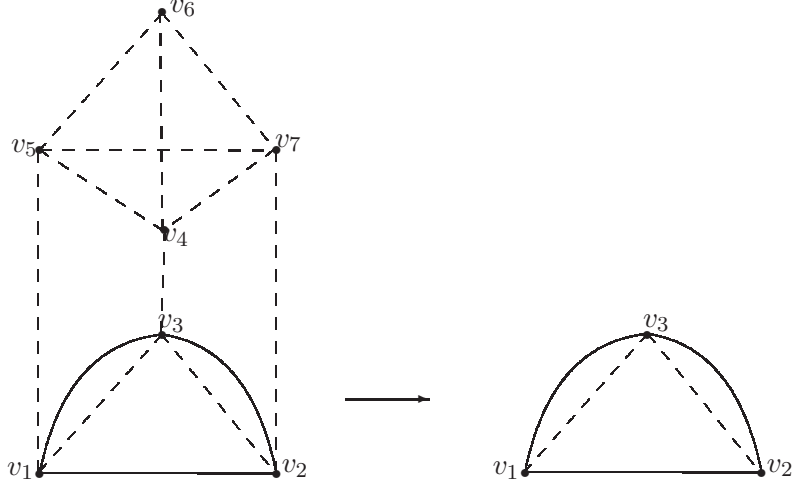
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Figure 2: The graph G on the left is direction reducible to the subgraph H on the right in two steps. Since the direction edge v_5v_7 is contained in the direction-pure circuit induced by $\{v_4, v_5, v_6, v_7\}$ we can delete v_5v_7 by (a). The graph we now obtain by contracting H to a single vertex is direction pure and is the union of two edge-disjoint spanning trees so we can reduce G to H by (b). Theorem 1.2 now tells us that a generic framework (G, p) is globally rigid if and only if $(H, p|_H)$ is globally rigid. Since H is an M-circuit, $(H, p|_H)$ is globally rigid by [8] or [2]. Hence (G, p) is globally rigid.

Theorem 1.3. *Suppose G is a direction irreducible mixed graph with at least two length edges. Then every generic realisation of G is globally rigid if and only if G is 2-connected, direction-balanced and redundantly rigid.*

The organisation of this paper by section is 1: Introduction, 2: Preliminaries, 3: Realising a graph as a direction framework, 4: Direction reduction, 5: Direction irreducible graphs, 6: Proof of Theorem 1.3, 7: Algorithmic considerations, 8: Closing Remarks.

2 Preliminaries

In this section we collect tools from diverse areas that we will use in our proofs.

2.1 Rigidity

Suppose (G, p) is a 2-dimensional direction-length framework. Its *rigidity matrix* is a $(|D| + |L|) \times 2|V|$ matrix $R(G, p)$, where each edge in $D \cup L$ corresponds to a row and each vertex in V corresponds to a pair of consecutive columns. We choose an arbitrary reference orientation for the edges, and use the notation $e = uv$ to mean that e has been oriented from u to v. Fix an edge e, a vertex x, and write $p(u) - p(v) = (a, b)$. Then the two entries in the rigidity matrix corresponding to e and x are as follows. If $e \in L$ we take (a, b) if $x = u$, $(-a, -b)$ if $x = v$, $(0, 0)$ otherwise. If $e \in D$ we take $(b, -a)$ if $x = u$, $(-b, a)$ if $x = v$, $(0, 0)$ otherwise.

We refer to vectors in the null space $Z(G, p)$ of $R(G, p)$ as *infinitesimal motions*. The labeling of the columns of $R(G, p)$ allows us to consider each infinitesimal motion z as a map from V to $\mathbb{R}^2$, with the properties that $z(u) - z(v)$ is perpendicular to $p(u) - p(v)$ if $e = uv \in L$, or

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parallel to $p(u) - p(v)$ if $e = uv \in D$. For any $a \in \mathbb{R}^2$ the translation given by $z(v) = a$ for all $v \in V$ is an infinitesimal motion, so $\dim Z(G,p) \geq 2$ and rank $R(G,p) \leq 2|V| - 2$. We can 'factor out' translations by restricting attention to realisations (G,p) that are in *standard position*, i.e. satisfy $p(v_0) = (0,0)$ for some fixed $v_0 \in V$. Write $R(G,p)_{v_0}$ for the matrix obtained from $R(G,p)$ by deleting the 2 columns corresponding to v_0 and let $Z(G,p)_{v_0}$ be its null space. Then $Z(G,p)_{v_0}$ is isomorphic to the subspace $Z(G,p)^*_{v_0}$ of $Z(G,p)$ consisting of all infinitesimal motions which fix v_0. Since all non-zero translations belong to $Z(G,p) \setminus Z(G,p)^*_{v_0}$ we have $\dim Z(G,p)_{v_0} = \dim Z(G,p) - 2$, so rank $R(G,p)_{v_0} = 2|V| - 2 - \dim Z(G,p)_{v_0} = 2|V| - \dim Z(G,p) = $ rank $R(G,p)$. We say that the framework (G,p) is *infinitesimally rigid* if rank $R(G,p) = 2|V| - 2$, and is *independent* if the rows of $R(G,p)$ are linearly independent.

A property P of frameworks is *generic* if whenever some generic realisation of a graph G has property P then all generic realisations of G have property P. If P is a generic property then we say that a graph G has property P if some generic realisation of G has property P (or equivalently all generic realisations of G have property P). Infinitesimal rigidity and independence are both generic properties, as the rank of $R(G,p)$ is the same for all generic realisations of G. Results from [9, 11], which will be described in Section 2.5, imply that infinitesimal rigidity and rigidity are equivalent properties for generic direction-length frameworks. Thus rigidity and redundant rigidity are also generic properties.

The rigidity matrix of (G,p) defines the *rigidity matroid* of (G,p): the ground set $D \cup L$ corresponds to rows of the rigidity matrix, and a subset is independent when the corresponding rows are linearly independent. Any two generic realisations of G have the same rigidity matroid, which we call the (2-dimensional) *rigidity matroid* $M(G)$ of G. (We refer the reader to [17] for an introduction to the theory of matroids.)

Servatius and Whiteley [18] characterised independence in the rigidity matroid of a mixed graph $G = (V; D, L)$: a set of edges $F \subseteq D \cup L$ is independent in $M(G)$ if and only if for all $\emptyset \neq F' \subseteq F$ we have $|F'| \leq 2|V(F')| - 2$, with strict inequality if F' is length or direction pure. This implies that F is a circuit of $M(G)$ if and only if $F - e$ is independent for all $e \in F$ and either F is mixed and $|F| = 2|V(F)| - 1$, or F is pure and $|F| = 2|V(F)| - 2$.

