The existence of designs II

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Abstract

We generalise the existence of combinatorial designs to the setting of subset sums in lattices with coordinates indexed by labelled faces of simplicial complexes. This general framework includes the problem of decomposing hypergraphs with extra edge data, such as colours and orders, and so incorporates a wide range of variations on the basic design problem, notably Baranyai-type generalisations, such as resolvable hypergraph designs, large sets of hypergraph designs and decompositions of designs by designs. Our method also gives approximate counting results, which is new for many structures whose existence was previously known, such as high dimensional permutations or Sudoku squares.

1 Introduction

The existence of combinatorial designs was proved in [15], to which we refer the reader for an introduction to and some history of the problem. There we obtained a more general result on clique decompositions of hypergraphs, that can be roughly understood as saying that under certain extendability conditions, the obstructions to decomposition can already be seen in two natural relaxations of the problem: the fractional relaxation (where we see geometric obstructions) and the integer relaxation (where we see arithmetic obstructions). The main theorem of this paper is an analogous result in a more general setting of lattices with coordinates indexed by labelled faces of simplicial complexes. There are many prerequisites for the statement of this result, so in this introduction we will first discuss several applications to longstanding open problems in Design Theory, which illustrate various aspects of the general picture, and give some indication of why it is more complicated than one might have expected given the results of [15].

1.1 Resolvable designs

In 1850, Kirkman formulated his famous 'schoolgirls problem':

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.

The general problem is to determine when one can find designs that are 'resolvable': the set of blocks can be partitioned into perfect matchings. Recall from [15] that a set S of q-subsets of an n-set X is a *design* with parameters (n, q, r, λ) if every r-subset of X belongs to exactly λ elements

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of S, and that such S exists for any $n > n_0(q,r,\lambda)$ satisfying the necessary divisibility conditions that $\binom{q-i}{r-i}$ divides $\lambda\binom{n-i}{r-i}$ for $0 \le i \le r-1$. For a resolvable design, a further necessary condition is that $q \mid n$, otherwise there is no perfect matching on X, let alone a partition of S into perfect matchings! We will show that these necessary conditions suffice for large n; the case $r = 2$ of this result was proved in 1973 by Ray-Chaudhuri and Wilson [26, 27].

Theorem 1.1. *Suppose $q \ge r \ge 1$ and λ are fixed and $n > n_0(q,r,\lambda)$ is large with $q \mid n$ and $\binom{q-i}{r-i} \mid \lambda\binom{n-i}{r-i}$ for $0 \le i \le r-1$. Then there is a resolvable (n,q,r,λ) design.*

It is convenient to generalise the problem and then translate the generalisation into an equivalent hypergraph decomposition problem. Suppose $G \in \mathbb{N}^{X_r}$ is a r-multigraph supported in an n-set X, where $q \mid n$, satisfying the necessary divisibility conditions for a K_q^r-decomposition, i.e. $\binom{q-i}{r-i}$ divides $\sum\{G_e : f \subseteq e\}$ for any i-set f with $0 \le i \le r$ (we say G is K_q^r-divisible). We also suppose that G is 'vertex-regular' in that $|G(x)|$ is the same for all $x \in X$ (this is clearly necessary for a resolvable decomposition). Under certain conditions (extendability and regularity) to be defined later, we will show that G has a K_q^r-decomposition that is resolvable, i.e. its q-sets can be partitioned into perfect matchings, each of which can be viewed as a K_q^r-tiling, i.e. n/q vertex-disjoint K_q^r's.

Now we set up an equivalent hypergraph decomposition problem. Let Y be a set of m vertices disjoint from X, where m is the least integer with[1] $\binom{m}{r-1} \ge q|G|/Qn$, with $Q = \binom{q}{r}$. Let J be an $(r-1)$-graph on Y with $|J| = q|G|/Qn = \binom{q-1}{r-1}^{-1}|G(x)|$ for all $x \in X$. Let G' be the r-multigraph obtained from G by adding as edges (with multiplicity one) all r-sets of the form $f \cup \{x\}$ where $f \in J$ and $x \in X$. Let H be the r-graph whose vertex set is the disjoint union of a q-set A and an $(r-1)$-set B, and whose edges consist of all r-sets in $A \cup B$ that are contained in A or have exactly one vertex in A.

We claim that a resolvable K_q^r-decomposition of G is equivalent to an H-decomposition of G'. To see this, suppose that an H-decomposition $\mathcal{H}$ of G' is given. Let $\mathcal{A}$ be the set of the restrictions to A of the copies of H in $\mathcal{H}$. Then $\mathcal{A}$ is a K_q^r-decomposition of G. Furthermore, for each $f \in J$ and $x \in X$ there is a unique copy of H in $\mathcal{H}$ containing $f \cup \{x\}$, so the elements of $\mathcal{A}$ corresponding to copies of H containing f form a perfect matching of X, and every element of $\mathcal{A}$ is thus obtained from a unique such f, so $\mathcal{A}$ is resolvable. Conversely, any resolvable K_q^r-decomposition of G can be converted into an H-decomposition of G' by assigning an edge of J to each perfect matching in the resolution and forming copies of H in the obvious way.

For general r-graphs H, Glock, Kühn, Lo and Osthus [9] solved the H-decomposition problem for certain structures that they call 'supercomplexes', which in particular solves the problem for r-graphs G that are 'typical' or have large minimum degree: they show that in this setting there is an H-decomposition if G is H-divisible, i.e. every i-set degree is divisible by the greatest common divisor of the i-set degrees in H. However, the bipartite form of the problem considered here is not covered by their framework; indeed, we will see later in more generality that there are additional complications in partite settings. Our general result on H-decompositions will be of the form discussed above, i.e. that under certain extendability conditions, the obstructions to decomposition appear in the fractional or integer relaxation. However, one should note that there can be additional obstructions in the integer relaxation besides the divisibility conditions mentioned above.

[1] We identify G with its edge set, so $|G|$ denotes the number of edges in G.

1.2 Baranyai-type designs

Next we develop the theme suggested by the construction in the previous section, namely that of obtaining variations on the basic design problem that are equivalent to certain partite hypergraph decomposition problems. We will call these Baranyai-type designs, after the classical result of Baranyai [1] that any complete r-graph K_n^r with $r \mid n$ can be partitioned into perfect matchings.

One natural question of this type is whether K_n^q can be decomposed into (n,q,r,λ)-designs; such a decomposition is known as a 'large set' of designs. Besides the necessary divisibility conditions discussed above for the existence of one such design, another obvious necessary condition is that the size $\lambda\binom{q}{r}^{-1}\binom{n}{r}$ of each design in the decomposition should divide the size $\binom{n}{q}$ of K_n^q; equivalently, we need $\lambda \mid \binom{n-r}{q-r}$. It is natural to conjecture that these necessary divisibility conditions should be sufficient, apart from a finite number of exceptions. Even in the very special case of Steiner Triple Systems, this was a longstanding open problem, settled in 1991 by Teirlinck [29]. Lovett, Rao and Vardy [22] extended the method of [18] to show that if $q > 9r$, $\ell \in \mathbb{N}$ and $n > n_0(q,r,\ell)$ satisfies the divisibility conditions then there is a large set of (n,q,r,λ)-designs, where $\binom{n}{q} = \ell\lambda\binom{q}{r}^{-1}\binom{n}{r}$. This settles the existence conjecture when the edge multiplicity λ is within a constant factor of the maximum possible multiplicity, but leaves it open otherwise (for example, it does not include the case of large sets of Steiner systems). We will prove the general form of the existence conjecture for large sets of designs.

Theorem 1.2. *Suppose $q \geq r \geq 1$ are fixed, $n > n_0(q,r)$ is large and $\lambda \mid \binom{n-r}{q-r}$ with all $\binom{q-i}{r-i} \mid \lambda\binom{n-i}{r-i}$. Then there is a large set of (n,q,r,λ) designs.*

As for resolvable designs, we can consider the more general problem of decomposing any q-multigraph G on an n-set X into (n,q,r,λ)-designs. This clarifies the general form of the divisibility conditions, as there are several conditions that collapse into one in the case that $G = K_n^q$. Indeed, for each $0 \leq i \leq r$ and i-set $I \subseteq [n]$ we need the degree $|G(I)|$ of I to be divisible by the number $Z_i := \lambda\binom{q-i}{r-i}^{-1}\binom{n-i}{r-i}$ of q-sets containing I in any (n,q,r,λ)-design. Furthermore, we clearly need G to be an 'r-multidesign', meaning that all $|G(e)|$ with $e \in [n]_r$ are equal.

Again we formulate an equivalent hypergraph decomposition problem. Let Y be a set of m vertices disjoint from X, where m is the least integer with $\binom{m}{q-r} \geq |G|/Z_0$. Let J be an $(q-r)$-graph on Y with $|J| = |G|/Z_0$. Let G' be the q-multigraph obtained from G by adding as edges with multiplicity λ all q-sets of the form $e \cup f$ with $e \subseteq X$ and $f \in J$. Let H be the q-graph whose vertex set is the disjoint union of a q-set A and a $(q-r)$-set B, and whose edges consist of A and all q-sets in $A \cup B$ that contain B. Then a decomposition of G into (n,q,r,λ)-designs is equivalent to an H-decomposition of G. Indeed, given an H-decomposition $\mathcal{H}$ of G, each edge of G appears as exactly one copy of A in $\mathcal{H}$, and for each $f \in J$ the copies of A within the copies of H that contain f form an (n,q,r,λ)-design. Conversely, any decomposition of G into (n,q,r,λ)-designs can be converted into an H-decomposition of G by assigning an edge of J to each design in the decomposition.

Another natural example of a Baranyai-type design is what we will call a 'complete resolution' of K_n^q: we partition K_n^q into Steiner $(n,q,q-1)$ systems, each of which is partitioned into Steiner $(n,q,q-2)$ systems, and so on, down to Steiner $(n,q,1)$ systems (which are perfect matchings). Again we show that this exists for $n > n_0(q)$ under the necessary divisibility conditions, which take the simple form $q-j \mid n-j$ for $0 \leq j < q$, i.e. $n = q \bmod lcm([q])$.

Theorem 1.3. *Suppose q is fixed and $n > n_0(q)$ is large with $n = q \bmod lcm([q])$. Then there is a complete resolution of K_n^q.*

To formulate an equivalent hypergraph decomposition problem, we consider disjoint sets of vertices X and Y where $|X| = n$ and Y is partitioned into Y_j, $0 \le j < q$ with $|Y_j| = \frac{n-j}{q-j}$. We let G' be the q-graph whose edges are all q-sets $e \subseteq X \cup Y$ such that $|e \cap Y_j| \le 1$ for all $0 \le j < q$, and if $e \cap Y_j \neq \emptyset$ then $e \cap Y_i \neq \emptyset$ for all $i > j$. Let H be the q-graph whose vertex set is the disjoint union of two q-sets A and $B = \{b_0, \dots, b_{q-1}\}$, whose edges are all q-sets $e \subseteq A \cup B$ such that if $b_j \in e$ then $b_i \in e$ for all $i > j$.

Then a complete resolution of K_n^q is equivalent to an H-decomposition of G'. Indeed, given an H-decomposition $\mathcal{H}$ of G', we note that for any $y_i \in Y_i$ for $j \le i \le q$ the set of copies of A in the copies of H in $\mathcal{H}$ that contain $\{y_j, \dots, y_q\}$ form a Steiner $(n, q, j-1)$ system, and as y_j ranges over Y_j we obtain a partition of the Steiner (n, q, j) system corresponding to the copies of H in $\mathcal{H}$ that contain $\{y_{j+1}, \dots, y_q\}$. Conversely, a complete resolution of K_n^q can be converted into an H-decomposition of G' by iteratively assigning vertices of Y_j to the Steiner $(n, q, j-1)$ systems that decompose each Steiner (n, q, j) system.

1.3 Partite decompositions

The above applications demonstrate the need for hypergraph decomposition in various partite settings. We defer our general statement and just give here some easily stated particular cases. First we consider the nonpartite setting and the typicality condition from [15].

Definition 1.4. Suppose G is an r-graph on $[n]$. The density of G is $d(G) = |G|\binom{n}{r}^{-1}$. We say that G is (c, s)-*typical* if for any set A of $(r-1)$-subsets of $V(G)$ with $|A| \le s$ we have $|\cap_{f \in A} G(f)| = (1 \pm |A|c)d(G)^{|A|}n$.

We show that any typical r-graph has an H-decomposition provided that it satisfies the necessary divisibility condition discussed above (this result was also proved in [9]).

Theorem 1.5. *Let H be an r-graph on $[q]$ and G be an H-divisible (c, h^q)-typical r-graph on $[n]$, where $n = |V(G)| > n_0(q)$ is large, $h = 2^{50q^3}$, $\delta = 2^{-10^3 q^5}$, $d(G) > 2n^{-\delta/h^q}$, $c < c_0 d(G)^{h^{30q}}$ where $c_0 = c_0(q)$ is small. Then G has an H-decomposition.*

Next we consider the other extreme in terms of partite settings.

Definition 1.6. Let H be an r-graph. We call an r-graph G an H-blowup if $V(G)$ is partitioned as $(V_x : x \in V(H))$ and each $e \in G$ is f-partite for some $f \in H$, i.e. $f = \{x : e \cap V_x \neq \emptyset\}$.

We write G_f for the set of f-partite $e \in G$. For $f \in H$ let $d_f(G) = |G_f| \prod_{x \in f} |V_x|^{-1}$. We call G a (c, s)-typical H-blowup if for any $s' \le s$ and distinct $e_1, \dots, e_{s'}$ where each e_j is f_j-partite for some $f_j \in V(H)_{r-1}$, and any $x \in \cap_{j=1}^{s'} H(f_j)$ we have

$$\left| V_x \cap \bigcap_{j=1}^{s'} G(e_j) \right| = (1 \pm s'c)|V_x| \prod_{j=1}^{s'} d_{f_j + x}(G).$$

We say G has a partite H-decomposition if it has an H-decomposition using copies of H with one vertex in each part V_x.

We say G is H-balanced if for every $f \subseteq V(H)$ and f-partite $e \subseteq V(G)$ there is some n_e such that $|G_{f'}(e)| = n_e$ for all $f \subseteq f' \in H$.

Note that if G has a partite H-decomposition then G is H-balanced; we establish the converse for typical H-blowups.

Theorem 1.7. *Let H be an r-graph on $[q]$ and G be an H-balanced (c, h^q)-typical H-blowup on $(V_x : x \in V(H))$, where each $n/h \le |V_x| \le n$ for some large $n > n_0(q)$ and $h = 2^{50q^3}$, $\delta = 2^{-10^3 q^5}$, $d_f(G) > d > 2n^{-\delta/h^q}$ for all $f \in H$ and $c < c_0 d^{h^{30q}}$, where $c_0 = c_0(q)$ is small. Then G has a partite H-decomposition.*

For example, if $H = K^r_{r+1}$ and $G = K^r_{r+1}(n)$ is the complete $(r+1)$-partite r-graph with n vertices in each part then Theorem 1.7 shows the existence of an object known variously as an r-dimensional permutation or latin hypercube. (It can be viewed as an assignment of 0 or 1 to the elements of $[n]^{r+1}$ so that every line has a unique 1, or as an assignment of $[n]$ to the elements of $[n]^r$ so that each line contains every element of $[n]$ exactly once.) The result for general G implies a lower bound on the number of r-dimensional permutations: we can estimate the number of choices for an almost H-decomposition by analysing a random greedy algorithm, and show that almost all of these can be completed to an (actual) H-decomposition (we omit the details of the proof, which are similar to those in [16]). In combination with the upper bound of Linial and Luria [21] we obtain the following answer to an open problem from [21].

Theorem 1.8. *The number of r-dimensional permutations of order n is $(n/e^r + o(n))^{n^r}$.*

More generally, many applications of our main theorem can be similarly converted to an approximate counting result, where the upper bound comes from a general bound by Luria [23] on the number of perfect matchings in a uniform hypergraph with small codegrees (for example, we could give such estimates for the number of resolvable designs or large sets of designs). Another example[2] is the following estimate for the number of (generalised) Sudoku squares (the theorem says nothing about the squares of the popular puzzle, in which $n = 3$).

Theorem 1.9. *The number of Sudoku squares with n^2 boxes of order n is $(n^2/e^3 + o(n^2))^{n^4}$.*

1.4 Colours and labels

There are many questions in design theory that are naturally expressed as a decomposition problem for hypergraphs with extra data associated to edges, such as colours or vertex labels. A decomposition theorem for coloured multidigraphs with several such applications was given by Lamken and Wilson [20]. Here we illustrate one such application and an example of a hypergraph generalisation. (There are many other such applications, but for the sake of brevity we leave a detailed study for future research.)

The Whist Tournament Problem (posed in 1896 by Moore [24]) is to find a schedule of games for $4n$ players, where in each game two players oppose two others, such that (1) the games are arranged into rounds, where each player plays in exactly one game in each of the rounds, (2) each player partners every other player exactly once and opposes every other player exactly twice. (There is also a similar problem for $4n + 1$ players in which one player sits out in each round.) Whist Tournaments exist for all n (see [5, Chapter VI.64]). If we remove condition (1) we obtain the Whist Table Problem. As observed in [20], we obtain an equivalent form of the latter by considering a red/blue coloured multigraph on the set of players, where between each pair of players there is one

[2] Alexey Pokrovskiy drew this to my attention. Our theorem does not apply to the construction given by Luria [23], but it is not hard to give a suitable alternative construction. For example, let H be the 4-graph with $V(H) = \{x_1, x_2, y_1, y_2, z_1, z_2\}$ and $E(H) = \{x_1x_2y_1y_2, x_1x_2z_1z_2, y_1y_2z_1z_2, x_1y_1z_1z_2\}$. Then an H-decomposition of the complete n-blowup of H can be viewed as a Sudoku square, where we represent rows by pairs (a_1, a_2), columns by (b_1, b_2), symbols by (c_1, c_2) and boxes by (a_1, b_1); a copy of H with vertices $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ represents a cell in row (a_1, a_2) and column (b_1, b_2) with symbol (c_1, c_2).

red edge ('partner') and two blue edges ('oppose'), and we seek a decomposition into copies of K_4 coloured as a blue C_4 with two red diagonals. The Whist Tournament Problem is equivalent to a partite decomposition problem that fits into our framework, but not that of [20] (which only covers the case of $4n+1$ players).

There are many ways to formulate similar problems with more complexities, such as larger teams and particular roles for players within teams. Here we describe a fictional illustration of this idea, which we may call a 'tryst tournament' (sports aficionados will no doubt be able to provide real examples). A tryst team consists of three players, one of whom is designated the captain. A tryst game is played by nine players divided into three tryst teams. The Tryst Table Problem is to find a schedule of tryst games for n players, such that (1) for every triple T of players and every $x \in T$ there is exactly one game in which T is a team and x is the captain, (2) for every triple T of players there is exactly one game in which T is the set of captains of the three teams in that game.[3]

Theorem 1.10. *The Tryst Table Problem has a solution for all sufficiently large n.*

We reformulate the Tryst Table Problem (somewhat vaguely at first) as follows. Form a 'structure' G on the set V of players by including a red triple ('captains') for each triple and a blue 'pointed' triple ('teams') for each triple T and $x \in T$. We want to decompose G by copies of a 'structure' H on 9 vertices, with 3 vertex-disjoint blue pointed triples, and a red triple consisting of the points of the blue triples.