Servatius and Whiteley also gave the following recursive construction for independent rigid mixed graphs, i.e. bases in the rigidity matroid of the 'complete mixed graph'. A 0-*extension* of G is a mixed graph obtained from G by adding a new vertex v and two edges at v, either of which may be a length edge or a direction edge, and which may go to the same vertex of G if they consist of one length edge and one direction edge. A 1-*extension* of G is a mixed graph obtained from G by adding a new vertex v, deleting an edge e of G, and adding three edges at v, such that the neighbours of v include both endpoints of e, and neither D nor L decrease in size. They showed that 0-extensions and 1-extensions preserve independence and rigidity, and conversely, any independent rigid mixed graph can be constructed starting from a single vertex by a sequence of 0-extensions and 1-extensions.

2.2 M-circuits and M-components

It is well known that a matroid can be expressed as the direct sum of its connected components, which are the equivalence classes of the relation $\sim$, where $e \sim f$ if $e = f$ or there is a circuit containing e and f. We define the M-*components* of a mixed graph $G = (V; D, L)$ to be the subgraphs induced by the edges in the connected components of its rigidity matroid $M(G)$. Similarly, we define the M-*circuits* of G to be the subgraphs induced by the edges of the circuits of $M(G)$. We can use the direct sum decomposition of the rigidity matroid $M(G)$ to calculate its

rank, which we will denote by $r(G)$. Indeed, if G has M-components $H_1, \ldots, H_m$ then we have $r(G) = \sum_{i=1}^{m} r(H_i)$, where $r(H_i)$ is $2|V(H_i)| - 3$ when H_i is pure and is $2|V(H_i)| - 2$ otherwise. We can use this fact to show that M-connectivity is equivalent to redundant rigidity when G is direction irreducible and satisfies the necessary conditions for generic global rigidity described in Section 1.

Lemma 2.1. *Suppose G is a direction irreducible, 2-connected, direction balanced mixed graph. Then G is M-connected if and only if G is redundantly rigid.*

Proof. We have already seen that redundant rigidity is a necessary condition for M-connectedness. To prove sufficiency we suppose that G is redundantly rigid but not M-connected. Let $H_1, H_2, \ldots, H_t$ be the M-components of G. Let $V_i = V(H_i)$, $X_i = V_i - \bigcup_{j \neq i} V_j$ and $Y_i = V_i - X_i$ for all $1 \leq i \leq t$. Since G is redundantly rigid, every edge of G is contained in some M-circuit. Hence $|V_i| \geq 3$ for all $1 \leq i \leq t$. Since G is 2-connected, $|Y_i| \geq 2$ for all $1 \leq i \leq t$, and since G is direction-balanced, $|Y_i| \geq 3$ when H_i is length-pure. Since G is direction irreducible, no direction edge of G is contained in a direction-pure M-circuit. This implies that each of the M-connected components is either mixed or length-pure. Without loss of generality, we may assume that $H_1, H_2, \ldots, H_\ell$ are length pure for some $0 \leq \ell \leq t$, and $H_{\ell+1}, H_{\ell+2}, \ldots, H_t$ are mixed. Then

$$r(G) = \sum_{i=1}^{\ell} (2|V_i| - 3) + \sum_{i=\ell+1}^{t} (2|V_i| - 2)$$

$$= \sum_{i=1}^{\ell} (2|X_i| + 2|Y_i| - 3) + \sum_{i=\ell+1}^{t} (2|X_i| + 2|Y_i| - 2)$$

$$\geq \sum_{i=1}^{t} (2|X_i| + |Y_i|),$$

since $|Y_i| \geq 2$ for all $1 \leq i \leq t$, with strict inequality when $1 \leq i \leq \ell$. Since the X_i are all disjoint, we have $\sum_{i=1}^{t} |X_i| = \left| \bigcup_{i=1}^{t} X_i \right|$. Also, since each element of Y_i is contained in at least one other Y_j with $j \neq i$, we have $\sum_{i=1}^{t} |Y_i| \geq 2 |\bigcup_{i=1}^{t} Y_i|$. Thus

$$r(G) \geq 2 \left(\left| \bigcup_{i=1}^{t} X_i \right| + \left| \bigcup_{i=1}^{t} Y_i \right| \right) = 2|V|.$$

This contradicts the fact that $r(G) \leq 2|V| - 2$. $\qquad\square$

2.3 Boundedness and global rigidity

Now we recall some results from [11, 12]. A direction-length framework (G, p) is *bounded* if there exists a real number K such that $\|q(u) - q(v)\| < K$ for all $u, v \in V$ whenever (G, q) is a framework equivalent to (G, p). It follows from [11] that boundedness is a generic property, so it makes sense to speak of boundedness for a mixed graph $G = (V; D, L)$. We say that G is *direction-independent* if D is independent in the direction-length rigidity matroid of G, i.e. the rows of $R(G, p)$ corresponding to D are linearly independent for any generic p. The facts that direction pure M-circuits are generically rigid (when considered as direction pure frameworks) and that generic rigidity and global rigidity are equivalent for direction pure frameworks, allow us reduce the problem of deciding if a mixed graph is bounded to the family of direction-independent mixed graphs. The following characterisation of boundedness for direction-independent mixed graphs follows from [11, Theorem 5.1 and Corollary 4.3].

Lemma 2.2. *Suppose that G is direction-independent. Then G is bounded if and only if G/L has two edge-disjoint spanning trees (where G/L is the graph obtained from G by contracting each edge in L and keeping all multiple copies of direction edges created by this contraction).*

We will also make use of the following property of unbounded mixed graphs. For any graph H, the *cycle matroid* $M_1(H)$ is defined by saying a set of edges is independent if and only if it is acyclic. Let $M_2(H)$ be the matroid union of two copies of $M_1(H)$, i.e. a set is independent in $M_2(H)$ if it is the union of two acyclic sets. Then Lemma 2.2 says that a direction-independent mixed graph G is bounded if and only if $M_2(G/L)$ achieves its maximum possible rank $2|V(G/L)| - 2$.

A *bounded component* is a maximal bounded subgraph of G. It is shown in [11] that each edge $e \in L$ lies in a bounded component and that the vertex sets of the bounded components partition V. The following lemma is implicit in [11]; for completeness we include a short proof.

Lemma 2.3. *[11] Suppose G is direction-independent and $\mathcal{S}$ is a set of bounded components of G with $|\mathcal{S}| \geq 2$. Then there are at most $2|\mathcal{S}| - 3$ edges of G joining distinct components in $\mathcal{S}$.*

Proof. Suppose on the contrary that $|\mathcal{S}| \geq 2$ and there are at least $2|\mathcal{S}| - 2$ edges of G that join two distinct components in $\mathcal{S}$. Suppose also that $\mathcal{S}$ is minimal with this property. Let G' be the subgraph of G spanned by $\cup_{C \in \mathcal{S}} C$. Let H be a graph with vertex set $\mathcal{S}$ with $2|\mathcal{S}| - 2$ edges, each of which correspond to an edge of G joining two distinct component in $\mathcal{S}$. Note that H only has direction edges, as length edges cannot join two distinct bounded components. Also, the minimality of $\mathcal{S}$ implies that $M_2(H)$ is independent and has rank $2|\mathcal{S}| - 2$, so H has two edge-disjoint spanning trees. By Lemma 2.2, for each component $C \in \mathcal{S}$, $C/L[C]$ has two edge-disjoint spanning trees. Combining these with those in H we obtain two edge-disjoint spanning trees in G'/L. But then G' is bounded, so is contained in a bounded component of G, contradicting $|\mathcal{S}| \geq 2$. $\bullet$

Now we can state the main result of [12] on global rigidity, which establishes that length-redundancy is a necessary condition for generic global rigidity and takes a first step towards understanding when direction-redundancy is necessary. A subgraph of a mixed graph is said to be *trivial* if it has exactly one vertex, otherwise it is *non-trivial*.

Theorem 2.4. *[12] Suppose that (G, p) is a globally rigid generic realisation of a mixed graph $G = (V; D, L)$ and e is an edge of G.*
(a) If $e \in L$ and $|L| \geq 2$ then $G - e$ is rigid.
(b) If $e \in D$ and $G - e$ has a non-trivial rigid subgraph then $G - e$ is either rigid or unbounded.

2.4 Substitution

The following subgraph substitution operation is an important tool which we will use throughout this paper. Suppose $G = (V; D, L)$ is a mixed graph, $U \subseteq V$, $H = G[U]$ is the subgraph of G induced by U, and H' is another mixed graph with vertex set U. Then the *substitution G' of H by H' in G* is obtained from G by deleting all edges of H and adding all edges of H'. We record the following properties.

Lemma 2.5. *If G, H and H' are rigid then G' is rigid.*

Proof. The ranks of G and G' are both equal to the rank of the graph obtained from G by joining all pairs of vertices of H by both a direction and a length edge. $\bullet$

Lemma 2.6. *Suppose* $p : V \to \mathbb{R}^2$ *is such that* (G, p) *and* $(H', p|_U)$ *are both globally rigid. Then* (G', p) *is globally rigid.*

Proof. Let (G', q) be an equivalent framework to (G', p). Since $(H', p|_U)$ is globally rigid, $q|_U$ is congruent to $p|_U$. In particular, $(H, q|_U)$ and $(H, p|_U)$ are equivalent. But G and G' agree on all edges not contained in U, so (G, q) and (G, p) are equivalent. Since (G, p) is globally rigid, q and p are congruent. Hence (G', p) is globally rigid. $\bullet$

2.5 Differential geometry and the framework space

Here we recall some basic concepts of differential geometry; we refer the reader to [15] for an introduction to this subject. Let X be a smooth manifold, $f : X \to \mathbb{R}^n$ be a smooth map, and k be the maximum rank of its derivative $df|_y$ over all $y \in X$. A point $x \in X$ is a *regular point* of f if rank $df|_x = k$. The Inverse Function Theorem states that if U is open in $\mathbb{R}^k$, $f : U \to \mathbb{R}^k$ is smooth, $x \in U$, and the derivative $df|_x : \mathbb{R}^k \to \mathbb{R}^k$ is non-singular, then f maps any sufficiently small open neighbourhood of x diffeomorphically onto an open subset of $\mathbb{R}^k$. The following lemma is a simple consequence of this (see [12, Lemma 3.3]).

Lemma 2.7. *Let* U *be an open subset of* $\mathbb{R}^m$, $f : U \to \mathbb{R}^n$ *be a smooth map and* $x \in U$ *be a regular point of* f. *Suppose that the rank of* $df|_x$ *is* n. *Then there exists an open neighbourhood* $W \subseteq U$ *of* x *such that* $f(W)$ *is an open neighbourhood of* $f(x)$ *in* $\mathbb{R}^n$.

The following function plays an important role in rigidity theory. For $v_1, v_2 \in V$ with $p(v_i) = (x_i, y_i)$ let $l_p(v_1, v_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$, and $s_p(v_1, v_2) = (y_1 - y_2)/(x_1 - x_2)$ whenever $x_1 \neq x_2$. Suppose $e = v_1 v_2 \in D \cup L$. We say that e is *vertical* in (G, p) if $x_1 = x_2$. The *length* of e in (G, p) is $l_p(e) = l_p(v_1, v_2)$, and the *slope* of e is $s_p(e) = s_p(v_1, v_2)$, whenever e is not vertical in (G, p). Let $V = \{v_1, v_2, \ldots, v_n\}$ and $D \cup L = \{e_1, e_2, \ldots, e_m\}$. We view p as a point $(p(v_1), p(v_2), \ldots, p(v_n))$ in $\mathbb{R}^{2n}$. Let T be the set of all points $p \in \mathbb{R}^{2n}$ such that (G, p) has no vertical direction edges. Then the *rigidity map* $f_G : T \to \mathbb{R}^m$ is given by $f_G(p) = (h(e_1), h(e_2), \ldots, h(e_m))$, where $h(e_i) = l_p(e_i)$ if $e_i \in L$ and $h(e_i) = s_p(e_i)$ if $e_i \in D$.

One can verify (see [12]) that each row in the Jacobian matrix of the rigidity map is a non-zero multiple of the corresponding row in the rigidity matrix, so these matrices have the same rank. Thus the rigidity matrix achieves its maximum rank at a realisation (G, p) when p is a regular point of the rigidity map. In particular, this is the case when (G, p) is generic.

The *framework space* $S_{G,p,v_0} \subseteq \mathbb{R}^{2|V|-2}$ consists of all q in standard position with respect to v_0 with (G, q) equivalent to (G, p). Here we recall that 'standard position' means that $q(v_0) = (0, 0)$, and we identify a realisation (G, q) with the vector in $\mathbb{R}^{2|V|-2}$ obtained by concatenating the vectors $q(v)$ for $v \in V \setminus \{v_0\}$. The proof of the following lemma is the same as that of [12, Theorem 1.3], omitting the part that proves $-p_0 \notin C$, as this is now an assumption.