To make sense of the undefined terms just used we now switch to a setting in which all edges come with labels on their vertices, so our fundamental object becomes a set of functions (instead of a hypergraph, which is a set of sets). For the Tryst Table Problem, we let G^* contain a red copy and a blue copy of each injection from [3] to V. We define a set H^* of red and blue injections from [3] to [9] as follows, in which we imagine that three teams are labelled 123, 456 and 789 with captains 1, 4 and 7. The red functions of H^* consist of all bijections from [3] to 147. The blue functions of H^* consist of all bijections from [3] to one of the teams 123, 456 or 789, such that 1 is mapped to the captain. A copy of H^* in G^* is defined by fixing any injection $\phi : [9] \to V$ and composing all functions in H^* with ϕ; the interpretation of this copy is a tryst game between teams $\phi(123)$, $\phi(456)$ and $\phi(789)$ with captains $\phi(1)$, $\phi(4)$ and $\phi(7)$. It is clear that the Tryst Table Problem is equivalent to finding an H^*-decomposition of G^*.

Our main theorem is a decomposition result for vectors where coordinates are indexed by functions and take values in some lattice $\mathbb{Z}^D$. The 'subcoordinates' in $\mathbb{Z}^D$ may be interpreted as colours, so e.g. we may think of $J_\psi = (2,3) \in \mathbb{Z}^2$ as saying that J has 2 red copies and 3 blue copies of some function ψ. This general framework includes all of the problems discussed above and many other variations thereupon (see subsection 2.4 for more examples).

One consequence of our main theorem is a generalisation of the hypergraph decomposition result alluded to above to decompositions of coloured multihypergraphs by coloured hypergraphs. It seems hard to describe the divisibility conditions in general, so here we will specialise to the setting of rainbow clique decompositions, for which the divisibility conditions are quite simple. We write $[\binom{q}{r}]K_n^r$ for the r-multigraph on $[n]$ in which there are $\binom{q}{r}$ copies of each r-set coloured by $[\binom{q}{r}] = \{1, \dots, \binom{q}{r}\}$. We ask when $[\binom{q}{r}]K_n^r$ can be decomposed into rainbow copies of K_q^r, i.e. copies of K_q^r in

[3] We choose these simple rules for simplicity of exposition, and there is no doubt a direct proof of Theorem 1.10 not using our main theorem. The point is that one can use the same method to analyse variations with more rules, such as a Tryst Tournament Problem (arranging the games into rounds) and/or constraining more triples, e.g. we could also ask for every triple T of players and every $x \in T$ to have exactly two games in which x captains a team and $T \setminus \{x\}$ is the set of non-captains in a different team.

$[\binom{q}{r}]K_n^r$ in which the colours of edges are all distinct. A stricter version of the question is to fix some rainbow colouring of K_q^r and only allow the decomposition to use copies of K_q^r that are isomorphic to the fixed rainbow colouring. We will answer both versions of the question.

First we consider the question in which we allow any rainbow K_q^r. Ignoring colours, we have the same necessary divisibility condition as before for the multigraph $\binom{q}{r}K_n^r$ to have a K_q^r-decomposition, namely $\binom{q-i}{r-i} \mid \binom{q}{r}\binom{n-i}{r-i}$ for $0 \leq i \leq r-1$. We will show that under the same conditions we even have a rainbow K_q^r-decomposition.

Theorem 1.11. *Suppose $n > n_0(q)$ is large and $\binom{q-i}{r-i} \mid \binom{q}{r}\binom{n-i}{r-i}$ for $0 \leq i \leq r-1$. Then $[\binom{q}{r}]K_n^r$ has a rainbow K_q^r-decomposition.*

Now suppose that we only allow copies of some fixed rainbow colouring. For convenient notation we identify the set of colours with $[q]_r := \{B \subseteq [q] : |B| = r\}$ and suppose that in the fixed colouring of $[q]_r$ we colour each set by itself. We write $[q]_rK_n^r$ for the corresponding relabelling of $[\binom{q}{r}]K_n^r$. Any injection $\phi : [q] \to [n]$ defines a copy of $[q]_r$ where for each $B \in [q]_r$ we use the colour B copy of $\phi(B)$. We say $[q]_rK_n^r$ has a $[q]_r$-decomposition if it can be decomposed into such copies.

Theorem 1.12. *Suppose $n > n_0(q)$ is large and $\binom{r}{i} \mid \binom{n-i}{r-i}$ for $0 \leq i \leq r-1$. Then $[q]_rK_n^r$ has a $[q]_r$-decomposition.*

The divisibility conditions in Theorem 1.12 are necessary for $r \leq q/2$ but not in general.[4] To see necessity, suppose $\mathcal{D}$ is a $[q]_r$-decomposition of $[q]_rK_n^r$. Identify each copy $\phi([q]_r)$ with a vector $v^\phi \in (\mathbb{Z}^{[q]_r})^{K_n^r}$ where each $v^\phi_{\phi(B)} = e_B$ (the standard basis vector for $B \in [q]_r$). For any $f \in [n]_i$ we have $\sum\{v_e^\phi : f \subseteq e \in K_n^r, \phi \in \mathcal{D}\} = \binom{n-i}{r-i}\underline{1} \in \mathbb{Z}^{[q]_r}$ equal to $\binom{n-i}{r-i}$ in each coordinate. On the other hand, the contribution of any given $\phi([q]_r)$ to this sum is $\sum\{e_B : \phi^{-1}(f) \subseteq B\}$, which is a row of the inclusion matrix $M_i^r(q)$: this has rows indexed by $[q]_i$, columns by $[q]_r$ and each $M_i^r(q)_{fe} = 1_{f\subseteq e}$. As $M_i^r(q)$ has full row rank (by Gottlieb's Theorem [10], using $r \leq q/2$), each row must appear the same number of times, say m, and then any $B \in [q]_r$ contributes $m\binom{r}{i}$ times, so $\binom{r}{i} \mid \binom{n-i}{r-i}$.

1.5 Decomposition lattices

Here we will give some indication of what new ideas are needed in the general setting besides those in [15]. We start by recalling the proof strategy in [15] for clique decompositions of hypergraphs that are extendable and have no obstruction (fractional or integer) to decomposition. The first step is a Randomised Algebraic Construction, which results in a partial decomposition (the 'template') that covers a constant fraction of the edge set, and carries a rich structure of possible local modifications. By the nibble and other random greedy algorithms, we can choose another partial decomposition that covers all edges not in the template, which also spills over slightly into the template, so that every edge is covered once or twice, and very few edges (the 'spill') are covered twice.

Next we find an 'integral decomposition' of the spill, and apply a 'clique exchange algorithm' that replaces the integral decomposition by a 'signed decomposition', i.e. two partial decompositions, called 'positive' and 'negative', such that the underlying hypergraph of the negative decomposition is contained in that of the positive decomposition, and the difference forms a 'hole' that is precisely equal to the spill. We also ensure that for each positive clique Q^+ there is a modification (a 'cascade') to the clique decomposition of the template so that it contains Q^+. Then deleting the positive cliques

[4] The precise divisibility conditions can be extracted from our characterisation of the divisibility lattice, but we omit this for the sake of a simple illustration.

and replacing them by the negative cliques eliminates one of the two uses of each edge in the spill, and we end up with a perfect decomposition.

In broad terms, the strategy of the proof in this paper is similar. Furthermore, much of the proof for designs can be adapted to the general setting (we give the details of this in section 3). However, the 'integral decomposition' step becomes much more difficult (and it is necessary to overcome these difficulties, as this is a relaxation of the problem we are trying to solve). There are in fact two aspects of this step, which both become much more difficult: (i) characterising the decomposition lattice (i.e. the set of $\mathbb{Z}$-linear combinations of the vectors that we allow in our decomposition), (ii) finding bounded integral decompositions. Regarding (ii), there is a 'local decoding' trick for designs that greatly simplifies the proof of [15, Lemma 5.12] (version 2), but in the general setting we do not have local decodability, so we revert to the original randomised rounding method of [15] (version 1).

As for (i), we give a cautionary example here to show that the decomposition lattice is in general not what one might guess given its simple structure for designs. Let us first recall the characterisation by Graver and Jurkat [12] and Wilson [34] of the set of $J \in \mathbb{Z}^{K_n^r}$ with an integral K_q^r-decomposition, i.e. the $\mathbb{Z}$-linear combinations of (characteristic vectors of) copies of K_q^r. Clearly, any such J is K_q^r-divisible (as defined above), and the converse is also true for $n \geq q + r$. For partite problems there may be additional 'balance' constraints (as in Theorem 1.7) but there may be further more subtle constraints.

Let us consider the decomposition lattice of the triangles of a rainbow K_4, defined as follows. Fix any bijection $b : E(K_4) \to [6]$ and colouring $c : E(K_n) \to [6]$. Let $\mathcal{B}$ be the set of all b-coloured copies of K_4^3, i.e. for each injection $\phi : [4] \to [n]$ such that all $c(\phi(i)\phi(j)) = b(ij)$ we include $\{\phi([4] \setminus \{i\}) : i \in [4]\}$. We wish to characterise the lattice $\langle \mathcal{B} \rangle$ generated by $\mathcal{B}$. Certainly any $J \in \langle \mathcal{B} \rangle$ must be K_4^3-divisible and supported on the set $\mathcal{T}$ of triangles that appear in $\mathcal{B}$. One might guess by analogy with the lattice of K_4^3's in a random 3-graph (see [15]) that if c is random then whp there would be no further condition.

However, we will now describe a K_4^3-divisible vector (a 'twisted octahedron') that is not in $\langle \mathcal{B} \rangle$. Suppose that $b(12) = 3$, $b(13) = 2$, $b(23) = 1$, $b(14) = 4$, $b(24) = 5$, $b(34) = 6$. Consider any octahedron, i.e. complete tripartite graph, with parts $\{x_0, x_1\}$, $\{y_0, y_1\}$, $\{z_0, z_1\}$ coloured so that all $c(y_i z_j) = 1$, $c(x_0 y_0) = c(x_0 y_1) = c(x_1 z_0) = c(x_1 z_1) = 2$, $c(x_1 y_0) = c(x_1 y_1) = c(x_0 z_0) = c(x_0 z_1) = 3$. Then every triangle $x_i y_j z_k$ is rainbow. Let $J \in \mathbb{Z}^{\mathcal{T}}$ be ± 1 on these triangles and 0 otherwise, with $J_{x_i y_j z_k} = (-1)^{i+j+k}$. Then J is null ($\sum\{J_e : f \subseteq e \in \mathcal{T}\} = 0$ whenever $|f| \leq 2$) so is K_4^3-divisible.

To see $J \notin \langle \mathcal{B} \rangle$ we use the colouring to define an algebraic invariant. For each triangle $T = xy_0z_0 \in \mathcal{T}$ containing y_0z_0 we let $f(T)$ be one of the standard basis vectors of $\mathbb{Z}^4$ according to the colouring of T: we let $f(T)$ be e_1, e_2, e_3 or e_4 according to whether $(c(xy_0), c(xz_0))$ is $(2, 3)$, $(3, 2)$, $(5, 6)$ or $(6, 5)$. We extend f to a $\mathbb{Z}$-linear map from $\mathbb{Z}^{\mathcal{T}}$ to $\mathbb{Z}^6$, where $f(T') = 0$ if T' does not contain y_0z_0. If $J' \in \langle \mathcal{B} \rangle$ then $f(J')$ lies in the lattice generated by $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$. However, we have $f(J) = (1, -1, 0, 0)$, so $J \notin \langle \mathcal{B} \rangle$. This example hints at the importance of vertex labels (even when not explicitly presented in a problem) when characterising decomposition lattices.

1.6 Organisation

The organisation of this paper is as follows. In the next section we set up our general framework of labelled complexes and vector-valued decompositions, and develop some basic theory of these definitions that will be used throughout the paper. In section 3 we state our main theorem and present those parts of the proof that are somewhat similar to those in [15]. We define and analyse the Clique Exchange Algorithm in section 4. We complete the proof of our main theorem by solving

the problems of integral decomposition (section 5) and bounded integral decomposition (section 6). Section 7 gives several applications of our main theorem.

1.7 Notation

Most of the following notation is as in [15]. Let $[n] = \{1, \dots, n\}$. Let $\binom{S}{r}$ denote the set of r-subsets of S. We write $Q = \binom{[q]}{r}$ and also $Q = \binom{q}{r}$ (the use will be clear from the context). We identify $Q = \binom{[q]}{r}$ with the edge set of K_q^r (the complete r-graph on $[q]$). We write $K_q^r(S)$ for the complete q-partite r-graph with parts of size $|S|$ where each part is identified with S. If $S = [s]$ we write $K_q^r(S) = K_q^r(s)$.

We use 'concatenation notation' for sets (xyz may denote $\{x, y, z\}$) and for function composition (fg may denote $f \circ g$).

We say E holds with high probability (whp) if $\mathbb{P}(E) = 1 - e^{-\Omega(n^c)}$ for some $c > 0$ as $n \to \infty$.

We write Y^X for the set of vectors with entries in Y and coordinates indexed by X, which we also identify with the set of functions $f : X \to Y$. For example, we may consider $v \in \mathbb{F}_p^q$ as an element of a vector space over $\mathbb{F}_p$ or as a function from $[q]$ to $\mathbb{F}_p$.

We identify $v \in \{0,1\}^X$ with the set $\{x \in X : v_x = 1\}$, and $v \in \mathbb{N}^X$ with the multiset in X in which x has multiplicity v_x (for our purposes $0 \in \mathbb{N}$). We often consider algorithms with input $v \in \mathbb{Z}^X$, where each $x \in X$ is considered $|v_x|$ times, with a sign attached to it (the same as that of v_x); then we refer to x as a 'signed element' of v.

If G is a hypergraph, $v \in \mathbb{Z}^G$ and $e \in G$ we define $v(e) \in \mathbb{Z}^{G(e)}$ by $v(e)_f = v_{e \cup f}$ for $f \in G(e)$.

We denote the standard basis vectors in $\mathbb{R}^d$ by $e_1, \dots, e_d$. Given $I \subseteq [d]$, we let e_I denote the I by $[d]$ matrix in which the row indexed by $i \in I$ is e_i.

We write $M \in \mathbb{F}_p^{q \times r}$ to mean that M is a matrix with q rows and r columns having entries in $\mathbb{F}_p$. For $I \in Q = \binom{[q]}{r}$ we let M_I be the square submatrix with rows indexed by I. Note that $M_I = e_I M$.

We will regard $\mathbb{F}_{p^a}$ as a vector space over $\mathbb{F}_p$. For $e \subseteq \mathbb{F}_{p^a}$ we write $\dim(e)$ for the dimension of the subspace spanned by the elements of e. For $e \in \mathbb{F}_{p^a}^d$ we write $\dim(e)$ for the dimension of the set of coordinates of e.

When we use 'big-O' notation, the implicit constant will depend only on q.

We write $a = b \pm c$ to mean $b - c \le a \le b + c$.

Throughout the paper we omit floor and ceiling symbols where they do not affect the argument.

We also use the following notation (not from [15]).

Let $Bij(B, B')$ denote the set of bijections from B to B'.

Let $Inj(B, V)$ denote the set of injections from B to V.

We extend our concatenation notation to sets of functions, e.g. $\phi\Upsilon = \{\phi \circ \psi : \psi \in \Upsilon\}$.

We let $\emptyset$ denote the empty set and also the unique function with empty domain.

For any set X write $X_j = \binom{X}{j}$ (convenient notation for use in exponents).

We write $[q](S)$ for the set of partite maps $f : [q] \to [q] \times S$.

We write $Im(\phi)$ and $Dom(\phi)$ for the image and domain of a function ϕ.

We write $\psi \subseteq \phi$ for functions ψ and ϕ if ψ is a restriction of ϕ. Then $\phi \setminus \psi$ denotes the restriction of ϕ to $Dom(\phi) \setminus Dom(\psi)$. Given functions ϕ_j on A_j for $j = 1, 2$ that agree on $A_1 \cap A_2$ we write $\phi_1 \cup \phi_2$ for the function on $A_1 \cup A_2$ that restricts to ϕ_j on A_j for $j = 1, 2$.

Given $\Gamma \subseteq \mathbb{R}$, the Γ-span of $S \subseteq \mathbb{R}^d$ is $\langle S \rangle_\Gamma = \{\sum_{x \in S} \Phi_x x : \Phi \in \Gamma^S\}$. We write $\langle S \rangle = \langle S \rangle_{\mathbb{Z}}$.

If $u \in (\mathbb{Z}^D)^X$ we write $|u| = \sum_{x \in X} \sum_{d \in [D]} |(u_x)_d|$.

2 Basic structures

In this section we define the basic objects needed for the statement of our main theorem and record some simple properties of them that will be used throughout the paper. We start in the first subsection with the labelled complex structure that is the functional analogue of a simplicial complex. The second subsection concerns embeddings and extensions of labelled complexes, and defines the extendability property mentioned in the introduction. In the third subsection we consider adapted complexes, in which we add the structure of a permutation group that acts on labellings: the orbits of this action play the role of edges in the example of hypergraph decomposition. Then in the fourth subsection we formalise our general decomposition problem with respect to a superimposed system of vector values on functions; here we also show how to realise several concrete examples within this general framework. We describe some basic properties of vector-valued decompositions in the fifth subsection and introduce some terminology (atoms and types) for them; we also define the regularity property that formalises the 'no fractional obstacle' assumption discussed above.

2.1 Complexes

We start by defining a structure that we call a labelled complex.[5]

Definition 2.1. (labelled complexes) We call $\Phi = (\Phi_B : B \subseteq R)$ an R-system on V if $\phi : B \to V$ is injective for each $\phi \in \Phi_B$. We call an R-system Φ an R-complex on V if whenever $\phi \in \Phi_B$ and $B' \subseteq B$ we have $\phi \mid_{B'} \in \Phi_{B'}$. Let $\Phi^\circ_B = \{\phi(B) : \phi \in \Phi_B\}$, $\Phi^\circ_j = \bigcup\{\Phi^\circ_B : B \in \binom{R}{j}\}$ and $\Phi^\circ = \bigcup\{\Phi^\circ_B : B \subseteq R\}$. We write $V(\Phi) = \Phi^\circ_1$.

Note that if $A \subseteq A' \in \Phi^\circ_{B'}$ then $A \in \Phi^\circ_B$ for some $B \subseteq B'$ (not necessarily unique); thus Φ° is a (simplicial) complex. We will now define some basic operations (forming restrictions and neighbourhoods) for working with labelled complexes.