Lemma 2.8. *Suppose* (G, p) *is a generic direction-length framework, e is a direction edge of G, G is rigid, and $H = G - e$ is bounded and not rigid. Let v_0 be a vertex of G, let p_0 be obtained from p by translating v_0 to the origin, and let C be the connected component of the framework space S_{H,p,v_0} containing p_0. Then C is diffeomorphic to a circle. Furthermore, if $-p_0 \notin C$ then (G, p) is not globally rigid.*

2.6 Field extensions and genericity

A mixed framework (G, p) is *quasi-generic* if it is a translation of a generic framework. We will be mostly concerned with quasi-generic frameworks in standard position, i.e. with one vertex positioned at the origin. Such frameworks are characterized by the following elementary lemma.

Lemma 2.9. *[9] Let (G, p) be a framework with vertices $\{v_1, v_2, ..., v_n\}$, $p(v_1) = (0,0)$ and $p(v_i) = (p_{2i-1}, p_{2i})$ for $2 \leq i \leq n$. Then (G, p) is quasi-generic if and only if $\{p_3, p_4, \ldots, p_{2n}\}$ is algebraically independent over $\mathbb{Q}$.*

Given a vector $p \in \mathbb{R}^d$, $\mathbb{Q}(p)$ denotes the field extension of $\mathbb{Q}$ by the coordinates of p. We say that p is *generic* in $\mathbb{R}^d$ if the coordinates of p are algebraically independent over $\mathbb{Q}$. Given fields K, L with $K \subseteq L$ the *transcendence degree* $td[L : K]$ of L over K is the size of the largest subset of L which is algebraically independent over K. (We refer the reader to [4] for an introduction to field extensions and transcendence degree.) A reformulation of Lemma 2.9 is that if (G, p) is a framework with n vertices, one of which is at the origin, then (G, p) is quasi-generic if and only if $td[\mathbb{Q}(p) : \mathbb{Q}] = 2n - 2$.

Recall that $G = (V; D, L)$ is independent if $D \cup L$ is independent in the (generic) rigidity matroid of G, and that f_G denotes the rigidity map of G, which is defined at all realisations (G, p) with no vertical direction edges. The next result relates the genericity of $f_G(p)$ to the genericity of p when G is independent.

Lemma 2.10. *[9] Suppose that G is an independent mixed graph and (G, p) is a quasi-generic realization of G. Then $f_G(p)$ is generic.*

We use $\overline{K}$ to denote the algebraic closure of a field K. Note that $td[\overline{K} : K] = 0$. We say that G is *minimally rigid* if it is rigid but $G - e$ is not rigid for any edge e; equivalently G is both rigid and independent. The following lemma relates $\overline{\mathbb{Q}(p)}$ and $\overline{\mathbb{Q}(f_G(p))}$ when G is minimally rigid.

Lemma 2.11. *[9] Let G be a minimally rigid mixed graph and (G, p) be a realization of G with no vertical direction edges and with $p(v) = (0,0)$ for some vertex v of G. If $f_G(p)$ is generic then $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_G(p))}$.*

Lemmas 2.10 and 2.11 imply the following result for rigid mixed graphs.

Corollary 2.12. *Let G be a rigid mixed graph and (G, p) be a quasi-generic realization of G with $p(v) = (0,0)$ for some vertex v of G. Then $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_G(p))}$.*

Proof. Let H be a minimally rigid spanning subgraph of G. By Lemma 2.10, $f_H(p)$ is generic. Hence Lemma 2.11 gives $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_H(p))}$. It is not difficult to see that $\overline{\mathbb{Q}(f_H(p))} \subseteq \overline{\mathbb{Q}(f_G(p))} \subseteq \overline{\mathbb{Q}(p)}$. Thus $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_G(p))}$. $\bullet$

We also need the following lemma, which implies that every realisation of a rigid mixed graph which is equivalent to a generic realisation is quasi-generic.

Lemma 2.13. *[9] Let (G, p) be a quasi-generic realisation of a rigid mixed graph G. Suppose that (G, q) is equivalent to (G, p) and that $p(v) = (0,0) = q(v)$ for some vertex v of G. Then $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(q)}$, so (G, q) is quasi-generic.*

3 Realisations of graphs with given direction constraints

Here we give a result concerning the realisation of a graph as a direction pure framework with given directions for its edges. We need the following concepts, introduced by Whiteley in [19]. A *frame* is a graph $G = (V, E)$ together with a map $q : E \to \mathbb{R}^2$. The *incidence matrix* of the frame (G, q) is an $|E| \times 2|V|$ matrix $I(G, q)$ defined as follows. We first choose an arbitrary reference orientation for the edges of E. Each edge in E corresponds to a row of $I(G, q)$ and each vertex of V to two consecutive columns. The submatrix of $I(G, q)$ with row labeled by $e = uv \in E$ and columns labeled by $x \in V$ is $q(e)$ if $x = u$, is $-q(e)$ if $x = v$, and is the 2-dimensional zero vector otherwise. It is known (see [19]) that when q is generic, $I(G, q)$ is a linear representation of $M_2(G)$ (the matroid union of two copies of the cycle matroid of G). Thus we may use the characterisation of independence in $M_2(G)$ given by Nash-Williams [16] to determine when $I(G, q)$ has linearly independent rows. For $X \subseteq V$, let $i_G(X)$ denote the number of edges of G between vertices in X.

Theorem 3.1. *Suppose $G = (V, E)$ is a graph and $q : E \to \mathbb{R}^2$ is generic. Then the rows of $I(G, q)$ are linearly independent if and only if $i_G(X) \leq 2|X| - 2$ for all $\emptyset \neq X \subseteq V$.*

We can use this result to show that a graph $G = (V, E)$ satisfying $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$ can be realised as a direction pure framework with a specified algebraically independent set of slopes for its edges, and that this realisation is unique up to translation and dilation when $|E| = 2|V| - 3$. Note that given any realisation of G, we can always translate a specified vertex z_0 to $(0, 0)$ and dilate to arrange any specified distance t between a specified pair of vertices x, y.