Definition 2.2. (restriction) Let Φ be an R-complex and Φ' an R-system. We let $\Phi[\Phi']$ be the R-system where $\Phi[\Phi']_B$ is the set of $\phi \in \Phi_B$ such that $\phi \mid_{B'} \in \Phi'_{B'}$ for all $B' \subseteq B$ such that $\Phi'_{B'}$ is defined (we allow some $\Phi'_{B'}$ to be undefined). If $\Phi'_x = U \subseteq V(\Phi)$ for all $x \in R$ and Φ'_B is undefined otherwise then we also write $\Phi[\Phi'] = \Phi[U] = \{\phi \in \Phi : Im(\phi) \subseteq U\}$.

Lemma 2.3. *$\Phi[\Phi']$ is an R-complex.*

Proof. Consider $\phi \in \Phi[\Phi']_B$ and $B^* \subseteq B$ such that Φ'_{B^*} is defined. We need to show $\phi \mid_{B^*} \in \Phi[\Phi']_{B^*}$. To see this, note that $\phi \mid_{B^*} \in \Phi_{B^*}$, and if $B' \subseteq B^*$ is such that $\Phi'_{B'}$ is defined then $(\phi \mid_{B^*}) \mid_{B'} = \phi \mid_{B'} \in \Phi'_{B'}$ as $\phi \in \Phi[\Phi']$. □

Definition 2.4. Let Φ be an R-complex and $\phi^* \in \Phi$. We write $\Phi \mid_{\phi^*} = \{\phi \in \Phi : \phi^* \subseteq \phi\}$.

Definition 2.5. (neighbourhoods) Let Φ be an R-complex and $\phi^* \in \Phi_{B^*}$. For $\phi \in (\Phi \mid_{\phi^*})_{B \cup B^*}$ with $B \subseteq R \setminus B^*$ let $\phi/\phi^* = \phi \mid_B$. We define an $(R \setminus B^*)$-system Φ/ϕ^* where each $(\Phi/\phi^*)_B$ consists of all ϕ/ϕ^* with $\phi \in (\Phi \mid_{\phi^*})_{B \cup B^*}$. For $J \in \Gamma^\Phi$ we define $J/\phi^* \in \Gamma^{\Phi/\phi^*}$ by $(J/\phi^*)_{\phi/\phi^*} = J_\phi$ whenever $\phi^* \subseteq \phi$.

Lemma 2.6. *Φ/ϕ^* is an $R \setminus B^*$-complex.*

Proof. Consider $B' \subseteq B \subseteq R \setminus B^*$ and $\phi/\phi^* \in (\Phi/\phi^*)_B$. We need to show $(\phi/\phi^*) \mid_{B'} \in (\Phi/\phi^*)_{B'}$. This holds as $\phi' := \phi \mid_{B^* \cup B'} \in \Phi_{B^* \cup B'}$ with $\phi' \mid_{B^*} = \phi^*$. □

[5] We suppress the term 'labelled' in our terminology, as the labels are indicated by the labelling set R.

2.2 Embeddings and extensions

Here we formulate our extendability property and show that it is maintained under taking neighbourhoods. We start by defining embeddings of labelled complexes.

Definition 2.7. Let H and Φ be R-complexes. Suppose $\phi : V(H) \to V(\Phi)$ is injective. We call ϕ a Φ-embedding of H if $\phi \circ \psi \in \Phi$ for all $\psi \in H$.

We will define extendability using the following labelled complex of partite maps.

Definition 2.8. Let $R(S)$ be the R-complex of all partite maps from R to $R \times S$, i.e. whenever $i \in B \subseteq R$ and $\psi \in R(S)_B$ we have $\psi(i) = (i, x)$ for some $x \in S$. If $S = [s]$ we write $R(S) = R(s)$.

The following extendability property can be viewed as a labelled analogue of that in [15].[6]

Definition 2.9. Suppose $H \subseteq R(S)$ is an R-complex and $F \subseteq V(H)$. Define $H[F] \subseteq R(S)$ by $H[F] = \{\psi \in H : Im(\psi) \subseteq F\}$. Suppose ϕ is a Φ-embedding of $H[F]$. We call $E = (H, F, \phi)$ a Φ-extension of rank $s = |S|$. We say E is simple if $|V(H) \setminus F| = 1$.

We write $X_E(\Phi)$ for the set or number of Φ-embeddings of H that restrict to ϕ on F. We say E is ω-dense (in Φ) if $X_E(\Phi) \geq \omega |V(\Phi)|^{v_E}$, where $v_E := |V(H) \setminus F|$. We say Φ is (ω, s)-extendable if all Φ-extensions of rank s are ω-dense.

We will also require the following extension of the previous definition that allows for a system of extra restrictions.

Definition 2.10. Let Φ be an R-complex and $\Phi' = (\Phi^t : t \in T)$ with each $\Phi^t \subseteq \Phi$. Let $E = (H, F, \phi)$ be a Φ-extension and $H' = (H^t : t \in T)$ for some mutually disjoint $H^t \subseteq H \setminus H[F]$; we call (E, H') a (Φ, Φ')-extension.

We write $X_{E,H'}(\Phi, \Phi')$ for the set or number of $\phi^* \in X_E(\Phi)$ with $\phi^* \circ \psi \in \Phi^t_B$ whenever $\psi \in H^t_B$ and Φ^t_B is defined. We say (E, H') is ω-dense in (Φ, Φ') if $X_{E,H'}(\Phi, \Phi') \geq \omega |V(\Phi)|^{v_E}$. We say (Φ, Φ') is (ω, s)-extendable if all (Φ, Φ')-extensions of rank s are ω-dense in (Φ, Φ').

When $|T| = 1$ we identify $\Phi' \subseteq \Phi$ with (Φ'). We also write $X_E(\Phi, \Phi') = X_{E,H\setminus H[F]}(\Phi, \Phi')$. For $L \subseteq \Phi^\circ$ we write $X_E(\Phi, L) = X_E(\Phi, \Phi')$ where $\Phi' = \{\phi \in \Phi : Im(\phi) \in L\}$; we also write $\Phi[L] = \Phi[\Phi']$, and say that (Φ, L) is (ω, s)-extendable if (Φ, Φ') is (ω, s)-extendable.

If $L \subseteq V(\Phi)$ we also say that (Φ, L) is (ω, s)-extendable wrt L if $X_{E,H'}(\Phi, \Phi') \geq \omega |L|^{v_E}$ for all (Φ, Φ')-extensions (E, H') of rank s.

Note that if $\Phi' \subseteq \Phi$ and (Φ, Φ') is (ω, s)-extendable then $\Phi[\Phi']$ is (ω, s)-extendable. In the next definition we combine the operations of taking neighbourhoods and restriction to an (unordered) hypergraph; the accompanying lemma shows that under the generalised extendability condition of the previous condition the resulting labelled complex is extendable. We note that if $L = \Phi^\circ_r$ the restriction has no effect, so $\Phi/\phi^* L = \Phi/\phi^*$, and in this case Lemma 2.12 states that if Φ is (ω, s)-extendable then Φ/ϕ^* is (ω, s)-extendable. A less trivial example is when $L \subseteq V(\Phi) = \Phi^\circ_1$; then $\Phi/\phi^* L = (\Phi/\phi^*)[L]$ is obtained by restricting Φ/ϕ^* to L.

Definition 2.11. Suppose Φ is an R-complex, $L \subseteq \Phi^\circ_r$ and $\phi^* \in \Phi_{B^*}$. Let $\Phi/\phi^* L$ be the set of all $\phi/\phi^* \in \Phi/\phi^*$ such that $e \in L$ for all $e \in Im(\phi)_r$ with $e \setminus Im(\phi^*) \neq \emptyset$.

Lemma 2.12. *If (Φ, L) is (ω, s)-extendable (wrt L) then $\Phi/\phi^* L$ is (ω, s)-extendable.*

[6] The unlabelled analogue of our assumption here is weaker than that in [15], as we only consider partite extensions.

Proof. Consider any Φ/ϕ^*L-extension $E = (H, F, \phi)$ where $H \subseteq (R \setminus B^*)(s)$. We need to show that E is ω-dense in Φ/ϕ^*L. To see this, we consider $H^+ \subseteq R(s)$ where for each $B \subseteq R \setminus B^*$, $B' \subseteq B^*$, $\psi \in H_B$ we include $\psi \cup B'$ in $H^+_{B \cup B'}$ defined by $(\psi \cup B') \mid_B = \psi$ and $(\psi \cup B')(i) = i = (i, 1)$ for $i \in B'$. Consider the Φ-extension $E^+ = (H^+, F^+, \phi \cup \phi^*)$ with $F^+ = F \cup B^*$.

As (Φ, L) is (ω, s)-extendable, we have $X_{E^+}(\Phi, L) > \omega|V(\Phi)|^{v_{E^+}}$, or $X_{E^+}(\Phi, L) > \omega|L|^{v_{E^+}}$ if (Φ, L) is (ω, s)-extendable wrt L. It remains to show that if $\phi^+ \in X_{E^+}(\Phi, L)$ then $\phi^+ \mid_{V(H)} \in X_E(\Phi/\phi^*L)$. For any $B \subseteq R \setminus B^*$, $\psi \in H_B$ as $\phi^+ \in X_{E^+}(\Phi)$ we have $\phi' := \phi^+ \circ (\psi \cup B^*) \in \Phi_{B \cup B^*}$, and as $\phi' \mid_{B^*} = \phi^*$ we have $\phi^+ \mid_{V(H)} \circ \psi = \phi' \mid_B = \phi'/\phi^* \in (\Phi/\phi^*)_B$. Furthermore, as $\phi^+ \in X_{E^+}(\Phi, L)$ we have $e \in L$ for any $e \in Im(\phi^+)_r$ with $e \setminus Im(\phi^*) \neq \emptyset$. Therefore $\phi^+ \mid_{V(H)} \in X_E(\Phi/\phi^*L)$. □

2.3 Adapted complexes

Next we introduce the setting of adapted complexes, where we have a permutation group acting on the functions in a labelled complex. We start with some notation for permutation groups; in particular, given a permutation group Σ on R we define an R-complex $\Sigma^{\leq}$ that consists of all restrictions of elements of Σ.

Definition 2.13. Suppose Σ is a permutation group on R. For $B, B' \subseteq R$ we write $\Sigma^{B'}_B = \{\sigma \mid_B : \sigma \in \Sigma, \sigma(B) = B'\}$, $\Sigma_B = \cup_{B'} \Sigma^{B'}_B$, $\Sigma^{B'} = \cup_B \Sigma^{B'}_B$, $\Sigma[B] = \cup_{B' \subseteq B} \Sigma^{B'}$, $\Sigma^{\leq} = \cup_{B,B'} \Sigma^{B'}_B$.

We let $\mathcal{P}^{\Sigma}$ be the equivalence classes of the relation $B \sim B' \leftrightarrow \Sigma^B_{B'} \neq \emptyset$. Note that $B \sim B'$ implies $|B| = |B'|$. We write $\mathcal{P}^{\Sigma}_j = \{C \in \mathcal{P}^{\Sigma} : B \in C \Rightarrow |B| = j\}$.

We will restrict attention to labelled complexes in which any function can be relabelled under the group action, as follows.

Definition 2.14. (adapted) Suppose Φ is an R-complex and Σ is a permutation group on R. For $\sigma \in \Sigma$ and $\phi \in \Phi_{\sigma(B)}$ let $\phi\sigma = \phi \circ \sigma \mid_B$. We say Φ is Σ-adapted if $\phi\sigma \in \Phi$ for any $\phi \in \Phi$, $\sigma \in \Sigma$.

Next we introduce some notation for the orbits of the action implicit in the previous definition; these will play the role of edges in hypergraph decompositions.

Definition 2.15. (orbits)

For $\psi \in \Phi_B$ with $B \subseteq R$ we define the orbit of ψ by $\psi\Sigma := \psi\Sigma^B = \{\psi\sigma : \sigma \in \Sigma^B\}$. We denote the set of orbits by Φ/Σ. We write $\Phi_C = \cup_{B \in C} \Phi_B$ for $C \in \mathcal{P}^{\Sigma}$. We write $Im(O) = Im(\psi)$ for $\psi \in O \in \Phi/\Sigma$. For $O, O' \in \Phi/\Sigma$ we write $O \subseteq O'$ if there are $\psi \in O$, $\psi' \in O'$ with $\psi \subseteq \psi'$.

Note that the orbits partition Φ and $\Phi_C = \bigcup\{\psi\Sigma : \psi \in \Phi_B\}$ for any $B \in C$. When we later consider functions on Φ we will decompose them by orbits as follows.

Definition 2.16. (orbit decomposition) Let Γ be an abelian group. For $J \in \Gamma^{\Phi_r}$ and $O \in \Phi_r/\Sigma$ we define J^O by $J^O_\psi = J_\psi 1_{\psi \in O}$. The orbit decomposition of J is $J = \sum_{O \in \Phi_r/\Sigma} J^O$.

Now we will illustrate the role of orbits with the two most obvious examples (see also subsection 2.4 for more examples).

Examples.

i. If $\Sigma = \{id_R\}$ is the trivial group then each equivalence class and orbit has size 1, and we can identify Φ with Φ/Σ. This choice of Σ is suitable for 'fully partite' hypergraph decompositions, in which every edge is uniquely labelled by the set of parts that it meets. We also denote $\Sigma^{\leq}$ by $\overrightarrow{R}$, or by $\overrightarrow{q}$ when $R = [q]$. Then $\overrightarrow{q}_B = \{id_B\}$ for all $B \subseteq [q]$.

ii. If Σ is the symmetric group S_R on R then the equivalence classes of $\mathcal{P}^\Sigma$ are $\binom{R}{r}$ for $0 \leq r \leq |R|$. We also denote $\Sigma^{\leq}$ by $R^{\leq}$. Then each $R^{\leq}_B = Inj(B, R)$ consists of all injections from B to R. We can identify Φ° with Φ/Σ, where $e \in \Phi^\circ$ is identified with $\{\psi \in \Phi : Im(\psi) = e\}$. This choice of Σ is suitable for nonpartite hypergraph decompositions, in which the labels play no essential role.

Next we show that adapted complexes have neighbourhoods that are also adapted complexes.

Definition 2.17. Let Σ be a permutation group on R. For $B^* \subseteq R$ and $\sigma \in \Sigma$ with $\sigma \mid_{B^*} = id_{B^*}$ we write $\sigma/B^* = \sigma \mid_{R\setminus B^*}$. We let Σ/B^* be the set of all such σ/B^*.

Note that Σ/B^* is a permutation group on $R \setminus B^*$.

Lemma 2.18. *Let Φ be a Σ-adapted R-complex and $\phi^* \in \Phi_{B^*}$. Then Φ/ϕ^* is a Σ/B^*-adapted $(R \setminus B^*)$-complex.*

Proof. Suppose $B \subseteq R \setminus B^*$, $\sigma = \sigma'/B^* \in \Sigma/B^*$ and $\psi = \psi'/\phi^* \in (\Phi/\phi^*)_B$. As Φ is Σ-adapted, $\psi'\sigma' \in \Phi$, so $\psi\sigma = \psi'\sigma'/\phi^* \in \Phi/\phi^*$. □

Next we introduce the labelled complex structure defined by embeddings of one labelled complex in another.

Definition 2.19. Given R-complexes Φ and A we let $A(\Phi)$ denote the set of Φ-embeddings of A. We let $A(\Phi)^{\leq}$ denote the $V(A)$-complex where each $A(\Phi)^{\leq}_F$ for $F \subseteq V(A)$ is the set of Φ-embeddings of $A[F]$.

In the next subsection we will apply Definition 2.19 with $A = \Sigma^{\leq}$; we conclude this subsection by showing that if Φ is Σ-adapted then we can identify the resulting complex of embeddings with Φ itself.

Lemma 2.20. *If Φ is Σ-adapted and $B \subseteq [q]$ then $\Sigma[B](\Phi) = \Phi_B$.*

Proof. Consider any $\phi \in \Phi_B$. As Φ is Σ-adapted, for any $\sigma \in \Sigma^B$ we have $\phi\sigma \in \Phi$. As Φ is a $[q]$-complex we deduce $\phi\sigma \in \Phi$ for any $\sigma \in \Sigma[B]$, so $\phi \in \Sigma[B](\Phi)$. Conversely, if $\phi \in \Sigma[B](\Phi)$ then $\phi = \phi\ id_B \in \Phi_B$. □

2.4 Vector-valued decompositions

Now we introduce our general framework for decomposing vectors with coordinates indexed by the functions of a labelled complex and entries in some abelian group. We follow the definition with several examples to show how it captures hypergraph decompositions and other related problems.

Definition 2.21. Let $\mathcal{A}$ be a set of R-complexes; we call $\mathcal{A}$ an R-complex family. If each $A \in \mathcal{A}$ is a copy of $\Sigma^{\leq}$ we call $\mathcal{A}$ a $\Sigma^{\leq}$-family. For $r \in \mathbb{N}$ we write $A_r = \bigcup\{A_B : B \in \binom{R}{r}\}$ and $\mathcal{A}_r = \cup_{A\in\mathcal{A}} A_r$. We let $\mathcal{A}(\Phi)^{\leq}$ denote the $V(A)$-complex family $(A(\Phi)^{\leq} : A \in \mathcal{A})$.

Let $\gamma \in \Gamma^{\mathcal{A}_r}$ for some abelian group Γ; we call γ a Γ-system for $\mathcal{A}_r$.

Let Φ be an R-complex. For $\phi \in A(\Phi)^{\leq}$ with $A \in \mathcal{A}$ we define $\gamma(\phi) \in \Gamma^{\Phi_r}$ by $\gamma(\phi)_{\phi\circ\theta} = \gamma_\theta$ for $\theta \in A_r$ (zero otherwise). We call $\gamma(\phi)$ a γ-molecule and let $\gamma(\Phi)$ be the set of γ-molecules.

Given $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ we define $\partial\Psi = \partial^\gamma\Psi = \sum_\phi \Psi_\phi\gamma(\phi) \in \Gamma^{\Phi_r}$. We also call Ψ an integral $\gamma(\Phi)$-decomposition of $G = \partial\Psi$ and call $\langle\gamma(\Phi)\rangle$ the decomposition lattice. If furthermore $\Psi \in \{0, 1\}^{\mathcal{A}(\Phi)}$ (i.e. $\Psi \subseteq \mathcal{A}(\Phi)$) we call Ψ a $\gamma(\Phi)$-decomposition of G.

Examples.

i. Suppose H and G are r-graphs with $V(H) = [q]$. Let Φ be the complete $[q]$-complex on $V(G)$, i.e. all $\Phi_B = Inj(B, V(G))$. Let $G^* = \{\psi \in \Phi_r : Im(\psi) \in G\}$. Let $\Sigma = S_{[q]}$, $\mathcal{A} = \{A\}$ with $A = \Sigma^{\leq}$, and $\gamma \in \{0,1\}^{A_r}$ with each $\gamma_\theta = 1_{Im(\theta)\in H}$. Note that Φ is Σ-adapted. For any $\phi \in A(\Phi) = \Phi_q$ and $\theta \in A_r$ we have $\gamma(\phi)_{\phi\theta} = \gamma_\theta = 1_{Im(\theta)\in H}$, so an (integral) H-decomposition of G is equivalent to a (an integral) $\gamma(\Phi)$-decomposition of G^*. Similarly, if $\mathcal{H}$ is a family of r-graphs, by introducing isolated vertices we may assume they all have vertex set $[q]$. Then an $\mathcal{H}$-decomposition of G is equivalent to a $\gamma(\Phi)$-decomposition of G^*, where now $\mathcal{A}$ contains A^H defined as above for each $H \in \mathcal{H}$, and γ_θ is as defined above whenever $\theta \in A_r^H$.

ii. We generalise the previous example (for simplicity we revert to one r-graph H). Now suppose H and G have coloured edges. Let the set of colours be $[D]$, let H^d and G^d be the edges in H and G of colour d. Let $e_1, \dots, e_D$ be the standard basis of $\mathbb{Z}^D$. Let Φ, Σ, $\mathcal{A}$ be as above. Define $G^* \in (\mathbb{N}^D)^{\Phi_r}$ by $G^*_\psi = e_d$ for all ψ with $Im(\psi) \in G^d$ and $G^*_\psi = 0$ otherwise. Define $\gamma \in (\mathbb{N}^D)^{A_r}$ by $\gamma_\theta = e_d$ for all $\theta \in A_r$ with $Im(\psi) \in H^d$ and $\gamma_\theta = 0$ otherwise. Then an H-decomposition of G that respects colours is equivalent to a $\gamma(\Phi)$-decomposition of G^*.

iii. Now suppose that G is q-partite, say with parts $V_1, \dots, V_q$. The previous examples will not be useful for finding an H-decomposition of G, as our main theorem requires (Φ, G) to be extendable, but if we allow Φ to disrespect the partition then we cannot extend all partial embeddings within G. Instead, we define Φ_B for $B \subseteq [q]$ to consist of all partite $\psi \in Inj(B, V(G))$, i.e. $\psi(i) \in V_i$ for all $i \in B$. We let $\Sigma = \{id\}$ be trivial, $\mathcal{A} = \{A\}$ with $A = \Sigma^{\leq}$, i.e. all $A_B = \{id_B\}$. Defining G^* and γ as in the first example, we again see that an H-decomposition of G is equivalent to a $\gamma(\Phi)$-decomposition of G^*.

iv. Next we consider the H-decomposition problem for G when we are given bipartitions (X, Y) of $V(G)$ and (A, B) of $V(H) = [q]$, and we only allow copies of H in which A maps into X and B into Y (recall that the problems of resolvable designs and large sets of designs are equivalent to such bipartite decomposition problems). We let Φ_F for $F \subseteq [q]$ consist of all $\psi \in Inj(F, V(G))$ such that $\psi(F \cap A) \subseteq X$ and $\psi(F \cap B) \subseteq Y$. As usual, we let $G^* = \{\psi \in \Phi_r : Im(\psi) \in G\}$. We let Σ be the group of all $\sigma \in S_q$ such that $\sigma(A) = A$ and $\sigma(B) = B$. Then Φ is Σ-adapted. As usual, we let $\mathcal{A} = \{A'\}$ with $A' = \Sigma^{\leq}$ and $\gamma \in \{0,1\}^{A'_r}$ with each $\gamma_\theta = 1_{Im(\theta)\in H}$. Then an H-decomposition of G is equivalent to a $\gamma(\Phi)$-decomposition of G^*.

v. In the above examples we were decomposing hypergraphs (sets of sets) and treating the labellings (sets of functions) as a convenient device, but many applications explicitly require labellings. An example that may have some topological motivation is that of decomposing the set of top-dimensional cells of an oriented simplicial complex. The standard definition of orientations fits very well with our framework: for r-graphs H and G, an orientation is defined by a bijective labelling of each edge by $[r]$, where two labellings are considered equivalent if they differ by an even permutation in S_r. Then we wish to decompose G by copies of H, where we only allow copies $\phi(H)$ such that for each edge e of H composing the labelling of e with ϕ gives a labelling of $\phi(e)$ equivalent to that in G. To realise this problem in our framework (consider for simplicity the nonpartite setting where Φ is the complete $[q]$-complex on $V(G)$ and $\Sigma = S_q$), for each $B \in Q = [q]_r$ we let $\pi_B \in Bij([r], B)$ be order preserving and let G^*_B consist of all $\psi \in \Phi_B$ such that $\psi' = \psi \circ \pi_B$ with $Im(\psi') \in G$ is correctly oriented. Similarly, we let $\mathcal{A} = \{A\}$ with $A = \Sigma^{\leq}$ and $\gamma \in \{0,1\}^{A_r}$ where for $\theta \in A_B$ we let γ_θ be 1 if $\theta' = \theta \circ \pi_B$ with $Im(\theta') \in H$ is correctly oriented, otherwise $\gamma_\theta = 0$. Then an oriented H-decomposition of G is equivalent to a $\gamma(\Phi)$-decomposition of G^*.

2.5 Atoms and types

In this subsection we introduce some structures and terminology for working with vector-valued decompositions, and make some preliminary observations regarding the decomposition lattice $\langle\gamma(\Phi)\rangle$. We also define the regularity property referred to above. Throughout we let $\Sigma \leq S_q$ be a permutation group, Φ be a Σ-adapted $[q]$-complex, $\mathcal{A}$ be a $\Sigma^{\leq}$-family and $\gamma \in \Gamma^{\mathcal{A}_r}$.

Definition 2.22. (atoms) For any $\phi \in \mathcal{A}(\Phi)$ and $O \in \Phi_r/\Sigma$ such that $\gamma(\phi)^O \neq 0$ we call $\gamma(\phi)^O$ a γ-atom at O. We write $\gamma[O]$ for the set of γ-atoms at O. We say γ is elementary if all γ-atoms are linearly independent. We define a partial order $\leq_\gamma$ on Γ^{Φ_r} where $H \leq_\gamma G$ iff $G - H$ can be expressed as the sum of a multiset of γ-atoms.

Note that if $J \in \langle\gamma(\Phi)\rangle$ then each J^O can be expressed as a $\mathbb{Z}$-linear combination of γ-atoms at O. Furthermore, if γ is elementary then this expression is unique, so if J is the sum of a multiset Z of γ-atoms then a $\gamma(\Phi)$-decomposition of J may be thought of as a partition of Z, where each part is the set of γ-atoms contained in some molecule $\gamma(\phi)$. This is a combinatorially natural condition, as it avoids arithmetic issues that arise e.g. for decompositions of integers (the Frobenius coin problem). In our main theorem we will assume that γ is elementary, but the proof also uses other vector systems derived from γ that are not necessarily elementary.

The following definition and accompanying lemma give various equivalent ways to represent atoms. The notation $\gamma(\phi)$ matches the notation for molecules in Definition 2.21 when $\phi \in \mathcal{A}(\Phi)$.

Definition 2.23. For $\psi \in \Phi_B$ and $\theta \in \mathcal{A}_B$ we define $\gamma[\psi]^\theta \in \Gamma^{\psi\Sigma}$ by $\gamma[\psi]^\theta_{\psi\sigma} = \gamma_{\theta\sigma}$.

For $\phi \in A(\Phi)^{\leq} = \Phi$ we define $\gamma(\phi) \in \Gamma^{\Phi_r}$ by $\gamma(\phi)_{\phi\theta} = \gamma_\theta$ whenever $\theta \in A_r$ with $Im(\theta) \subseteq Dom(\phi)$.

Lemma 2.24. *Suppose $\phi \in A(\Phi)$ and $\psi \in A(\Phi)^{\leq}_B = \Phi_B$.*

i. If $\psi = \phi\theta$ with $\theta \in A_r$ then $\gamma(\phi)^{\psi\Sigma} = \gamma[\psi]^\theta$.
Furthermore, if $\theta \in A_B$ and $\sigma \in \Sigma^B$ then $\gamma[\psi]^\theta = \gamma[\psi\sigma]^{\theta\sigma}$.

ii. If $\psi \subseteq \phi$ then $\gamma(\phi)^{\psi\Sigma} = \gamma[\psi]^{id_B} = \gamma(\psi)$.

Proof. For (i), by Definitions 2.21 and 2.23, for any $\sigma \in \Sigma^B$ we have $\gamma(\phi)_{\psi\sigma} = \gamma(\phi)_{\phi\theta\sigma} = \gamma_{\theta\sigma} = \gamma[\psi]^\theta_{\psi\sigma}$, i.e. $\gamma(\phi)^{\psi\Sigma} = \gamma[\psi]^\theta$. Furthermore, if $\theta \in A_B$, $\sigma \in \Sigma^B_{B'}$, $\sigma' \in \Sigma^{B'}$ then $\gamma[\psi]^\theta_{\psi\sigma\sigma'} = \gamma_{\theta\sigma\sigma'} = \gamma[\psi\sigma]^{\theta\sigma}_{\psi\sigma\sigma'}$. For (ii), we have $\psi = \phi\ id_B$, so $\gamma(\phi)^{\psi\Sigma} = \gamma[\psi]^{id_B}$ by (i). Also, for any $\sigma \in \Sigma^B$ we have $\gamma(\psi)_{\psi\sigma} = \gamma_\sigma = \gamma[\psi]^{id_B}_{\psi\sigma}$, so $\gamma[\psi]^{id_B} = \gamma(\psi)$. □

The following definition will be used for the extendability assumption on $(\Phi, \gamma[G])$ in our main theorem, which gives a lower bound on extensions such that all atoms belong to G.

Definition 2.25. For $G \in \Gamma^{\Phi_r}$ we let $\gamma[G] = (\gamma[G]^A : A \in \mathcal{A})$ where each $\gamma[G]^A$ is the set of $\psi \in A(\Phi)^{\leq}_r = \Phi_r$ such that $\gamma(\psi) \leq_\gamma G$.

When using the notation $\gamma[\psi]^\theta$ for an atom, there may be several choices of θ that give rise to the same atom; this defines an equivalence relation that we will call a type. To illustrate the following definition, we recall example i (nonpartite H-decomposition) from subsection 2.4. In this case, there are two types for each r-set B of labels: for any $\theta \in A_B$, if $Im(\theta) \in H$ then γ^θ is the all-1 vector (we think of this type as an edge), whereas $Im(\theta) \notin H$ then γ^θ is the all-0 vector (the zero type, which we think of as a 'non-edge').

Definition 2.26. (types) For $\theta \in \mathcal{A}_B$ with $B \in Q$ we define $\gamma^\theta \in \Gamma^{\Sigma^B}$ by $\gamma^\theta_\sigma = \gamma_{\theta\sigma}$.

A type $t = [\theta]$ in γ is an equivalence class of the relation $\sim$ on any $\mathcal{A}_B$ with $B \in Q$ where $\theta \sim \theta'$ iff $\gamma^\theta = \gamma^{\theta'}$. We write T_B for the set of types in $\mathcal{A}_B$.

For $\theta \in t \in T_B$ and $\psi \in \Phi_B$ we write $\gamma^t = \gamma^\theta$ and $\gamma[\psi]^t = \gamma[\psi]^\theta$.

If $\gamma^t = 0$ call t a zero type and write $t = 0$.

If $\phi \in \mathcal{A}(\Phi)$ with $\gamma(\phi)^{\psi\Sigma} = \gamma[\psi]^t$ we write $t_\phi(\psi) = t$.

The next lemma shows that $\gamma[\psi]^t$ is well-defined.

Definition 2.27. For $B \in C \in \mathcal{P}_r^\Sigma$ and $J \in \Gamma^{\Phi_C}$ we define $f_B(J) \in (\Gamma^{\Sigma^B})^{\Phi_B}$ by $(f_B(J)_\psi)_\sigma = J_{\psi\sigma}$.

Lemma 2.28. *If $J = \gamma[\psi]^t$ for some $\psi \in \Phi_B$, $t \in T_B$ then $f_B(J)_\psi = \gamma^t$.*

Proof. For any $\sigma \in \Sigma^B$ and $\theta \in t$ we have $(f_B(J)_\psi)_\sigma = J_{\psi\sigma} = \gamma[\psi]^\theta_{\psi\sigma} = \gamma_{\theta\sigma} = \gamma^\theta_\sigma = \gamma^t_\sigma$. □

We also see from Lemma 2.28 that γ is elementary iff for any $B \in Q$ the set of nonzero γ^t with $t \in T_B$ is linearly independent. Next we introduce certain group actions that will be important in section 5; the following lemma records their effect on types.

Definition 2.29. For any set X we define a right Σ_B^B action on X^{Σ^B} by $(v\tau)_\sigma = v_{\tau\sigma}$ whenever $v \in X^{\Sigma^B}$, $\tau \in \Sigma_B^B$, $\sigma \in \Sigma^B$.

Note that Definition 2.29 is indeed a right action, as for $\tau_1, \tau_2 \in \Sigma_B^B$ we have $((v\tau_1)\tau_2)_\sigma = (v\tau_1)_{\tau_2\sigma} = v_{\tau_1\tau_2\sigma} = (v(\tau_1\tau_2))_\sigma$. For future reference we also note the linearity $(v + v')\tau = v\tau + v'\tau$; indeed, for $\sigma \in \Sigma^B$ we have $((v + v')\tau)_\sigma = (v + v')_{\tau\sigma} = v_{\tau\sigma} + v'_{\tau\sigma} = (v\tau)_\sigma + (v'\tau)_\sigma$.

Lemma 2.30. *If $\theta \in \mathcal{A}_B$ and $\tau \in \Sigma_B^B$ then $\gamma^\theta\tau = \gamma^{\theta\tau}$.*

Proof. For any $\sigma \in \Sigma^B$ we have $(\gamma^\theta\tau)_\sigma = \gamma^\theta_{\tau\sigma} = \gamma_{\theta\tau\sigma} = \gamma^{\theta\tau}_\sigma$. □

The next definition and accompanying lemma restate and provide notation for the earlier observation that any vector in the decomposition lattice can be expressed as a $\mathbb{Z}$-linear combination of atoms (we omit the trivial proof).

Definition 2.31. Let $\mathcal{L}_\gamma^-(\Phi)$ be the set of $J \in \Gamma^{\Phi_r}$ such that $J^O \in \langle\gamma[O]\rangle$ for all $O \in \Phi_r/\Sigma$.

Lemma 2.32. $\langle\gamma(\Phi)\rangle \subseteq \mathcal{L}_\gamma^-(\Phi)$.

Next we define two notions of symmetry, one for vectors and the other for subsets.

Definition 2.33. We call $v \in (\Gamma^{\Sigma^B})^{\Phi_B}$ symmetric if $v_\psi\tau = v_{\psi\tau}$ whenever $\psi \in \Phi_B$, $\tau \in \Sigma_B^B$.

We call $H \subseteq \Gamma^{\Sigma^B}$ symmetric if $g\tau \in H$ whenever $g \in H$, $\tau \in \Sigma_B^B$.

Note that

$$G^B := \{\gamma^t : t \in T_B\} \quad \text{and} \quad \gamma^B := \langle G^B\rangle \leq \Gamma^{\Sigma^B}$$

are symmetric by Lemma 2.30. Now we use types to give an alternative description of the lattice from Definition 2.31.

Lemma 2.34. *Let $B \in C \in \mathcal{P}_r^\Sigma$ and $J \in \Gamma^{\Phi_C} \cap \mathcal{L}_\gamma^-(\Phi)$. Then $f_B(J) \in (\gamma^B)^{\Phi_B}$ is symmetric.*

Proof. By linearity, we can assume J is a γ-atom, say $J = \gamma[\psi]^\theta$ with $\psi \in \Phi_B$, $\theta \in \mathcal{A}_B$. For any $\tau \in \Sigma_B^B$ we have $J = \gamma[\psi\tau]^{\theta\tau}$ by Lemma 2.24.i and $f_B(J)_{\psi\tau} = \gamma^{\theta\tau} = \gamma^\theta\tau = f_B(J)_\psi\tau$ by Lemmas 2.28, 2.30 and 2.28 again. □

For future reference (in section 4) we also note the following lemma which will allow us to split any linear dependence of γ-atoms into constant sized pieces. If γ is elementary then $Z_B(\gamma) = \{0\}$ so there is nothing to prove; in this case we let $C_0 = 1$. We call C_0 the lattice constant.

Lemma 2.35. *There is $C_0 = C_0(\gamma)$ such that for any $n \in Z_B(\gamma) := \{n \in \mathbb{Z}^{\mathcal{A}_r} : \sum_\theta n_\theta\gamma^\theta = 0\}$, there are $n^i \in Z_B(\gamma)$ for $i \in [t]$ for some $t \le C_0|n|$ with each $|n^i| \le C_0$ and $n = \sum_{i\in[t]} n^i$.*

Proof. Let $\mathcal{X}$ be an integral basis for $Z_B(\gamma)$. Let Z be the matrix with columns $\mathcal{X}$. We will find an integral solution v of $n = Zv$ and then for each $X \in \mathcal{X}$ take $|v_X|$ of the n^i equal to $\pm X$ (with the sign of v_X). The following explicit construction implies the required bound for $|v|$ in terms of $|n|$. We can put Z in 'diagonal form' via elementary row and column operations: there are unimodular (integral and having integral inverses) matrices P and Q such that $D = PZQ$ has $D_{ij} \ne 0 \Leftrightarrow i = j$. To solve $n = Zv$ we need to solve $Pn = DQ^{-1}v$. Let R be the set of nonzero rows of D and let $(Pn)_R$ and D_R denote the corresponding restrictions of Pn and D. Then $D_R^{-1}(Pn)_R$ is integral (as $n \in \langle\mathcal{X}\rangle$) so $v = QD_R^{-1}(Pn)_R$ is an integral solution of $n = Zv$. □

Next we introduce some notation for the coefficients that arise from decomposing a vector into atoms.

Definition 2.36. (atom decomposition)

Suppose γ is elementary and $J \in \mathcal{L}_\gamma^-(\Phi)$. For $\psi \in \Phi_B$ with $|B| = r$ we define integers J_ψ^t for all nonzero $t \in T_B$ by $J^{\psi\Sigma} = \sum_{0\ne t\in T_B} J_\psi^t\gamma[\psi]^t$. Any choice of orbit representatives $\psi^O \in \Phi_{B^O}$ for each orbit $O \in \Phi_r/\Sigma$ defines an atom decomposition $J = \sum_{O\in\Phi_r/\Sigma}\sum_{0\ne t\in T_{B^O}} J_{\psi^O}^t\gamma[\psi^O]^t$.

We need one final definition before stating our main theorem in the next section; note that the coefficients G_ψ^t are as in the previous definition.