Theorem 3.2. *Let $G = (V, E)$ be a graph such that $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$. Let s be an injection from E to $\mathbb{R}$ such that $\{s_e\}_{e \in E}$ is generic. Suppose $x_0, y_0, z_0 \in V$ and $t \neq 0$ is a real number. Then there exists an injection $p : V \to \mathbb{R}^2$ such that $\|p(x_0) - p(y_0)\| = t$, $p(z_0) = (0, 0)$ and, for all $e = uv \in E$, $p(u) - p(v) \in \langle (1, s_e) \rangle$. Furthermore, if $|E| = 2|V| - 3$, then p is unique up to dilation by -1 through $(0, 0)$.*

Proof. We will construct p as a combination of vectors in the nullspaces of certain frames. First consider a generic frame q on G such that $q(e)$ is a scalar multiple of $(-s_e, 1)$ for every $e \in E$. Then for any p in the nullspace of $I(G, q)$ and $e = uv \in E$ we have $p(u) - p(v) \in \langle (1, s_e) \rangle$. However, p need not be injective. To address this issue, we instead fix any pair of vertices x, y, and consider the graph H obtained by adding the edge $f = xy$ to G (it may be parallel to an existing edge). Let (H, q) be a generic frame such that $q(e)$ is a scalar multiple of $(-s_e, 1)$ for every edge e of G. For all $X \subseteq V$ with $|X| \geq 2$, we have $i_H(X) \leq i_G(X) + 1 \leq 2|X| - 2$ by hypothesis. Theorem 3.1 now implies that the incidence matrix $I(H, q)$ of the frame has linearly independent rows. Thus rank $I(H, q) = $ rank $I(G, q|_E) + 1$. Writing Z_H for the null space of $I(H, q)$ and Z_G for the null space of $I(G, q|_E)$, we have $\dim Z_G = \dim Z_H + 1$, so we can choose $p_f \in Z_G \setminus Z_H$. Now taking a suitable linear combination of the vectors p_f, for all possible new edges $f = xy$, $x, y \in V$, we may construct a vector p in Z_G with $p(x) \neq p(y)$ for all $x, y \in V$. Since $p_f(u) - p_f(v) \in \langle (1, s_e) \rangle$ for each f we also have $p(u) - p(v) \in \langle (1, s_e) \rangle$. Furthermore, as noted before the proof, we can translate and dilate to satisfy the other conditions, thus constructing the required map p.

We next show uniqueness when $|E| = 2|V| - 3$. We have $\dim Z_G = 2|V| - $ rank $I(G, q|_E) = 2|V| - |E| = 3$. Define $p_1, p_2 : V \to \mathbb{R}^2$ by $p_1(v) = (1, 0)$ and $p_2(v) = (0, 1)$ for all $v \in V$. Note that $p_1, p_2 \in Z_G$. Also, p, p_1, p_2 are linearly independent, since $p(z_0) = (0, 0)$, $p_1(z_0) = (1, 0)$ and $p_2(z_0) = (0, 1)$, so $\{p, p_1, p_2\}$ is a basis for Z_G. Now suppose that $p' : V \to \mathbb{R}^2$ has the properties described in the first part of the lemma. Then $p' \in Z_G$ so $p' = ap + bp_1 + cp_2$ for some $a, b, c \in \mathbb{R}$.

Since $p'(z_0) = p(z_0) = (0,0)$ we have $b = c = 0$. Since $\|p'(x_0) - p'(y_0)\| = t = \|p(x_0) - p(y_0)\|$ we have $p' \in \{p, -p\}$. ●

The uniqueness part of this lemma gives the following two results of Whiteley, and Servatius and Whiteley.

Lemma 3.3. [20] *Suppose that (G, p) is a generic direction-pure framework. Then (G, p) is direction globally rigid if and only if it is direction-rigid.*

Lemma 3.4. [18] *Suppose that (G, p) is a generic realisation of a mixed graph $G = (V; D, L)$. If G is rigid and $|L| = 1$ then (G, p) is globally rigid.*

The following consequence of Theorem 3.2 will be needed in the next section.

Lemma 3.5. *Let $G = (V; D, L)$ be a rigid mixed graph and $H = (U; F, L)$ be a rigid proper induced subgraph of G which contains L. Suppose $G' = (V; D', L')$ is obtained from G by substituting H by $H' = (U; F', L')$, and that G' and H' are minimally rigid with exactly one length edge. Suppose (G, p) is a quasi-generic realisation with $p(u) = (0,0)$ for some vertex $u \in U$ and $(H, \tilde{q})$ is a realisation of H which is equivalent to $(H, p|_U)$. Then there is a realisation (G', q) of G' which is equivalent to (G', p) and satisfies $q|_U = \tilde{q}$.*

Proof. Define $f : D' \cup L' \to \mathbb{R}$ by $f(e) = s_{\tilde{q}}(e)$ for $e \in F'$, $f(e) = l_{\tilde{q}}(e)$ for $e \in L'$, and $f(e) = s_p(e)$ for $e \in D' \setminus F'$. Thus f is the evaluation of the rigidity map $f_{G'}$ at the required realisation (G, q), if it exists. To apply Theorem 3.2, we need to show that $f|_{D'}$ is generic. We will prove the stronger result that f is generic by showing that $td[\overline{\mathbb{Q}(f)} : \mathbb{Q}] = |D'| + |L'| = 2|V| - 2$. We have $td[\overline{\mathbb{Q}(p)} : \mathbb{Q}] = 2|V| - 2$, as p is quasi-generic, so it suffices to prove that $\overline{\mathbb{Q}(f)} = \overline{\mathbb{Q}(p)}$. Since G is rigid, Corollary 2.12 gives $\overline{\mathbb{Q}(f_G(p))} = \overline{\mathbb{Q}(p)}$. Also, f is obtained from $f_G(p)$ by replacing the values $f_H(p|_U)$ by the values $f_{H'}(\tilde{q})$, so we need to show that these generate the same algebraic closure over $\mathbb{Q}$. Since $(H, \tilde{q})$ is equivalent to $(H, p|_U)$, Lemma 2.13 gives $\overline{\mathbb{Q}(\tilde{q})} = \overline{\mathbb{Q}(p|_U)}$. Since $p|_U$ is quasi-generic, it follows that $\tilde{q}$ is quasi-generic. Then, since H and H' are rigid, two applications of Corollary 2.12, give $\overline{\mathbb{Q}(f_H(p|_U))} = \overline{\mathbb{Q}(p|_U)}$ and $\overline{\mathbb{Q}(f_{H'}(\tilde{q}))} = \overline{\mathbb{Q}(\tilde{q})}$. Putting these three equalities together gives

$$\overline{\mathbb{Q}(f_{H'}(\tilde{q}))} = \overline{\mathbb{Q}(\tilde{q})} = \overline{\mathbb{Q}(p|_U)} = \overline{\mathbb{Q}(f_H(p|_U))},$$

which is what we needed to prove $\overline{\mathbb{Q}(f)} = \overline{\mathbb{Q}(p)}$. Therefore f is generic. Now we can apply Theorem 3.2, with $x_0 y_0$ equal to the unique length edge of G', to obtain a realisation (G', q) with $f_{G'}(q) = f$. By construction $(H', q|_U)$ is equivalent to $(H', \tilde{q})$. But H' is globally rigid by Lemma 3.4, so $q|_U$ is congruent to $\tilde{q}$. Hence we can apply a translation, and possibly a dilation by -1, to obtain $q|_U = \tilde{q}$. ●

4 Direction reduction

In this section we prove Theorem 1.2. We also prove a lemma which determines when a rigid direction-independent mixed graph is direction reducible. For the proof of Theorem 1.2 we deal with the two reduction operations separately.