Definition 2.37. (regularity) Suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$ and $G \in (\mathbb{Z}^D)^{\Phi_r}$. Let

$$\mathcal{A}(\Phi, G) = \{\phi \in \mathcal{A}(\Phi) : \gamma(\phi) \le_\gamma G\}.$$

We say G is (γ, c, ω)-regular (in Φ) if there is $y \in [\omega n^{r-q}, \omega^{-1}n^{r-q}]^{\mathcal{A}(\Phi,G)}$ such that for all $B \in [q]_r$, $\psi \in \Phi_B$, $0 \ne t \in T_B$ we have

$$\partial^t y_\psi := \sum_{\phi:t_\phi(\psi)=t} y_\phi = (1 \pm c)G_\psi^t.$$

Note that if G and y are as in the previous definition and $\psi \in O \in \Phi_r/\Sigma$ then

$$(\partial^\gamma y)^O = \sum_\phi y_\phi\gamma(\phi)^O = \sum_{0\ne t\in T_B}\sum_{\phi:t_\phi(\psi)=t} y_\phi\gamma[\psi]^t = \sum_{0\ne t\in T_B}(1 \pm c)G_\psi^t\gamma[\psi]^t = (1 \pm c)G^O.$$

We note for future reference that this implies an upper bound on the use (see Definition 3.13 below) of any orbit $O \in \Phi_r/\Sigma$, namely $U(G)_O < 2|\mathcal{A}|\omega^{-1}$ (if $c < 1/2$). Also, summing over O we obtain $\partial^\gamma y = (1 \pm c)G$, i.e. $\sum_\phi y_\phi\gamma(\phi)_{\psi,d} = (1 \pm c)G_{\psi,d}$ for all $\psi \in \Phi_r$, $d \in [D]$.

3 Main theorem

Now we can state our main theorem. We will give the proof in this section, assuming Lemmas 3.27 and 3.18, which will be proved in sections 4 and 6. The parts of the proof given in this section are those that are somewhat similar to the proof in [15], so we will be quite concise in places where they are similar, and give more details at points of significant difference. To apply Theorem 3.1 we also need a concrete description of the decomposition lattice $\langle\gamma(\Phi)\rangle$; this will be given in section 5.

Theorem 3.1. *For any $q \geq r$ and D there are ω_0 and n_0 such that the following holds for $n > n_0$, $h = 2^{50q^3}$, $\delta = 2^{-10^3q^5}$, $n^{-\delta} < \omega < \omega_0$ and $c \leq \omega^{h^{20}}$. Let $\mathcal{A}$ be a $\Sigma^{\leq}$-family with $\Sigma \leq S_q$. Suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$ is elementary. Let Φ be a Σ-adapted $[q]$-complex on $[n]$. Let $G \in \langle\gamma(\Phi)\rangle$ be (γ, c, ω)-regular in Φ such that $(\Phi, \gamma[G]^A)$ is (ω, h)-extendable for each $A \in \mathcal{A}$. Then G has a $\gamma(\Phi)$-decomposition.*

Throughout this section we let Σ, $\mathcal{A}$, γ, Φ and G be as in the statement of Theorem 3.1. We note that the assumption that γ is elementary bounds $|\mathcal{A}|$ as a function of q and D, say $|\mathcal{A}| < (Dq)^{q^q}$. For convenient reference, we list here several parameters used throughout the paper.

$$Q = \tbinom{q}{r}, \quad z = h = 2^{50q^3}, \quad \delta = 2^{-10^3q^5}, \quad n^{-\delta} < \omega < \omega_0(q, D), \quad \omega_q := \omega^{(9q)^{q+5}},$$
$$p \text{ is a prime with } 2^{8q} < p < 2^{9q}, \quad a \in \mathbb{N} \text{ with } p^{a-2} < n \leq p^{a-1},$$
$$\gamma = np^{-a}, \quad \rho = \omega z^{-Q}|\mathcal{A}|^{-1}(q)_r^{-Q}\gamma^{q-r}, \text{ where } (q)_r = q!/(q-r)!,$$
$$c = \omega^{h^{20}}, \quad c_1 = (2Qc)^{1/2Q}, \quad c_{i+1} = \omega^{-h^3}c_i \text{ for } i \in [4].$$

3.1 Probabilistic methods

We briefly recall two concentration inequalities (see [15, Lemmas 2.4 and 2.11]).

Definition 3.2. Suppose Y is a random variable and $\mathcal{F} = (\mathcal{F}_0, \dots, \mathcal{F}_n)$ is a filtration. We say that Y is (C, μ)-*dominated (wrt $\mathcal{F}$)* if we can write $Y = \sum_{i=1}^n Y_i$, where Y_i is $\mathcal{F}_i$-measurable, $|Y_i| \leq C$ and $\mathbb{E}[|Y_i| \mid \mathcal{F}_{i-1}] < \mu_i$ for $i \in [n]$, where $\sum_{i=1}^n \mu_i < \mu$.

Lemma 3.3. *If Y is (C, μ)-dominated then $\mathbb{P}(|Y| > (1+c)\mu) < 2e^{-\mu c^2/2(1+2c)C}$.*

Definition 3.4. Let $a = (a_1, \dots, a_n)$ and $a' = (a'_1, \dots, a'_n)$, where $a_i \in \mathbb{N}$ and $a'_i \in [a_i]$ for $i \in [n]$, and $\Pi(a, a')$ be the set of $\pi = (\pi_1, \dots, \pi_n)$ where $\pi_i : [a'_i] \to [a_i]$ is injective. Suppose $f : \Pi(a, a') \to \mathbb{R}$ and $b = (b_1, \dots, b_n)$ with $b_i \geq 0$ for $i \in [n]$. We say that f is b-*Lipschitz* if for any $i \in [n]$ and $\pi, \pi' \in \Pi(a, a')$ such that $\pi_j = \pi'_j$ for $j \neq i$ and $\pi_i = \tau \circ \pi'_i$ for some transposition $\tau \in S_{a_i}$ we have $|f(s) - f(s')| \leq b_i$. We also say that f is B-*varying* where $B = \sum_{i=1}^n a'_i b_i^2$.

Lemma 3.5. *Suppose $f : \Pi(a, a') \to \mathbb{R}$ is B-varying and $X = f(\pi)$, where $\pi = (\pi_i) \in \Pi(a, a')$ is random with $\{\pi_i : i \in [n]\}$ independent and π_i uniform whenever $a'_i > 1$. Then $\mathbb{P}(|X - \mathbb{E}X| > t) \leq 2e^{-t^2/2B}$.*

The following lemma will be used to pass from fractional matchings to almost perfect matchings. The statement and proof are similar[7] to those given by Kahn [14], so we omit the details. Call a hypergraph H a $k^{\leq}$-graph if all edges have size at most k.

[7] The constraint set P acts on the edges of H in [14], whereas here it is more convenient to use vertices. It is assumed to be of constant size in [14], which allows for a simple second moment argument, but we need to allow P to grow polynomially in α^{-1}, which can be achieved by proving exponential tails on the failure probabilities (which follows

Lemma 3.6. *Suppose H is a $k^{\leq}$-graph and w is a fractional matching in H with $\sum_{\{x,y\}\subseteq e\in H} w_e < \alpha < \alpha_0(k)$ sufficiently small for all $\{x,y\} \subseteq V(H)$. Let $P \subseteq \mathbb{R}^{V(H)}$ with $|P| < \alpha^{-k}$ and $\max_v p_v < (\log\alpha)^{-2}\sum_v p_v$ for all $p \in P$. Then there is a matching M of H such that*

$$\sum_{v\in\bigcup M} p_v = (1\pm\alpha^{1/2k})\sum_{e\in H} w_e \sum_{v\in e} p_v \quad \text{for all } p \in P.$$

We will apply Lemma 3.6 to a hypergraph whose vertices can be identified with γ-atoms, we have $\alpha = O(n^{-1})$, and elements of P indicate atoms that 'use' a given ordered $(r-1)$-tuple from $[n]$; the conclusion will be that there is a matching with 'bounded leave' (see Definition 3.13 and Lemma 3.15 below).

3.2 Template

Recalling the proof strategy discussed in the introduction, we start by describing the template. This will be determined by some $M^* \subseteq \mathcal{A}(\Phi)$ such that $G^* := \sum_{\phi\in M^*}\phi(Q) \subseteq \Phi_r^\circ$, i.e. G^* is an r-graph (with no multiple edges) contained in Φ_r° and $\{\phi(Q) : \phi \in M^*\}$ is a K_q^r-decomposition of G^*. Thus for each $e \in \Phi_r^\circ$ there is at most one orbit $O \in \Phi_r/\Sigma$ with $Im(O) = e$ and $O \subseteq \phi\Sigma$ for some $\phi \in M^*$, and given such O with representative $\psi^O \in \Phi_B$ the use of O by ϕ has a unique type $t_\phi(\psi^O) = t \in T_B$ (which may be the zero type).

As in [15], we fix $M \in \mathbb{F}_p^{q\times r}$ as a $q \times r$ matrix over $\mathbb{F}_p$ that is generic, in that every square submatrix of M is nonsingular.

As G is (γ, c, ω)-regular, there is $y \in [\omega n^{r-q}, \omega^{-1}n^{r-q}]^{\mathcal{A}(\Phi,G)}$ with $\partial^t y_\psi = (1\pm c)G_\psi^t$ for all $B \in [q]_r$, $\psi \in \Phi_B$, $0 \neq t \in T_B$. We activate each $\phi \in \mathcal{A}(\Phi)$ independently with probability $y_\phi\omega n^{q-r}$.

Let $f = (f_j : j \in [z])$, with $z = h = 2^{50q^3}$, where we choose independent uniformly random injections $f_j : [n] \to \mathbb{F}_{p^a}$. Given f, for each $e \in \Phi_r^\circ$ we let

$$\mathcal{T}_e = \{j \in [z] : \dim(f_j(e)) = r\}.$$

We abort if any $|\mathcal{T}_e| \leq z - 2r$, which occurs with probability $O(n^{-r})$. We assume without further comment that the template does not abort.

We choose $T_e \in [z]$ for all $e \in \Phi_r^\circ$ independently and uniformly at random. We say $\phi \in \mathcal{A}(\Phi)$ is compatible with j if $T_e = j \in \mathcal{T}_e$ for all $e \in \phi(Q)$ and for some $y \in \mathbb{F}_{p^a}^r$ we have $f_j(\phi(i)) = (My)_i$ for all $i \in [q]$.

Let $\pi = (\pi_e : e \in \Phi_r^\circ)$ where we choose independent uniformly random injections $\pi_e : e \to [q]$. We say $\phi \in \Phi$ is compatible with π if $\pi_e\phi(i) = i$ whenever $\phi(i) \in e \in \phi(Q)$ (for brevity we write this as $\pi_e\phi = id$).

We choose independent uniformly random $A_\phi \in \mathcal{A}$ for each injection $\phi : [q] \to [n]$.

Definition 3.7. Let M_j^* be the set of all activated $\phi \in A_\phi(\Phi)$ compatible with j and π such that $\gamma(\phi) \leq_\gamma G$. The template is $M^* = \cup_{j\in[z]} M_j^*$.

The underlying r-graph of the template is $G^* = \cup_{j\in[z]} G_j^*$, where each $G_j^* = \cup_{\phi\in M_j^*}\phi(Q)$.

from the same proof by applying standard concentration inequalities). Also, the error term in the conclusion is not explicitly given in [14], whereas we state a polynomial dependence on α (the proof gives $\alpha^{1/2k}$), which is needed if one desires counting versions of our results, as in [16]. A final comment is that it is not essential to consider edge weights, as we anyway reduce to the case that all w_e are equal, but it is convenient to leave the weights in the statement, and this also facilitates comparison with the statement in [14].

Note that G^* is the edge-disjoint union of the G_j^*, and each $\{\phi(Q) : \phi \in M_j^*\}$ is a K_q^r-decomposition of G_j^* (see [15, Lemma 3.3]).

We introduce the following further notation that will be used in the analysis of the template.

Definition 3.8. For $e \in G^*$ let $\phi^e \in M^*$ be such that $e \in \phi^e(Q)$. We write $M^*(e) = \phi^e(Q)$.

For $J \subseteq G^*$ let $M^*(J) = \sum_{e \in J} M^*(e) \in \mathbb{N}^{G^*}$.

For $e \in G^*$ we write $\gamma(e) = \gamma(\phi^e)^{O_e}$ where $O_e = \pi_e^{-1}\Sigma$. We call $\gamma(e)$ an M^*-atom.

Note that $\gamma(e)$ may be zero in Definition 3.8. If $\gamma(e) \neq 0$ then $\gamma(e)$ is also a γ-atom. Furthermore, for any γ-atom $\gamma(\psi) \leq_\gamma \partial^\gamma M^*$ we have $\gamma(\psi) = \gamma(e)$ where $Im(\psi) = e$.

If $0 \leq_\gamma J \leq_\gamma \partial^\gamma M^*$ with $J = \sum Z$ for some set Z of γ-atoms we write $J^\circ = \{e \in G^* : \gamma(e) \in Z\}$. For example, $G^* = (\partial^\gamma M^*)^\circ$.

3.3 Extensions

Next we give estimates on the probability that certain γ-atoms appear in the template and deduce that the template is whp extendable. Our estimates are conditional on the following local events (defined similarly to [15]).

Definition 3.9. (local events) Suppose $e \in \Phi_r^\circ$. We reveal $T_e = j$ and $f_j \,|_e = \alpha$. If $\dim(\alpha) < r$ then $\mathcal{E}^e$ is the event that $T_e = j$ and $f_j \,|_e = \alpha$, which witnesses $e \notin G^*$.

Now suppose $\dim(\alpha) = r$, reveal π_e, and let $y \in \mathbb{F}_{p^a}^r$ with $f_j(x) = (My)_i$ for all $x \in e$, $\pi_e(x) = i$. We reveal $f_j^{-1}((My)_i)$ for all $i \in [q] \setminus \pi_e(e)$, and let $\phi : [q] \to [n]$ be such that $f_j\phi = My$, and reveal A_ϕ. If we do not have $\gamma(\phi) \leq_\gamma G$ with $\phi \in A_\phi(\Phi)$ then $\mathcal{E}^e$ is the event that $T_e = j$ and $f_j\phi = My$, which witnesses $e \notin G^*$.

Finally, if $\gamma(\phi) \leq_\gamma G$ we reveal whether ϕ is activated, and reveal $(T_{e'}, \pi_{e'})$ for all $e' \in \phi(Q) \setminus \{e\}$. Then $\mathcal{E}^e$ is defined by all the above information, which determines whether $e \in G^*$: given $T_e = j$, $f_j\phi = My$, $\phi \in A_\phi(\Phi)$, $\gamma(\phi) \leq_\gamma G$ we have $e \in G^*$ iff ϕ is activated and $T_{e'} = j$ and $\pi_{e'}\phi = id$ for all $e' \in \phi(Q)$.

We say that a vertex x is touched by $\mathcal{E}^e$ if $f_j(x)$ is revealed by $\mathcal{E}^e$.

We say that an edge e' is touched by $\mathcal{E}^e$ if $T_{e'}$ is revealed by $\mathcal{E}^e$.

The following lemma is analogous to [15, Lemma 3.6]. Let

$$\rho := \omega z^{-Q} |\mathcal{A}|^{-1} (q)_r^{-Q} \gamma^{q-r}.$$

Lemma 3.10. *Let $S \subseteq \Phi_r^\circ$ with $|S| < h = z$ and $\mathcal{E} = \cap_{f \in S} \mathcal{E}^f$. Let $\psi \in \Phi_B$ and $t \in T_B$ with $t \neq 0$ and $\gamma[\psi]^t \leq_\gamma G$ (i.e. $G_\psi^t > 0$). Suppose $e := Im(\psi)$ is not touched by $\mathcal{E}$ and $j \in [z] \setminus \{T_f : f \in S\}$. Then $\mathbb{P}(\gamma[\psi]^t \leq_\gamma \partial^\gamma M_j^* \mid \mathcal{E}) = (1 \pm 1.1c)\rho G_\psi^t$.*

Proof. We fix any $\phi \in A(\Phi)$ with $\gamma(\phi) \leq_\gamma G$ and $t_\phi(\psi) = t$, and estimate the probability that $\phi \in M_j^*$. We have $\mathbb{P}(A_\phi = A) = |\mathcal{A}|^{-1}$. We activate ϕ with probability $y_\phi \omega n^{q-r}$. We can assume every $e' \in \phi(Q)$ is not touched by $\mathcal{E}$, as this excludes $O(n^{q-r-1})$ choices. Then all $T_{e'} = j$ with probability z^{-Q}. With probability $(q)_r^{-Q}$ all $\pi_{e'}\phi = id$. We condition on $f_j \,|_e$ such that $\dim(f_j(e)) = r$; this occurs with probability $1 - O(n^{-1})$. There is a unique $y \in \mathbb{F}_{p^a}^r$ such that $(My)_i = f_j(x)$ for all $x \in e$, $i = \pi_e(x)$. With probability $(1 + O(n^{-1}))(p^{-a})^{q-r}$ we have $f_j(\phi(i)) = (My)_i$ for all $i \in [q] \setminus \pi_e(e)$. Therefore $\mathbb{P}(\phi \in M_j^* \mid \mathcal{E}) = (1 + O(n^{-1}))|\mathcal{A}|^{-1} y_\phi \omega n^{q-r} z^{-Q} (q)_r^{-Q} (p^{-a})^{q-r}$. The lemma follows by summing over ϕ, using $\partial^t y_\psi = (1 \pm c) G_\psi^t$. □

Remark 3.11. The same proof shows

i. $\mathbb{P}(\gamma[\psi]^t \leq_\gamma \partial^\gamma M_j^* \mid \mathcal{E} \cap \{T_e = j\}) = (1 \pm 1.1c)z\rho G_\psi^t$ for any $j \in [z] \setminus \{T_f : f \in S\}$,

ii. $\mathbb{P}(\{\gamma[\psi]^t \leq_\gamma \partial^\gamma M_j^*\} \cap \{\pi_e = \pi\} \cap \{A_{\phi^e} = A\} \mid \mathcal{E}) > \omega^2 \rho G_\psi^t$, for any $A \in \mathcal{A}$ and injection $\pi : e \to [q]$ such that $\psi' = \pi^{-1} \in A(\Phi)^{\leq}$ satisfies $\gamma[\psi]^t = \gamma(\psi')$.

Note that (ii) is weaker than the corresponding bound in [15] as we cannot permute ϕ (this may change $\gamma(\phi)$). Instead, we use extendability to see that there are at least ωn^{q-r} choices of ϕ with $\gamma(\phi) \leq_\gamma G$ containing ψ', and each has $y_\phi > \omega n^{r-q}$.

The following lemma is analogous to [15, Lemma 3.8].