Lemma 4.1. *Suppose (G, p) is a generic realisation of a mixed graph $G = (V; D, L)$ and that $e = uv \in D$ belongs to a direction-pure M-circuit $H = (U; F, \emptyset)$ of G. Then (G, p) is globally rigid if and only if $(G - e, p)$ is globally rigid.*

Proof. If $(G - e, p)$ is globally rigid then (G, p) is clearly globally rigid. Conversely, suppose that (G, p) is globally rigid and $(G - e, q)$ is equivalent to $(G - e, p)$. Since H is a direction-pure circuit, both $(H, p|_U)$ and $(H - e, p|_U)$ are rigid direction-pure frameworks. Hence $(H - e, p|_U)$ is a globally rigid direction-pure framework by Lemma 3.3. Thus $q(u) - q(v)$ is a scalar multiple of $p(u) - p(v)$, and hence (G, q) is equivalent to (G, p). Since G is globally rigid, q is congruent to p. This shows that $(G - e, p)$ is globally rigid. ●

Lemma 4.2. *Let (G, p) be a quasi-generic realisation of a rigid mixed graph $G = (V; D, L)$. Suppose that G has a proper induced subgraph H such that the graph F obtained by contracting H to a single vertex (and deleting all edges contained in H) has only direction edges and is the union of two edge-disjoint spanning trees. Then (G, p) is globally rigid if and only if $(H, p|_H)$ is globally rigid.*

Proof. First suppose that $(H, p|_H)$ is globally rigid. Let G' be constructed from G by substituting H by a rigid graph H' with exactly one length edge. Then G' is rigid by Lemma 2.5. Since G' is rigid and has exactly one length edge, (G', p) is globally rigid by Lemma 3.4. Thus (G, p) is globally rigid by Lemma 2.6.

Conversely, suppose that $(H, p|_H)$ is not globally rigid. Then there exists an equivalent but non-congruent realisation $(H, \tilde{q})$ of H. Without loss of generality we may suppose that $p(u) = (0, 0) = \tilde{q}(u)$ for some $u \in V(H)$. Let $E \subseteq D$ be the set of edges of F and m be the number of vertices of F. Then $|E| = 2m - 2$, as F is the union of two edge-disjoint spanning trees. Since G is rigid we have

$$2|V| - 2 = r(G) \le |E| + r(H) \le 2m - 2 + 2|V(H)| - 2 = 2|V| - 2.$$

Thus equality must hold throughout. In particular, $r(H) = 2|V(H)| - 2$, so H is rigid.

Let $G' = (V, D', L')$ obtained from G by replacing H by a minimally rigid mixed graph H' with exactly one length edge. Then G' is rigid by Lemma 2.5. Since G' has $|E| + 2|V(H)| - 2 = 2|V| - 2$ edges, G' is minimally rigid and has exactly one length edge. By Lemma 3.5, there is a realisation (G', q) of G' which is equivalent to (G', p) and satisfies $q|_H = \tilde{q}$. Undoing the substitution, we have a realisation (G, q) of G which is equivalent to (G, p) and satisfies $q|_U = \tilde{q}$. Furthermore, p is not congruent to q, since $p|_U$ is not congruent to $\tilde{q}$. Thus (G, p) is not globally rigid. ●

Theorem 1.2 follows immediately from Lemmas 4.1 and 4.2. For future reference we also prove the following result, which determines when a rigid direction-independent mixed graph G is direction reducible to a given subgraph H.

Lemma 4.3. *Suppose $G = (V; D, L)$ is a rigid direction-independent mixed graph and $H = (V'; D', L')$ is an induced proper subgraph of G. Then G is direction reducible to H if and only if $L' = L$ and $|D \setminus D'| = 2|V \setminus V'|$.*

Proof. Since G is direction-independent, G is direction reducible to H if and only if $L' = L$ and the graph $F = (V''; D'', \emptyset)$ obtained by contracting H to a single vertex v_H is the union of two edge-disjoint spanning trees. Thus, if G is direction reducible to H, then we have $L' = L$ and $|D \setminus D'| = |D''| = 2|V''| - 2 = 2|V \setminus V'|$.

We next assume that $L' = L$ and $|D \setminus D'| = 2|V \setminus V'|$ and show that F is the union of two edge-disjoint spanning trees. Suppose not. Then by a theorem of Nash-Williams [16], there exists $X \subseteq V''$ with $|X| \ge 2$ and $i_F(X) \ge 2|X| - 1$. If $v_H \notin X$ then the fact that G is direction-independent

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implies that $i_F(X) = i_G(X) \le 2|X| - 3$. Thus $v_H \in X$. Since $|D''| = |D \setminus D'| = 2|V''| - 2$ there are at most $(2|V''| - 2) - (2|X| - 1) = 2|V'' \setminus X| - 1$ edges in F which are not induced by X. It follows that

$$r(G) \le r(G[V \setminus (V'' \setminus X)]) + (2|V'' \setminus X| - 1) \le (2|V \setminus (V'' \setminus X)| - 2) + (2|V'' \setminus X| - 1) = 2|V| - 3.$$

This contradicts the hypothesis that G is rigid. Thus F is the union of two edge-disjoint spanning trees and G is direction reducible to H. ●

5 Direction irreducible mixed graphs

Theorem 1.2 enables us to reduce the problem of characterising globally rigid generic direction-length frameworks to the case when the underlying graph is direction irreducible. In this section we prove a structural lemma for direction irreducible mixed graphs which have a globally rigid generic realisation even though they are not redundantly rigid. (Our aim is to gather enough structural information to enable us to show that such graphs do not exist.)

Lemma 5.1. *Let $G = (V; D, L)$ be a direction irreducible mixed graph which has $|L| \ge 2$ and is not redundantly rigid. Suppose that (G, p) is a globally rigid generic realisation of G. Then*
(a) $G - e$ is bounded for all $e \in D$,
(b) $r(G - e) = r(G) - 1$ for all $e \in D$, and
(c) every length edge of G belongs to a length-pure M-circuit of G.

Proof. (a) First note that G is direction-independent, since G is direction irreducible. Now suppose for a contradiction that $G - e$ is not bounded for some $e \in D$. We will show that G has a direction reduction. Let $H_1, H_2, \ldots, H_m$ be the bounded components of $G - e$. Then each length edge of G is contained in one of the subgraphs H_i. Let $D^* \subseteq D$ be the set of all edges of G joining distinct subgraphs H_i and H be the graph obtained from G by contracting each H_i to a single vertex. Since G is rigid, G is bounded. Since G is direction-independent, Lemma 2.2 now implies that the graph G/L obtained from G by contacting each length edge has two edge-disjoint spanning trees. Since H can be obtained from G/L by contracting a (possibly empty) set of direction edges, H also has two edge-disjoint spanning trees. In particular, $|D^*| \ge 2m - 2$. On the other hand, Lemma 2.3 implies that $|D^* - e| \le 2m - 3$. Thus $e \in D^*$, $|D^*| = 2m - 2$, and H is the union of two edge-disjoint spanning trees. Since G is rigid we have

$$2|V| - 2 = r(G) \le |D^*| + \sum_{i=1}^{m} r(H_i) \le 2m - 2 + \sum_{i=1}^{m} (2|V(H_i)| - 2) = 2|V| - 2.$$

Thus equality must hold throughout. In particular, $r(H_i) = 2|V(H_i)| - 2$ for each i, so each subgraph H_i is rigid.

Let $G' = (V, D', L')$ be obtained from G by substituting each non-trivial subgraph H_i by a minimally rigid graph H_i' with exactly one length edge. Each framework $(H_i', p|_{H_i'})$ is globally rigid by Lemma 3.4. Thus repeated applications of Lemma 2.6 imply that (G', p) is globally rigid. On the other hand, $|D'| + |L'| = |D^*| + \sum_{i=1}^{m} r(H_i) = 2m - 2 + \sum_{i=1}^{m} (2|V(H_i)| - 2) = 2|V| - 2$, so G' is minimally rigid. Theorem 2.4(a) now implies that G' has exactly one length edge. Since H_i' contains a length edge whenever H_i is non-trivial, $G - e$ has exactly one non-trivial bounded component, H_1 say. This implies that G is direction reducible to H_1, which contradicts the hypothesis that G is direction irreducible.

(b) Suppose that $r(G - e) = r(G)$ for some $e \in D$. Then e is contained in an M-circuit H of G. Since G is direction-independent, H must be a mixed M-circuit. Since G is not redundantly rigid, $G - f$ is not rigid for some $f \in D \cup L$. Theorem 2.4(a) implies that $f \in D$. Clearly f is not an edge of H and hence H is a non-trivial rigid subgraph of $G - e$. Theorem 2.4(b) now implies that $G - f$ is unbounded, contradicting (a).

(c) Choose $e \in L$. Then e belongs to an M-circuit H of G by Theorem 2.4(a). By (b), H cannot be a mixed M-circuit. Hence H is length-pure.   •

6   Proof of Theorem 1.3

Since every generic realisation of a direction irreducible, 2-connected, direction balanced, redundantly rigid graph is globally rigid by Theorem 1.1 and Lemma 2.1, we only need to show necessity in Theorem 1.3. Hence we may suppose that G is a direction irreducible mixed graph and that every generic realisation of G is globally rigid. Then G is 2-connected and direction balanced by [8]. We will complete the proof by applying Theorem 6.1 below to deduce that G must also be redundantly rigid. The proof idea is to show that, if G is not redundantly rigid, then for any given generic realisation (G, p), we can construct a sequence of generic realisations $q_0, q_1, \ldots, q_t$ such that $t \leq |D|$ and (G, q_t) is not globally rigid. We construct this sequence from (G, p) by first reflecting (G, p) in the x-axis to obtain (G, q_0), and then recursively "correcting" the changed direction constraints back to their original value in (G, p). Every time we "correct" a direction constraint, we obtain a new realisation in our sequence.

Theorem 6.1. *Let $G = (V; D, L)$ be a direction irreducible mixed graph with $|L| \geq 2$ such that G is not redundantly rigid. Then some generic realisation of G is not globally rigid.*

Proof. We shall proceed by contradiction. Assume that all generic realisations of G are globally rigid. By Lemma 5.1(b) and (c), every length edge of G is contained in a length-pure circuit in the rigidity matroid of G, and no direction edge of G is contained in any circuit. Let $D = \{d_1, d_2, \ldots, d_k\}$, let $G_1 = (V_1; \emptyset, L_1)$ be a non-trivial M-connected component of G and let $v_0 \in V_1$.

Let (G, p) be a quasi-generic realisation of G with $p(v_0) = (0, 0)$ and let (G, q_0) be the quasi-generic realisation obtained by reflecting (G, p) in the x-axis. Then $(G - D, p)$ is equivalent to $(G - D, q_0)$. In addition we have $s_{q_0}(d_i) = -s_p(d_i)$ for all $d_i \in D$, so (G, p) and (G, q_0) are not equivalent.

Claim 6.2. *For all $j \in \{0, 1, \ldots, k\}$ there exists a quasi-generic framework (G, q_j) with $q_j(v_0) = (0, 0)$, with rigidity map $f_G(q_j) = (h_{q_j}(e))_{e \in E}$ given by*

$$h_{q_j}(e) = \begin{cases} s_{q_0}(e) & \text{when } e \in \{d_{j+1}, d_{j+2}, \ldots, d_k\} \\ h_p(e) & \text{otherwise,} \end{cases}$$

and with the property that that $(G_1, q_j|_{V_1})$ can be obtained from $(G_1, q_0|_{V_1})$ by a rotation about the origin.

Proof. We proceed by induction on j. If $j = 0$ then the claim holds trivially for (G, q_0). Hence suppose that the required framework (G, q_j) exists for some $0 \leq j < k$. The quasi-generic framework $(G - d_j, q_j)$ is bounded but not rigid by Lemma 5.1(a) and (b) (since boundedness and rigidity are generic properties). Since (G, q_j) is globally rigid by assumption, Lemma 2.8 implies

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that we can continuously move $(G_j - d_j, q_j)$ to form $(G_j - d_j, -q_j)$ whilst keeping v_0 fixed at the origin and maintaining all edge constraints. During this motion, the direction of the missing edge $d_{j+1} = u_{j+1}v_{j+1}$ changes continuously from $q_j(v_{j+1}) - q_j(u_{j+1})$ to $-(q_j(v_{j+1}) - q_j(u_{j+1}))$, a rotation by $180°$. So at some point in this motion we must pass through a realisation $(G - d_{j+1}, q_{j+1})$ at which the slope of this missing edge is $s_p(d_{j+1})$. We can now add the edge d_j back to this realisation to obtain the desired framework (G, q_{j+1}). Note that since G_1 is a length rigid subgraph of $G - d_j$ and the motion of $(G_j - d_j, q_j)$ is continuous and keeps v_0 fixed at the origin, $(G_1, q_{j+1}|_{V_1})$ can be obtained from $(G_1, q_j|_{V_1})$ by a rotation about the origin.