Lemma 3.12. *Suppose $E = (\phi, F, H)$ is a Φ-extension with $|H| \leq h/3$. Let $A \in \mathcal{A}$ and $H' \subseteq H_r \setminus H[F]$. Then whp $X_{E,H'}(\Phi, \gamma[\partial^\gamma M^*]^A) > \omega n^{v_E}(z\rho/2)^{|H'|}$.*

Proof. As $(\Phi, \gamma[G]^A)$ is (ω, h)-extendable, there are at least ωn^{v_E} choices of $\phi^+ \in X_{E,H'}(\Phi, \gamma[G]^A)$, i.e. $\phi^+ \in X_E(\Phi)$ with $\phi^+\psi' \in \gamma[G]_B^A$ for all $\psi' \in H'_B$. We fix any such ϕ^+ and estimate $\mathbb{P}(\phi^+ \in X_{E,H'}(\Phi, \gamma[\partial^\gamma M^*]^A))$ by repeated application of Lemma 3.10. For any $\psi = \phi^+\psi'$ with $\psi' \in H'_B$, we condition on the intersection $\mathcal{E}$ of all previously considered local events, and estimate $p_\psi^j = \mathbb{P}(\gamma[\psi]^t \leq_\gamma \partial^\gamma M_j^* \mid \mathcal{E})$, where $t \in T_B$ contains $id_B \in A$ and we can assume $Im(\psi)$ is not touched by $\mathcal{E}$. If $t = 0$ then $p_\psi^j = 1$; otherwise, there are at least $2z/3$ choices of $j \in [z]$ not used by any previous edge such that Lemma 3.10 applies to give $p_\psi^j = (1 \pm 1.1c)\rho G_\psi^t$. Multiplying all conditional probabilities and summing over ϕ^+ gives $\mathbb{E}X_{E,H'}(\Phi, \gamma[\partial^\gamma M^*]^A) > \omega n^{v_E}(0.6z\rho)^{|H'|}$. The proof of concentration is similar to that in [15, Lemma 3.8], noting that the effect of changing any A_ϕ has a similar effect to that of changing whether ϕ is activated. □

3.4 Approximate decomposition

Similarly to [15, section 4] we will now complete the template to an approximate decomposition, namely $M' \subseteq \mathcal{A}(\Phi)$ such that $\partial^\gamma M'$ is almost equal to G, except that some (suitably bounded) set of M^*-atoms are each covered one time too many. First we introduce some notation and terminology that will be used throughout the rest of the paper.

Definition 3.13. For $J \in (\mathbb{Z}^D)^{\Phi_r}$ and $\psi \in \Phi_r$, we define the *use* $U(J)_\psi$ of ψ by J as the minimum possible value of $\sum_{w \in W} |x_w|$ where W is the set of γ-atoms at $O = \psi\Sigma$ and $x \in \mathbb{Z}^W$ with $J^O = \sum x_w w$. If there is no such x then $U(J)_\psi$ is undefined. For $\psi' \in \Phi$ we let $U(J)_{\psi'} = \sum\{U(J)_\psi : \psi' \subseteq \psi \in \Phi_r\}$. We note that use is a property of orbits, so $U(J)_{\psi\Sigma} = U(J)_\psi$ is well-defined. We say J is θ-bounded if $U(J)_\psi < \theta|V(\Phi)|$ whenever $\psi \in \Phi_{r-1}$.

Note that as γ is elementary the use of 'minimum' in Definition 3.13 is redundant, as J^O has a unique atom decomposition; however, we will also need this definition for other γ that are not necessarily elementary. We will also sometimes use the following definition that ignores the edge labellings and atom structure (so is analogous to that used in [15]).

Definition 3.14. Suppose $J \in (\mathbb{Z}^D)^{\Phi_r^\circ}$. We let $U(J)_e = \sum_{d \in [D]} |(J_e)_d|$ for $e \in \Phi_r^\circ$ and $U(J)_f = \sum\{U(J)_e : f \subseteq e \in \Phi_r^\circ\}$ for $f \in \Phi^\circ$. We say J is θ-bounded if $U(J)_f < \theta|V(\Phi)|$ for all $f \in \Phi_{r-1}^\circ$.

Now we find a $\gamma(\Phi)$-decomposition of almost all of $G - \partial^\gamma M^*$, such that the leave is bounded in the sense of Definition 3.13. The following lemma is analogous to [15, Lemma 4.1].

Lemma 3.15. *There is $M^n \subseteq \mathcal{A}(\Phi)$ with c_1-bounded* leave $L := G - \partial^\gamma M^* - \partial^\gamma M^n \geq_\gamma 0$.

Proof. We define $\Phi' \subseteq \mathcal{A}(\Phi)$ randomly as follows. Consider any $\phi \in \mathcal{A}(\Phi)$ such that $\gamma(\phi) \leq_\gamma G$ and reveal the local events $\mathcal{E}^e$ for each $e \in Q' := \phi(Q)$. If ϕ is not activated or $T_e = T_{e'}$ for any $e \neq e'$ in Q' then we do not include ϕ in Φ'. For each $e \in Q'$ we fix ψ_e with image e such that $\gamma(\psi_e) \leq_\gamma \gamma(\phi)$, and let $t_e = t_\phi(\psi_e)$, so $\gamma(\psi_e) = \gamma[\psi_e]^{t_e}$. For $v \in \{0,1\}^{Q'}$ let $\mathcal{E}_v^\phi$ be the event that all $v_e = 1_{\gamma(\psi_e) \leq_\gamma \partial^\gamma M^*}$. If ϕ is activated, all T_e for $e \in Q'$ are distinct and $\mathcal{E}_v^\phi$ holds then we include ϕ in Φ' independently with probability $p_\phi^v = \prod_{e \in Q'}(1 - v_e q_e)$, where $q_e = 1/G_{\psi_e}^{t_e}$ if $t_e \neq 0$ or $q_e = z\rho$ if $t_e = 0$.

Now we fix any ψ and $t \neq 0$ with $G_\psi^t > 0$ and estimate the number X of $\phi \in \Phi'$ with $t_\phi(\psi) = t$. We consider any activated ϕ with $t_\phi(\psi) = t$, let $e' = Im(\psi)$, $Q' = \phi(Q)$ and condition on the local event $\mathcal{E}^{e'}$ and any event $\mathcal{C} = \cap_{e \in Q'}\{T_e = j_e\}$ such that all j_e are distinct (the latter occurs with probability $(z)_Q z^{-Q}$).

For any $v \in \{0,1\}^{Q'}$ with $v_{e'} = (\partial^\gamma M^*)_\psi^t$, by repeated application of Remark 3.11.i we have $\mathbb{P}(\mathcal{E}_v^\phi \mid \mathcal{E}^{e'} \cap \mathcal{C}) = (1 \pm Qc)\prod_{e \in Q' \setminus \{e'\}} p_e^{v_e}$, where if $t_e = 0$ we let $p_e^1 = 1$ and $p_e^0 = 0$, and otherwise $p_e^1 = z\rho G_{\psi_e}^{t_e}$ and $p_e^0 = 1 - z\rho G_{\psi_e}^{t_e}$. Then $p_e^0 + p_e^1(1 - q_e) = 1 - z\rho$, so as in the proof of [15, Lemma 4.1], $\mathbb{P}[\phi \in \Phi' \mid \mathcal{E}^{e'} \cap \mathcal{C}] = (1 \pm Qc)(1 - (\partial^\gamma M^*)_\psi^t / G_\psi^t)(1 - z\rho)^{Q-1}$.

We activate each ϕ with probability $y_\phi \omega n^{q-r}$, so as $\partial^t y_\psi = (1 \pm c)G_\psi^t$ we deduce $\mathbb{E}X = (z)_Q z^{-Q} \cdot (1 \pm Qc)(1 - (\partial^\gamma M^*)_\psi^t / G_\psi^t)(1 - z\rho)^{Q-1} \cdot (1 \pm c)\omega n^{q-r} G_\psi^t = (1 \pm 1.1Qc)d' n^{q-r}(G - \partial^\gamma M^*)_\psi^t$, where $d' = (z)_Q z^{-Q}\omega(1 - z\rho)^{Q-1}$. As in the proof of [15, Lemma 4.1], by Lemma 3.5 whp $X = (1 \pm 1.2Qc)d' n^{q-r}(G - \partial^\gamma M^*)_\psi^t$.

Finally, we consider the following hypergraph H, where $V(H)$ is the disjoint union of sets V_ψ^t of size $(G - \partial^\gamma M^*)_\psi^t$ corresponding to the γ-atoms of $G - \partial^\gamma M^*$ counted with multiplicity. For each $\phi \in \Phi'$ we let $\mathcal{V}_\phi$ be the set of all V_ψ^t with $\gamma[\psi]^t \leq_\gamma \gamma(\phi)$, and include as an edge a uniformly random set e^ϕ with one vertex in each $V \in \mathcal{V}_\phi$. Then whp every $v \in V(H)$ has degree $(1 \pm 2Qc)d' n^{q-r}$. Also any $\{u, v\} \subseteq V(H)$ is contained in $O(n^{q-r-1})$ edges. Let $P \subseteq \mathbb{R}^{V(H)}$ where for each $\psi' \in \Phi_{r-1}$ and we include $p^{\psi'}$ where $p_v^{\psi'}$ is 1 if v is in some V_ψ^t with $\psi'\Sigma \subseteq \psi\Sigma$, otherwise 0. Then $\sum_v p_v^{\psi'}$ counts the number of γ-atoms of $G - \partial^\gamma M^*$ on orbits containing $\psi'\Sigma$.

By Lemma 3.6, applied with uniform weights $w_e = ((1 + 2Qc)d' n^{q-r})^{-1}$, there is a matching M^n in H with $\sum_{v \in \bigcup M^n} p_v = (1 \pm (1.1Qc)^{1/2Q}) \sum_{e \in H} w_e \sum_{v \in e} p_v = (1 \pm (1.2Qc)^{1/2Q}) \sum_{v \in V(H)} p_v$ for all $p \in P$. We can also view M^n as a subset of $\mathcal{A}(\Phi)$. Then $L := G - \partial^\gamma M^* - \partial^\gamma M^n$ contains at most $(1.2Qc)^{1/2Q}$ proportion of the γ-atoms of $G - \partial^\gamma M^*$ on orbits containing $\psi'\Sigma$, for any $\psi' \in \Phi_{r-1}$. Recalling that $U(G)_O < 2|\mathcal{A}|\omega^{-1}$ and $|\mathcal{A}| < D^{q^{2q}}$ we see that L is c_1-bounded. □

To complete the approximate decomposition, we choose a partial $\gamma(\Phi)$-decomposition that exactly matches the leave on $\Phi_r^\circ \setminus G^*$, but has some 'spill' S in G^* that we will need to correct for later. The following lemma is analogous to [15, Lemma 4.2].

Lemma 3.16. *Suppose $0 \leq_\gamma L \leq_\gamma G - \partial^\gamma M^*$ is c_1-bounded. Then there is $M^c \subseteq \mathcal{A}(\Phi)$ such that $\gamma(\phi) \leq_\gamma L + \partial^\gamma M^*$ for all $\phi \in M^c$ and $\partial^\gamma M_\psi^c = L_\psi$ for all $\psi \in \Phi_r$ with $Im(\psi) \notin G^*$, with* spill *$S := G^* \cap \sum_{\phi \in M^c} \phi(Q)$, such that $M^*(S)$ is a set and c_2-bounded.*

Proof. We order the γ-atoms of L as $(\gamma(\psi_i) : i \in [n_L])$, where each $\psi_i \in A^i(\Phi)_{B^i}^{\leq} = \Phi_{B^i}$. For each i we consider the Φ-extension $E_i = (\overrightarrow{q}, B^i, \psi_i)$ and let $H' = \overrightarrow{q}_r \setminus \{B^i\}$. We apply a random greedy algorithm to select $\phi_i \in X_{E_i,H_i'}(\Phi, \gamma[\partial^\gamma M^*]^{A^i})$, where we write $S_i = G^* \cap \cup_{i'<i}\phi_{i'}(Q)$ and choose ϕ_i uniformly at random such that $M^*(\phi_i(Q))$ is a set disjoint from $M^*(S_i)$. For each i we add $\phi_i \in A^i(\Phi)$ to M^c. The remainder of the proof is very similar to that of [15, Lemma 4.2]. We show that whp M^c has the stated properties. At any step i before $M^*(S)$ fails to be c_2-bounded at

most half of the choices of ϕ_i are forbidden, as $X_{E_i,H_i'}(\Phi, \gamma[\partial^\gamma M^*]^{A^i}) > \omega(z\rho/2)^Q n^{q-r}$ by Lemma 3.12. Then for all $e \in G^*$ we estimate $r_e := \sum_{i\in[n_L]} \mathbb{P}'(e \in M^*(\phi_i(Q))) < 2(2q)^{2q}\omega^{-1}(z\rho/2)^{-Q}c_1$, so by Lemma 3.3 whp $M^*(S)$ is c_2-bounded. □

3.5 Proof modulo lemmas

In this subsection we give the proof of Theorem 3.1, assuming two lemmas that will be proved later. First we need to define use and boundedness for vectors indexed by $\mathcal{A}(\Phi)$.

Definition 3.17. For $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ and $\psi \in \Phi$ the use of ψ by Ψ is $U(\Psi)_\psi = \sum\{|\Psi_\phi| : \psi\Sigma \subseteq \phi\Sigma\}$. We also write $U(\Psi)_{\psi\Sigma} = U(\Psi)_\psi$. We say Ψ is θ-bounded if $\sum\{U(\Psi)_\psi : \psi' \subseteq \psi \in \Phi_r\} < \theta|V(\Phi)|$ whenever $\psi' \in \Phi_{r-1}$.

The following is a bounded integral decomposition lemma, analogous to [15, Lemma 5.1], which will be proved in section 6. Recall $\omega_q := \omega^{(9q)^{q+5}}$.

Lemma 3.18. *Let $\mathcal{A}$ be a $\Sigma^{\leq}$-family with $\Sigma \leq S_q$ and $|\mathcal{A}| \leq K$ and suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be an (ω, h)-extendable Σ-adapted $[q]$-complex on $[n]$, where $n^{-h^{-3q}} < \omega < \omega_0(q, D, K)$ and $n > n_0(q, D, K)$. Suppose $J \in \langle\gamma(\Phi)\rangle$ is θ-bounded, with $n^{-(5hq)^{-r}} < \theta < 1$. Then there is some $\omega_q^{-2h}\theta$-bounded $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial^\gamma\Psi = J$.*

The following lemma takes as input a bounded integral decomposition as produced by Lemma 3.18 and produces a signed decomposition analogous to that in [15, Lemma 8.1]. In the next subsection we will reduce Lemma 3.19 to Lemma 3.27, which will be proved in section 4. Let $c_2' = \omega^{-h^2}c_2$.

Lemma 3.19. *Suppose $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ is c_2'-bounded with $0 \leq_\gamma \partial^\gamma\Psi \leq_\gamma \partial^\gamma M^*$. Let $S = (\partial^\gamma\Psi)^\circ$ and suppose $M^*(S)$ is a set. Then there is $M^o \subseteq M^*$ and $M^i \subseteq \mathcal{A}(\Phi)$ such that $\partial^\gamma M^o = \partial^\gamma M^i + \partial^\gamma\Psi$.*

The proof of Theorem 3.1 is now quite short given these lemmas.

Proof of Theorem 3.1. Fix a template M^* as in Definition 3.7 that satisfies all of the whp statements in the paper. Let M^n be obtained from Lemma 3.15 and M^c and S from Lemma 3.16. Let $J = \partial^\gamma(M^* + M^n + M^c) - G$ and note that $J \in \langle\gamma(\Phi)\rangle$, $0 \leq_\gamma J \leq_\gamma \partial^\gamma M^*$ and $J^\circ = S$. Then J is c_2-bounded, so by Lemma 3.18 there is some c_2'-bounded $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial^\gamma\Psi = J$. Then we can apply Lemma 3.19 to obtain $M^o \subseteq M^*$ and $M^i \subseteq \mathcal{A}(\Phi)$ such that $\partial^\gamma M^o = \partial^\gamma M^i + J$. Now $M = M^n \cup M^c \cup (M^* \setminus M^o) \cup M^i$ is a $\gamma(\Phi)$-decomposition of G. □

3.6 Absorption

In this subsection, we establish the algebraic absorbing properties of the template, and so reduce Lemma 3.19 to Lemma 3.27. Following [15, section 6], with appropriate modifications for the more general setting here, we will define absorbers, cascades and cascading cliques, then estimate the number of cascades for any cascading cliques. We will always be concerned with absorbing maps that are compatible with the template (possibly with one 'bad edge'), as in the following definition.

Definition 3.20. Let $\phi \in A(\Phi)$ for some $A \in \mathcal{A}$. We say that ϕ is M^*-compatible if ϕ is π-compatible and $\phi^e \in A(\Phi)$ for all $e \in \phi(Q) \cap G^*$. Also, for $e' \in \phi(Q)$ and $Q^* = \phi(Q) \cap G^* \setminus \{e'\}$, we say that ϕ is M^*-compatible bar e' if $\pi_e\phi = id$ and $\phi^e \in A(\Phi)$ for all $e \in Q^*$.

Note that if ϕ is M^*-compatible and $\phi(Q) \subseteq G^*$ then $\gamma(e) = \gamma(\psi)$ whenever $e = Im(\psi) \in G^*$ with $\psi \subseteq \phi$. To define absorbers we require some notation: let $Ker := \{a \in \mathbb{F}_p^q : aM = 0\}$ and

$$v_a^w = MM_{[r]}^{-1}(e_{[r]} + a)w, \quad v'^w_a = w + Maw \quad \text{for } a \in Ker^r \text{ and } w \in \mathbb{F}_{p^a}^q.$$

Definition 3.21. (absorbers) Let $\phi \in A(\Phi)$ be M^*-compatible with $\phi(Q) \subseteq G_j^*$ and $\dim(w) = q$, where $w := f_j\phi \in \mathbb{F}_{p^a}^q$. Suppose $\phi^w : [q] \times Ker \to [n]$ such that

i. $f_j\phi^w((i,a)) = w_i + a \cdot w$ for each $i \in [q]$, $a \in Ker$,

ii. if $\phi' \in [q](Ker)$ with $f_j\phi^w\phi' = v_a^w$ for some $a \in Ker^r$ then $A_{\phi^w\phi'} = A$ and $\phi^w\phi' \in M_j^*$.

We say that ϕ is absorbable and call ϕ^w the absorber for ϕ. We also refer to the subgraph[8] $\Theta^{\phi(Q)} = \Theta^w = \phi^w(K_q^r(Ker))$ of G^* as the absorber for $\phi(Q)$.