It remains to show that (G, q_{j+1}) is quasi-generic. Let H be a minimally rigid spanning subgraph of G. Since $h_{q_{j+1}}(e) = \pm h_p(e)$ for all $e \in E(G)$ we have $\mathbb{Q}(f_H(q_{j+1})) = \mathbb{Q}(f_H(p))$. Since $f_H(p)$ is generic by Lemma 2.10, Lemma 2.11 implies that

$$\text{td}[\overline{\mathbb{Q}(q_{j+1})}, \mathbb{Q}] = \text{td}[\overline{\mathbb{Q}(f_H(q_{j+1}))}, \mathbb{Q}] = \text{td}[\overline{\mathbb{Q}(f_H(p))}, \mathbb{Q}] = 2|V| - 2.$$

We can now use Lemma 2.9 to deduce that (H, q_{j+1}), and hence also (G, q_{j+1}), are quasi-generic. $\qquad\square$

Applying Claim 6.2 with $j = k$, we obtain a quasi-generic realisation q_k of G which is equivalent to (G, p), has $q_k(v_0) = (0, 0)$, and is such that $(G_1, q_k|_{V_1})$ can be obtained from $(G_1, q_0|_{V_1})$ by a rotation about the origin. Since q_0 was obtained from p by reflecting V_1 across the x-axis, we have

$$q_k(v) = RZp(v) \quad \text{for all } v \in V_1$$

where R and Z are the 2×2 matrices representing this rotation and reflection. Since $(G_1, p|_{V_1})$ is a quasi-generic framework with at least four vertices and RZ acts on $\mathbb{R}^2$ as a reflection in some line through the origin, we have $q_k(v) \neq \pm p(v)$ for some $v \in V_1$. Hence $q_k|_{V_1}$ is not congruent to $p|_{V_1}$, and q_k is not congruent to p. This implies that (G, p) is not globally rigid and contradicts our initial assumption that all generic realisations of G are globally rigid. $\qquad\square$

7 Algorithmic considerations

In this section we describe a polynomial algorithm which decides if every generic realisation of a given mixed graph $G = (V; D, L)$ is globally rigid. If $|L| \leq 1$ then we need only determine whether G is rigid and this can be accomplished using an orientation algorithm as in [1] or a pebble game algorithm as in [14]. Hence we may suppose that $|L| \geq 2$.

We first consider the case when G is direction irreducible. In this case Theorem 1.3 tells us we need only determine whether G is 2-connected, direction balanced and redundantly rigid. The first two properties can be checked using the connectivity algorithm of [5], and the third by an orientation or pebble game algorithm.

It remains to show how we can reduce G to the direction irreducible case when G is direction reducible. We do this in two stages. In the first stage we reduce G to a direction independent graph $G' = (V; D', L)$ by choosing D' to be a maximal subset of D which is independent in $M(G)$. This may again be accomplished using an orientation or pebble game algorithm.

Our final step is to find a direction reduction for a direction independent graph. We accomplish this by using the following lemma combined with the algorithm for determining the bounded components in a mixed graph given in [11].

Lemma 7.1. *Suppose $G = (V; D, L)$ is a rigid, direction independent mixed graph. Then G is direction reducible if and only if $G - e$ is unbounded and has exactly one nontrivial bounded*

component for some $e \in D$. Furthermore, if $G - e$ is unbounded and has exactly one nontrivial bounded component H for some $e \in D$, then G is direction reducible to H.

Proof. We first suppose that G is direction reducible to a subgraph $G' = (V'; D', L)$. By Lemma 4.3, $|D \setminus D'| = 2|V \setminus V'|$. It follows that, for any $e \in D \setminus D'$, the graph obtained from $G - e$ by contracting $E(G')$ has $2|V \setminus V'| - 1$ edges and $|V \setminus V'| + 1$ vertices, so does not have two edge-disjoint spanning trees. This implies that $(G - e)/L$ does not have two edge-disjoint spanning trees so $G - e$ is unbounded by Lemma 2.2. In addition, we have

$$2|V| - 2 = r(G) \leq r(G') + |D \setminus D'| \leq 2|V'| - 2 + 2|V \setminus V'|$$

and equality must hold throughout. In particular, $r(G') = 2|V'| - 2$, so G' is rigid, and hence bounded. Since $L \subseteq E(G')$, G' must be the unique nontrivial bounded component of G.

We next suppose that $G - e$ is unbounded and has exactly one nontrivial bounded component $H = (V'; D', L)$ for some $e \in D$. Then $|(D - e) \setminus D'| \leq 2(|V \setminus V'| + 1) - 3$ by Lemma 2.3, so $|D \setminus D'| \leq 2|V \setminus V'|$. Strict inequality cannot hold since G is rigid, and hence bounded. Thus $|D \setminus D'| = 2|V \setminus V'|$ and G is direction reducible to H by Lemma 4.3. $\qquad\bullet$

8   Closing remarks

The question of deciding whether global rigidity is a generic property of direction-length frameworks remains open. Theorem 1.1 shows that it is a generic property when the underlying graph is M-connected, and the necessary conditions for global rigidity given in [8] show that it is also a generic property if the underlying graph is not both 2-connected and direction balanced. Theorem 1.2 and Lemma 2.1 reduce the question to the case when the underlying graph is direction irreducible and is not redundantly rigid. Theorem 1.3 tells us that a direction irreducible mixed graph G which is not redundantly rigid has a generic realisation which is not globally rigid, but it is still conceivable that G may also have a generic realisation which is globally rigid. We believe that this is not the case:

Conjecture 8.1. *Suppose (G, p) is a generic realisation of a direction irreducible mixed graph G with at least two length edges. Then (G, p) is globally rigid if and only if G is 2-connected, direction-balanced, and redundantly rigid.*

We use Lemma 5.1 to obtain some evidence in support Conjecture 8.1 in the arXiv version of this paper. In particular we show that the conjecture holds when the length edges of the mixed graph induce a length rigid subgraph.

Theorem 8.2. *[3] Let (G, p) be a generic realisation of a rigid graph $G = (V; D, L)$. Suppose that L induces a length-rigid subgraph of G with at least two edges and $r(G - e) = r(G) - 1$ for all $e \in D$. Then (G, p) is not globally rigid.*

This result allows us to obtain the following simple characterisation of global rigidity for generic realisations of mixed graphs with $2|V| - 1$ edges.

Theorem 8.3. *[3] Suppose (G, p) is a generic realisation of a mixed graph $G = (V; D, L)$ with $|D| + |L| \leq 2|V| - 1$. Then (G, p) is globally rigid if and only if G is rigid and either $|L| = 1$ or $|D| + |L| = 2|V| - 1$ and the subgraph of G induced by the unique circuit in $M(G)$ is mixed, direction-balanced, and contains L.*

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