We denote the edges of Θ^w by e_a^w where $f_j(e_a^w) = (e_I + a)w$ for some $a \in Ker^I$, $I \in Q$. As in [15, Lemma 6.3], each edge has full dimension under the relevant embedding: we have $\dim(f_j(e_a^w)) = r$. As in [15], we also view $[q](Ker)$ as a subset of $\mathbb{F}_p^{q\times q}$, and then we can write Definition 3.21.i as $f_j\phi^w\phi' = w + \phi' w$. We define

$$\phi^a := MM_{[r]}^{-1}(e_{[r]} + a) - I, \quad \text{and} \quad \phi'^a = Ma,$$

so that $f_j\phi^w\phi^a = w + \phi^a w = v_a^w$ and $f_j\phi^w\phi'^a = w + \phi'^a w = v'^w_a$. We write

$$\Psi(\phi^w) = \{\phi^w\phi^a : a \in Ker^r\} \quad \text{and} \quad \Psi'(\phi^w) = \{\phi^w\phi'^a : a \in Ker^r\},$$

noting that $\Psi(\phi^w) \subseteq M_j^*$ and $\phi \in \Psi'(\phi^w)$. By [15, Lemma 6.4], $\Psi(\phi^w)$ and $\Psi'(\phi^w)$ both give K_q^r-decompositions of $\Theta^{\phi(Q)}$. Furthermore, $\partial^\gamma\Psi(\phi^w) = \partial^\gamma\Psi'(\phi^w) \leq_\gamma \partial^\gamma M^*$.

Next we recall [15, Lemma 6.5].

Lemma 3.22. *There are K_q^r-decompositions Υ and Υ' of $\Omega = K_q^r(p)$ such that*

i. $|V(f) \cap V(f')| \leq r$ for all $f \in \Upsilon$ and $f' \in \Upsilon'$,

ii. if $f \in \Upsilon$ and $\{f', f''\} \subseteq \Upsilon'$ with $|V(f) \cap V(f')| = |V(f) \cap V(f'')| = r$ then $(V(f') \setminus V(f)) \cap (V(f'') \setminus V(f)) = \emptyset$.

We identify Υ and Υ' with subsets of $[q](p)$, i.e. the set of partite maps from $[q]$ to $[q] \times [p]$. We identify $[q]$ with $\{(i,1) : i \in [q]\} \subseteq V(\Omega)$ and with the corresponding map $id_{[q]}$; by relabelling we can assume $[q] \in \Upsilon$. For $U' \subseteq U \subseteq [n]$ we say that U is j-generic for U' if $\dim(f_j(U)) = \dim(f_j(U')) + |U| - |U'|$. Now we can define cascades.

Definition 3.23. (cascades) Let $\phi \in A(\Phi)$ be M^*-compatible. Suppose ϕ^c is an embedding of $K_q^r(p)$ in G_j^* where $\phi^c\, id_{[q]} = \phi$ and $Im(\phi^c)$ is j-generic for $Im(\phi)$, such that each $\phi^c\phi'$ with $\phi' \in \Upsilon'$ has $A_{\phi^c\phi'} = A$ and is absorbable, with absorber $\Theta^{\phi^c\phi'(Q)} = \phi^{w_{\phi'}}(K_q^r(Ker))$, and $C_{\phi^c} = \sum\{\Theta^{\phi^c\phi'(Q)} : \phi' \in \Upsilon'\}$ is a set (without multiple elements). We call C_{ϕ^c} a cascade for ϕ.

To flip a cascade C_{ϕ^c} we replace

$$\Psi(C_{\phi^c}) := \bigcup\{\Psi(w_{\phi'}) : \phi' \in \Upsilon'\} \qquad \text{by}$$

$$\Psi'(C_{\phi^c}) := \{\phi^c\phi' : \phi' \in \Upsilon\} \cup \bigcup\{\Psi'(w_{\phi'}) \setminus \{\phi^c\phi'\} : \phi' \in \Upsilon'\}.$$

This modifies M^* so as to include ϕ. Next we define the class of cliques for which we will show that there are many cascades.

[8] Here we use 'Θ' rather than the natural 'A' used in [15] to avoid clashes with other uses of 'A' in the paper.

Definition 3.24. (cascading cliques) Let $\mathcal{Q}^* = \cup_{j\in[z]}\mathcal{Q}_j$, where each $\mathcal{Q}_j$ is the set of all M^*-compatible $\phi \in \mathcal{A}(\Phi)$ with $\dim(f_j\phi) = q$ and $\phi(Q) \subseteq G_j^*$ (we call ϕ cascading).

The following is [15, Lemma 6.10].

Lemma 3.25. *Suppose $\phi \in \mathcal{Q}^*$, $Q' = \phi(Q)$ and $e \in G^*$ with $|e \setminus V(M^*(Q'))| = r'$. Let $\phi' \in \Upsilon'$, $I \in Q$, $a \in Ker^I$. Then there are at most $p^{2q}n^{q(p-1)-r'}$ cascades C_{ϕ^c} for ϕ such that the absorber $\Theta^{\phi^c\phi'(Q)} = \phi^{w_{\phi'}}(K_q^r(Ker))$ for $\phi^c\phi'$ satisfies $e_a^{w_{\phi'}} = e$.*

The following is analogous to [15, Lemma 6.11].

Lemma 3.26. *whp for any cascading $\phi \in \mathcal{Q}^*$ there are at least $\omega^{p^{q^2}}n^{q(p-1)}$ cascades for ϕ.*

Proof. We follow the proof in [15], indicating the necessary modifications. Suppose $\phi \in A(\Phi)$, let $Q' = \phi(Q)$, and condition on local events $\mathcal{E} = \cap_{e\in Q'}\mathcal{E}^e$ such that $\phi \in \mathcal{Q}_j$. Let U be the set of vertices touched by $\mathcal{E}$. As ϕ is M^*-compatible, each $e \in \phi(Q)$ has $\phi^e \in A(\Phi) \cap M^*$ with $e \in \phi^e(Q)$ and $\pi_e\phi^e = \pi_e\phi = id$.

Next we specify the combinatorial structure of a potential cascade for ϕ. For the base of the cascade, we fix a Φ-embedding ϕ^c of $[q](p)$ with $\phi^c\ id_{[q]} = \phi$ and $Im(\phi^c) \setminus Im(\phi)$ disjoint from U, such that for all $\phi^c\psi \in A(\Phi)_r^{\leq}$ where $\psi \in [q](p)_r$ we have $\gamma(\phi^c\psi) \leq_\gamma G$. For the absorbers in the cascade, for each $\phi' \in \Upsilon'$ we fix any Φ-embedding $\phi^{\phi'}$ of $[q](Ker)$ with $\phi^{\phi'}\phi'^0 = \phi^c\phi'$, such that for all $\phi^{\phi'}\psi \in A(\Phi)_r^{\leq}$ where $\psi \in [q](Ker)_r$ we have $\gamma(\phi^{\phi'}\psi) \leq_\gamma G$. If ϕ' is some ϕ'_e with $\phi(Im(\phi'_e) \cap [q]) = e \in \phi(Q)$ we have the additional constraint $\phi^{\phi'_e}\phi^{a_e} = \phi^e$, where $a_e \in Ker^r$ is such that $Im(\pi_e) \subseteq Im(\phi^{a_e})$. We choose absorbers as disjointly as possible, i.e. with 'private vertices' $I_{\phi'}$ that are pairwise disjoint and disjoint from $U \cup Im(\phi^c)$, where $I_{\phi'} = Im(\phi^{\phi'}) \setminus Im(\phi^c\phi')$ if ϕ' is not some ϕ'_e or $I_{\phi'_e} = Im(\phi^{\phi'_e}) \setminus (Im(\phi^c\phi'_e) \cup Im(\phi^e))$.

As $(\Phi, \gamma[G]^A)$ is (ω, h)-extendable, the number of such choices for ϕ^c and $\phi^{\phi'}$ given ϕ and $\mathcal{E}$ is at least $0.9\omega n^{q(p-1)+v_+}$, where $v_+ = \sum_{\phi'}|I_{\phi'}| = p^rq(p^{q-r}-1) - Q(q-r)$.

Now we consider the algebraic constraints that must be satisfied for the cascade described above to appear in the template. We condition on $f_j\phi^c$ such that $Im(\phi^c)$ is j-generic for $Im(\phi)$ and define $w_{\phi'} = f_j\phi^c\phi'$ for $\phi' \in \Upsilon'$. Then each $\dim(w_{\phi'}) = q$. Now ϕ^c will define a cascade $C_{\phi^c} = \sum\{\Theta^{\phi^c\phi'(Q)} : \phi' \in \Upsilon'\}$ with each $\Theta^{\phi^c\phi'(Q)} = \phi^{w_{\phi'}}(K_q^r(Ker)) = \phi^{\phi'}(K_q^r(Ker))$ as in Definitions 3.21 and 3.23 if

i. $f_j\phi^{\phi'}((i,a)) = (w_{\phi'})_i + a \cdot w_{\phi'}$ for each $\phi' \in \Upsilon'$, $i \in [q]$, $a \in Ker$, and
ii. $\phi^{\phi'}\phi^a$ is activated, $A_{\phi^{\phi'}\phi^a} = A$, and $T_e = j$ and $\pi_e\phi^{\phi'}\phi^a = id$ for all $\phi' \in \Upsilon'$, $a \in Ker^r$, $e \in \phi^{\phi'}\phi^a(Q)$.

We have the same bound on the probability of these events as in [15], as the estimates there were sufficiently crude to absorb the extra factor of $|\mathcal{A}|^{p^{r(q-r+1)}}$ here for the events $A_{\phi^{\phi'}\phi^a} = A$. The proof of concentration of the number of cascades is also the same (the effect of changing A_ϕ is analysed in the same way as the effect of changing whether ϕ is activated). □

The proof of [15, Lemma 8.1] (the cascade random greedy algorithm) now applies to show that Lemma 3.19 follows from the following lemma.

Lemma 3.27. *Suppose $\Psi^0 \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ is c_2'-bounded with $0 \leq_\gamma \partial^\gamma\Psi^0 \leq_\gamma \partial^\gamma M^*$. Let $S = (\partial^\gamma\Psi^0)^\circ$ and suppose $M^*(S)$ is a set. Then there are $M^\pm \subseteq \mathcal{A}(\Phi)$ such that every $\phi \in M^+$ is cascading, $M^*(\sum_{\phi\in M^+}\phi(Q))$ is a set and $3c_4$-bounded, and $\partial^\gamma M^+ = \partial^\gamma M^- + \partial^\gamma\Psi^0$.*

We will prove Lemma 3.27 in section 4.

4 Clique Exchange Algorithm

In this section we define our Clique Exchange Algorithm, which has three applications in this paper, namely to the proofs of Lemmas 4.1 and 3.27 (in this section) and Lemma 6.8 (in section 6). The following lemma will allow us to modify an integer decomposition so as to avoid unforced uses of bad sets. For the statement we recall the lattice constant $C_0 = C_0(\gamma)$ from Lemma 2.35, and that $C_0 = 1$ if γ is elementary.

Lemma 4.1. *Let $\mathcal{A}$ be a $\Sigma^{\leq}$-family with $\Sigma \leq S_q$ and $|\mathcal{A}| \leq K$ and suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be an (ω, h)-extendable Σ-adapted $[q]$-complex on $[n]$, where $\omega < \omega_0(q, D, K)$ and $n > n_0(q, D, K)$. Suppose $\Psi^0 \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ is θ-bounded with $n^{-1/2} < \theta < \omega^4$. Let $B^k \subseteq \Phi^\circ_k$ be η-bounded for $r \leq k \leq q$, where $\eta = (9q)^{-2q}\omega$. Then there is some $M_2\theta$-bounded $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$, where $M_2 = q^{(2q)^{2q}} C_0^2 \omega^{-2}$, with $\partial^\gamma \Psi = J := \partial^\gamma \Psi^0$ such that*

i. if $k > r$ then $U(\Psi)_\psi \leq 1$ for all $\psi \in \Phi_k$, and $U(\Psi)_\psi = 0$ if $Im(\psi) \in B^k$,

ii. $U(\Psi)_\psi \leq U(J)_\psi + C_0 + 1$ for all $\psi \in \Phi_r$, and $U(\Psi)_\psi = U(J)_\psi$ if $Im(\psi) \in B^r$.

Lemma 4.1 and Lemma 3.18 immediately imply the following lemma, which will be used (with smaller q) in the inductive proof of Lemma 3.18 in section 6.

Lemma 4.2. *Let $\mathcal{A}$ be a $\Sigma^{\leq}$-family with $\Sigma \leq S_q$ and $|\mathcal{A}| \leq K$ and suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be an (ω, h)-extendable Σ-adapted $[q]$-complex on $[n]$, where $n^{-h^{-3q}} < \omega < \omega_0(q, D, K)$ and $n > n_0(q, D, K)$. Suppose $J \in \langle \gamma(\Phi) \rangle$ is θ-bounded with $n^{-(5hq)^{-r}} < \theta < \omega^4$. Suppose $B^k \subseteq \Phi^\circ_k$ is η-bounded for $r \leq k \leq q$, where $\eta = (9q)^{-2q}\omega$. Then there is some $\omega_q^{-3h}\theta$-bounded $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial\Psi = J$ such that*

i. if $k > r$ then $U(\Psi)_\psi \leq 1$ for all $\psi \in \Phi_k$, and $U(\Psi)_\psi = 0$ if $Im(\psi) \in B^k$,

ii. $U(\Psi)_\psi \leq U(J)_\psi + C_0 + 1$ for all $\psi \in \Phi_r$, and $U(\Psi)_\psi = U(J)_\psi$ if $Im(\psi) \in B^r$.

4.1 Splitting Phase

Now we start the proof of Lemma 4.1. Suppose $\Psi^0 \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ is θ-bounded. We will obtain the desired Ψ by an algorithm similar to that in [15] (but with several significant differences).

To define the first phase of the algorithm, we recall the K^r_q-decompositions Υ and Υ' of $\Omega = K^r_q(p)$ given by Lemma 3.22, and write $\Omega' = K^r_q(p) \setminus Q$.

Algorithm 4.3. (Splitting Phase) Let $(\phi_i : i \in |\Psi^0|)$ be any ordering of the signed elements of Ψ^0, i.e. $\Psi^0 = \sum_i s_i\{\phi_i\}$ with each $s_i \in \pm 1$ and $\phi_i \in A^i(\Phi)$ for some $A^i \in \mathcal{A}$. We apply a random greedy algorithm to choose $\phi_i^* \in X_{E_i}(\Phi)$ for each i, where $E_i = ([q](p), [q], \phi_i)$. We say ϕ_i^* uses $e \in \Phi^\circ$ if $e \subseteq Im(\phi_i^*\phi)$ for some $\phi \in \Upsilon' \cup \Upsilon$ and $e \setminus Im(\phi_i) \neq \emptyset$. Let F_i be the set of used $e \in \Phi^\circ_r$. We choose $\phi_i^* \in X_{E_i}(\Phi)$ uniformly at random subject to not using F_i or $B = \cup_k B^k$.

Lemma 4.4. *whp Splitting Phase does not abort and $F_{|\Psi^0|}$ is $M_1\theta$-bounded, where $M_1 = 2^{r+3}(pq)^q\omega^{-1}$.*

Proof. For $i \in [|\Psi^0|]$ we let $\mathcal{B}_i$ be the bad event that F_i is not $M_1\theta$-bounded. Let τ be the smallest i for which $\mathcal{B}_i$ holds or the algorithm aborts, or ∞ if there is no such i. It suffices to show whp $\tau = \infty$. We fix $i_0 \in [|\Psi^0|]$ and bound $\mathbb{P}(\tau = i_0)$ as follows.

We claim that for any $i < i_0$ the restrictions on ϕ_i^* forbid at most half of the possible choices of $\phi_i^* \in X_{E_i}(\Phi)$. To see this, first note that $X_{E_i}(\Phi) > \omega n^{pq-q}$ as Φ is (ω, s)-extendable. As F_i is $M_1\theta$-bounded and each B^k is η-bounded, at most $(pq)^q(q\eta + M_1\theta)n^{pq-q}$ choices use $F_i \cup B$; the claim follows.

Now for each $e \in \Phi_r^\circ$ let $r_e = \sum_{i<i_0} \mathbb{P}'(e \in \phi_i^*(\Omega'))$, where $\mathbb{P}'$ denotes conditional probability given the choices made before step i.

For any $i < i_0$, writing $r' = |e \setminus Im(\phi_i)|$, there are at most $(pq)^q n^{pq-q-r'}$ choices of ϕ_i^* such that $e \in \phi_i^*(\Omega')$, so by the claim $\mathbb{P}'(e \in \phi_i^*(\Omega')) < 2(pq)^q \omega^{-1} n^{-r'}$. Also, given $r' \in [r]$, as Ψ^0 is θ-bounded there are at most $2\binom{r}{r'}\theta n^{r'}$ choices of i such that $|e \setminus Im(\phi_i)| = r'$. Therefore $r_e < 2^{r+2}(pq)^q\omega^{-1}\theta$.

Now fix any $f \in \Phi_{r-1}^\circ$ and write $U(F_i)_f = \sum_{i<i_0} X_i$, where X_i is the number of $e \in \Phi_r^\circ$ with $f \subseteq e \in \phi_i^*(\Omega')$. Then each $|X_i| < |\Omega'|$ and $\sum_{i<i_0} \mathbb{E}'X_i = \sum\{r_e : f \subseteq e\} < 2^{r+2}(pq)^q\omega^{-1}\theta n$, so by Lemma 3.3 whp $U(F_i)_f < M_1\theta n$, so F_i is $M_1\theta$-bounded, so $\tau > i_0$. Taking a union bound over i_0, whp $\tau = \infty$, as required. □

We let[9] $\Psi^1 = \Psi^0 + \sum_{i\in[|\Psi^0|]} s_i(A^i(\Phi[\phi_i^*\Upsilon']) - A^i(\Phi[\phi_i^*\Upsilon]))$. Then $\partial^\gamma\Psi^1 = \partial^\gamma\Psi^0 = J$, and all signed elements of Ψ^0 are cancelled, so Ψ^1 is supported on maps added during Splitting Phase.

We classify maps added during Splitting Phase as near or far, where the near maps are those of the form $\phi_i^*\phi$ for $\phi \in \Upsilon'$ with $|Im(\phi) \cap [q]| = r$. Also, for each pair (O, ϕ') where ϕ' is added during Splitting Phase, $O \in \Phi_r/\Sigma$ and $O \subseteq \phi'\Sigma$, we call (O, ϕ') near if $\phi' = \phi_i^*\phi$ is near and $Im(O) = \phi_i^*(Im(\phi) \cap [q])$, otherwise we call (O, ϕ') far. We fix orbit representatives $\psi^O \in O$ and say that (O, ϕ') has type θ where $\gamma(\phi')^O = \gamma[\psi^O]^\theta = \gamma(\psi^O\theta^{-1})$ (we fix any such θ for each γ-atom at O). We also classify maps and near pairs as positive or negative according to their sign in Ψ^1.

Note that for each orbit O such that there is some far pair on O there are exactly two such far pairs $(O, \phi^\pm)$ and $\gamma(\phi^-)^O = -\gamma(\phi^+)^O$. For each $O \in \Phi_r/\Sigma$ we let Ψ^O be the sum of all $\pm\{\phi\}$ where $\pm(O, \phi)$ is a signed near pair in Ψ^1. Then $(\partial^\gamma\Psi^O)^O = (\partial^\gamma\Psi^1)^O = J^O$.

4.2 Grouping Phase

Now we will organise the near pairs on each O into some cancelling groups and $U(J)_O$ ungrouped near pairs. To do so we will introduce some additional near pairs in which we add and subtract some given element of $\mathcal{A}(\Phi)$ (which has no net effect on Ψ^1).

Consider any orbit O with $\psi^O \in \Phi_B$, $B \in Q$. As $J = \partial^\gamma\Psi^0 \in \langle\gamma(\Phi)\rangle = \mathcal{L}_\gamma(\Phi) \subseteq \mathcal{L}_\gamma^-(\Phi)$, we have $f_B(J)_{\psi^O} \in \gamma^B = \langle\gamma^\theta : \theta \in \mathcal{A}_B\rangle$. By definition of $U(J)_O$ we can express J^O as a sum of $U(J)_O$ signed γ-atoms, i.e. $f_B(J)_{\psi^O} = \sum_\theta n_\theta^O\gamma^\theta$, so $J^O = \sum_\theta n_\theta^O\gamma(\psi^O\theta^{-1})$, where $n^O \in \mathbb{Z}^{\mathcal{A}_B}$ with $|n^O| = U(J)_O$. Let $m_\theta^{O\pm}$ be the number of near pairs on O of type θ of each sign. As $J^O = (\partial^\gamma\Psi^O)^O$ we have $f_B(J)_{\psi^O} = \sum_\theta(m_\theta^{O+} - m_\theta^{O-})\gamma^\theta$, so $n^O - m^{O+} + m^{O-} \in Z_B(\gamma)$. By Lemma 2.35 we have $n^{Oj} \in Z_B(\gamma)$ for $j \in [t^O]$ for some $t^O \le C_0(|n^O| + |m^{O+}| + |m^{O-}|)$ with each $|n^{Oj}| \le C_0$ and $n^O - m^{O+} + m^{O-} = \sum_{j\in[t^O]} n^{Oj}$.

We will assign the near pairs to cancelling groups and ungrouped near pairs so that for each such O and θ, there are $|n_\theta^O|$ (correctly signed) ungrouped near pairs on O of type θ, and the jth group has $|n_\theta^{Oj}|$ such near pairs.

Let $d_\theta^O = (\sum_j |n_\theta^{Oj}|) - m_\theta^{O+} = ((|n_\theta^O| + \sum_j |n_\theta^{Oj}|) - (m_\theta^{O+} + m_\theta^{O-}))/2$. If $d_\theta^O > 0$ then we need to introduce d_θ^O new near pairs of type θ on O in the Grouping Phase below. If $d_\theta^O \le 0$ then we do not need to introduce any new near pairs of type θ on O. If $d_\theta^O < 0$ then we have $2|d_\theta^O|$ unassigned near pairs of type θ on O, with which we form $|d_\theta^O|$ additional cancelling groups each containing one positive and one negative near pair.

Algorithm 4.5. (Grouping Phase) Let $S^J = \{(O^i, \theta^i) : i \in [|S^J|]\}$ be such that each (O, θ) with $d_\theta^O > 0$ appears d_θ^O times. We apply a random greedy algorithm to choose $\phi_i \in X_{E_i}(\Phi)$ with

[9] Note that (e.g.) $A^i(\Phi[\phi_i^*\Upsilon]) = \{\phi_i^*\psi \in A^i(\Phi) : \psi \in \Upsilon\}$.

$E_i = (\overrightarrow{q}, Im(\theta^i), \psi_{O^i}(\theta^i)^{-1})$. We say ϕ_i uses $e \in \Phi_r^\circ$ if $Im(O^i) \neq e \subseteq Im(\phi_i)$. Let F_i' be the set of used $e \in \Phi_r^\circ$. We choose ϕ_i uniformly at random subject to not using $F_i' \cup F_{|\Psi^0|} \cup B$.

Similarly to Lemma 4.4, whp Grouping Phase does not abort and $F'_{|S^J|}$ is $M_1\theta$-bounded. We create new near pairs by adding and subtracting each ϕ_i, and then organise the near pairs into cancelling groups and ungrouped near pairs as described above.

4.3 Elimination Phase

In the Elimination Phase we replace Ψ^1 by Ψ^2 so as to remove all cancelling groups while preserving $\partial\Psi^2 = \partial\Psi^1 = J$. We start by recalling [15, Definition 6.15].

Definition 4.6. Let Ω_1 and Ω_2 be two copies of Ω. Fix $f \in \Upsilon$ and $f' \in \Upsilon'$ with $|V(f) \cap V(f')| = r$. For $j = 1, 2$ we denote the copies of Υ, Υ', f, f' in Ω_j by Υ_j, Υ'_j, f_j, f'_j. Let Ω^* be obtained by identifying Ω_1 and Ω_2 so that $f'_1 = f'_2$. Let $\Upsilon^+ = \Upsilon_1 \cup (\Upsilon'_2 \setminus \{f'_2\})$ and $\Upsilon^- = \Upsilon_2 \cup (\Upsilon'_1 \setminus \{f'_1\})$. Then Υ^+ is a K_q^r-decomposition of Ω^* containing f_1 and Υ^- is a K_q^r-decomposition of Ω^* containing f_2.

Next we introduce some notation for octahedra and their associated signed characteristic vectors.

Definition 4.7. (octahedra) Let Φ be an R-complex and $B \subseteq R$. The B-octahedron is $O^B = B(2)$. For $x \in [2]^B$ we define the sign of x by $s(x) = (-1)^{\sum_i (x_i - 1)}$. For $\psi \in O_B^B$ such that $\psi(i) = (i, x_i)$ for all $i \in B$ we also write $s(\psi) = s(x)$. Let $\mathcal{O}^B(\Phi)$ be the set of Φ-embeddings of O^B. For $\phi \in \mathcal{O}^B(\Phi)$ we let $\chi(\phi)$ denote the 'signed characteristic vector' in $\mathbb{Z}^{\Phi_B}$, where $\chi(\phi)_{\phi\circ\psi} = s(\psi)$ for $\psi \in O_B^B$, and all other entries of $\chi(\phi)$ are zero.

The following definition and lemma implement octahedra as signed combinations of cliques.

Definition 4.8. For $x = (x_i) \in [s]^q$ we identify x with the partite map $x : [q] \to [q] \times [s]$ where each $x(i) = (i, x_i)$, and also with the image of this map. We write 1 for the map with all $1(i) = 1$ and identify $[q]$ with $1([q])$.

For $e \in [q](s)_r$ let $X_e = \{x : e \subseteq x\}$. Suppose $w \in \{-1, 0, 1\}^{[s]^q}$.

We say $e \in [q](s)_r$ is bad for w if $|\{x \in X_e : w_x = 1\}| > 1$ or $|\{x \in X_e : w_x = -1\}| > 1$.

We say w is simple if no e is bad for w. We define $\partial w \in \mathbb{Z}^{[q](s)_r}$ by $\partial w_e = \sum_{x \in X_e} w_x$.

Lemma 4.9. *Let $s = (2q)^r$. Then for any $B \in Q$ there is a simple $w^B \in \{-1, 0, 1\}^{[s]^q}$ with $\partial w^B = \chi(O^B)$. Let w_r^B denote the set of $e \in [q](s)_r$ such that $w_x^B \neq 0$ for some $x \in X_e$. We can choose w^B with $w_1^B = 1$ so that $w_r^B[V(O^B) \cup [q]] = O_B^B \cup Q$.*

Proof. We start by setting $w_x^B = (-1)^{\sum_{i=1}^r (x_i - 1)}$ if $x_i \in [2]$ for $i \in B$ and $x_i = 1$ for $i \in [q] \setminus B$, otherwise $w_x^B = 0$. Then $\partial w_e^B = \chi(O^B)$. We will repeatedly apply transformations to w^B that preserve $\partial w^B = \chi(O^B)$ until w^B becomes simple. Suppose w^B is not simple. Fix e bad for w^B and $x, x' \in X_e$ with $w_x^B = 1$ and $w_{x'}^B = -1$. Fix a $[q](s)$-embedding ϕ of Ω^* as in Definition 4.6, where $\phi(f_1) = x$, $\phi(f_2) = x'$ and if $a \in \phi(V(\Omega^*)) \setminus (x \cup x')$ then $w_y^B = 0$ whenever $a \in y$. We modify w^B by adding -1 to each $\phi(g)$ where $g \in \Psi^+$ and 1 to each $\phi(g')$ where $g' \in \Psi^-$. This preserves $\partial w_e^B = \chi(O^B)$ and reduces the sum of $|w_x^B|$ over $x \in X_e$ with e bad for w^B. The process terminates with w^B that is simple, and we can relabel so that the other properties hold. □

Let $w^B = [q](s)[w_r^B]$ be the $[q]$-complex obtained by restricting $[q](s)$ to w_r^B.

Algorithm 4.10. (Elimination Phase) Let $(C^i : i \in [P])$ be any ordering of the cancelling groups, where each $C^i = \{(O^i, \phi^i_j) : j \in [|C^i|]\}$ for some orbit O^i with representative $\psi_{O^i} \in \Phi_{B^i}$ and each $\phi^i_j \in A^i_j(\Phi)$. Our random greedy algorithm will make several choices at step i. First we choose $\psi^*_i \in X_{E^i}(\Phi)$ where $E^i = (B^i(2), B^i, \psi_{O^i})$; we say that this choice has type 1. Then for each $j \in [|C^i|]$ we make type 2 choices $\phi^{*i}_j \in X_{E^i_j}(\Phi)$ where $E^i_j = (w^{B^i_j}, F^i_j, \phi'^i_j)$, $F^i_j = [q] \cup (B^i_j \times [2])$, $B^i_j = \theta^i_j(B^i)$, $\psi_{O^i} = \phi^i_j\theta^i_j$, $\phi'^i_j\,|_{[q]} = \phi^i_j$, $\phi'^i_j(\theta^i_j(x), y) = \psi^*_i(x, y)$ for $x \in B^i$, $y \in [2]$.

We let $\Omega'_i = B^i(2)_r \setminus \{id_{B^i}\}$ and $\Omega^i_j = w_r^{B^i_j} \setminus (B^i_j(2) \cup \overrightarrow{q}_r)$. We say that ψ^*_i uses $e \in \Phi^\circ_r$ with type 1 if $e = Im(\psi^*_i\psi)$ for some $\psi \in \Omega'_i$ (we also write $e \in \psi^*_i(\Omega'_i)$). We say that ϕ^{*i}_j uses $e \in \Phi^\circ_r$ with type 2 if $e = Im(\phi^{*i}_j\psi)$ for some $\psi \in \Omega^i_j$ or if $|e| > r$ and $e \subseteq Im(\phi^{*i}_j x)$ for some $x \in [s]^q \setminus \{1\}$ with $w_x^{B^i_j} \neq 0$.

For $\alpha = 1, 2$ let F^α_i be the set of $e \in \Phi^\circ_r$ used with type α. We make each choice at step i uniformly at random subject to not using $F^1_i \cup F^2_i \cup F_{|\Psi^0|} \cup F'_{|S^J|} \cup B$.

We will obtain Ψ from Ψ^1 by adding $\sum_x w_x^{B^i_j} A^i_j(\Phi[\phi^{*i}_j x])$ for each $i \in [P]$ and $j \in [|C^i|]$ with the opposite sign to that of the near pair (O^i, ϕ^i_j). This cancels all cancelling groups and preserves $\partial^\gamma\Psi = \partial^\gamma\Psi^1 = J$ by the following lemma, which shows that the construction for each cancelling group has no total effect on $\partial^\gamma\Psi$, using $\sum_{j\in[|C^i|]} \gamma^{\theta^i_j} = 0$.

Lemma 4.11. *With notation as in Algorithm 4.10, we have*

$$\partial^\gamma \sum_x w_x^{B^i_j} A^i_j(\Phi[\phi^{*i}_j x]) = \sum \{\chi(\psi^*_i)_e \gamma(\phi^{*i}_j e(\theta^i_j)^{-1}) : e \in \psi^*_i O^{B^i}_{B^i}\}.$$

Proof. By Lemma 4.9, we have $(\partial^\gamma \sum_x w_x^{B^i_j} A^i_j(\Phi[\phi^{*i}_j x]))_\psi$ equal to zero unless $\psi\Sigma = e\Sigma$ with $e \in \psi^*_i O^{B^i}_{B^i}$, in which case, writing $e' = e \circ (\theta^i_j)^{-1}$, it equals $(\partial w_{e'}^{B^i_j})\gamma(\phi^{*i}_j e') = \chi(\psi^*_i)_e\gamma(\phi^{*i}_j e')$. □

Recall $M_1 = 2^{r+3}(pq)^q\omega^{-1}$, $M_2 = q^{(2q)^{2q}} C_0^2\omega^{-2}$ and note that $M_2\theta < \omega^{1.5}$ for $\omega < \omega_0(q, D, K)$.

Lemma 4.12. *whp Elimination Phase does not abort and $F^1_P \cup F^2_P$ is $M_2\theta/2$-bounded.*

Proof. For $i \in [P]$ we let $\mathcal{B}_i$ be the bad event that F^1_i is not $2C_0M_1\theta$-bounded or F^2_i is not $M_2\theta/4$-bounded. Let τ be the smallest i for which $\mathcal{B}_i$ holds or the algorithm aborts, or ∞ if there is no such i. It suffices to show whp $\tau = \infty$. We fix $i_0 \in [P]$ and bound $\mathbb{P}(\tau = i_0)$ as follows.

Consider any $i < i_0$. Then F^1_i is $2C_0M_1\theta$-bounded and F^2_i is $M_2\theta/4$-bounded. As Φ is (ω, h)-extendable, each $X_{E_i}(\Phi) > \omega n^{v_{E_i}}$ and $X_{E^i_j}(\Phi) > \omega n^{v_{E^i_j}}$. As $F_{|\Psi^0|}$ and $F'_{|S^J|}$ are $M_1\theta$-bounded, and each B^k is η-bounded, at most half of the choices for ψ^*_i or ϕ^{*i}_j are forbidden due to using $F^1_i \cup F^2_i \cup F_{|\Psi^0|} \cup F'_{|S^J|} \cup B$.

Next we fix $e \in \Phi^\circ_r$ and estimate the probability of using e at step i with each type. For uses of type 1 there are at most $2^r n^{v_{E_i} - |e\setminus Im(O^i)|}$ choices of ψ^*_i with $e \in \psi^*_i(\Omega'_i)$, so $\mathbb{P}'(e \in \psi^*_i(\Omega'_i)) \leq 2^{r+1}\omega^{-1}n^{-|e\setminus Im(O^i)|}$. Similarly, for uses of type 2 we have $\mathbb{P}'(e \in \phi^{*i}_j(\Omega^i_j)) < 2|\Omega^i_j|\omega^{-1}n^{-r^i_j(e)}$, where $r^i_j(e)$ is the minimum $|e \setminus Im(\psi')|$ with $\psi' \subseteq \phi^i_j$ or $\psi' = \psi^*_i\psi$ with $\psi \in B^i_j(2)$.

For any $r' \in [r]$, as Φ^0 is θ-bounded, by construction of the cancelling groups in Grouping Phase, there are at most $\binom{r}{r'} C_0\theta n^{r'}$ choices of i with $|Im(O^i) \setminus e| = r'$, so $r^1_e := \sum_{i<i_0} \mathbb{P}'(e \in \psi^*_i(\Omega'_i)) < 2^{2r}\omega^{-1}C_0\theta$. Similarly, as $F_{|\Psi^0|}$ and $F'_{|S^J|}$ are $M_1\theta$-bounded and F^1_i is $2C_0M_1\theta$-bounded, there are

at most $4C_0M_1\binom{r}{r'}\theta n^{r'}$ choices of i with $r^i_j(e) = r'$, so $r^2_e := \sum_{i<i_0}\sum_{j\in[|C^i|]}\mathbb{P}'(e \in \phi^{*i}_j(\Omega^i_j)) < C_0|\Omega^i_j|2^{r+3}C_0M_1\theta$. By Lemma 3.3 we deduce whp F^1_i is $2C_0M_1\theta$-bounded and F^2_i is $M_2\theta/4$-bounded, so $\tau > i_0$. Taking a union bound over i_0, whp $\tau = \infty$, as required. □

For any $\psi \in \Phi_{r-1}$ we have $U(\Psi)_\psi \le U(\Psi^0)_\psi + |(F_{|\Psi^0|} + F'_{|S^J|} + F^1_P + F^2_P)\,|_\psi\,| < M_2\theta n$, so Ψ is $M_2\theta$-bounded. For $k > r$ we avoided using any $\psi \in \Phi_k$ more than once or B^k at all, so $U(\Psi)_\psi \le 1$ for all $\psi \in \Phi_k$, and $U(\Psi)_\psi = 0$ if $Im(\psi) \in B^k$. For any $\psi \in \Phi_r$ we have a contribution of $U(J)_\psi$ from ungrouped near pairs to $U(\Psi)_\psi$. If $\psi \in B^r$ there are no other uses, so $U(\Psi)_\psi = U(J)_\psi$, and otherwise there are at most[10] $C_0 + 1$ other uses by a cancelling group, so $U(\Psi)_\psi \le U(J)_\psi + C_0 + 1$. This completes the proof of Lemma 4.1.

4.4 Proof of Lemma 3.27

The proof of Lemma 3.27 is very similar to that of Lemma 4.1, so we will just show the necessary modifications. We consider $\Psi^0 \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ that is c'_2-bounded, where $c'_2 = \omega^{-h^2}c_2$. There are no bad sets B. We also suppose $0 \le_\gamma \partial^\gamma\Psi^0 \le_\gamma \partial^\gamma M^*$ and $M^*(S)$ is a set, where $S = (\partial^\gamma\Psi^0)^\circ$. We require the following definitions for Splitting Phase.

Definition 4.13. Consider any extension $E(\phi) = ([q](p), [q], \phi)$ where $\phi \in A(\Phi)$ with $\gamma(\phi) \le_\gamma G$. Let $H(\phi) = [q](p)_r \setminus \overrightarrow{q}_r$. We let $X^*_{E(\phi),H(\phi)}$ be the set or number of extensions $\phi^+ \in X_{E(\phi),H(\phi)}(\Phi, \gamma[\partial^\gamma M^*]^A)$ such that

i. ϕ^+ is *rainbow*: $j \ne j'$ whenever $\{\psi, \psi'\} \subseteq [q](p)_r \setminus \overrightarrow{q}_r$, $Im(\phi^+\psi) \in G^*_j$, $Im(\phi^+\psi') \in G^*_{j'}$, and
ii. each $\phi^+\phi'$ with $\phi' \in \Upsilon'$ is M^*-compatible if $|Im(\phi') \cap [q]| < r$ or M^*-compatible bar $\phi^+(e)$ if $Im(\phi') \cap [q] = e \in Q$.

We claim whp

$$X^*_{E(\phi),H(\phi)} > \omega n^{pq-q}(\omega^2 z\rho/2)^{Qp^r}. \tag{1}$$

Indeed, the proof of Lemma 3.12 already gives rainbow extensions ϕ^+, and by Remark 3.11.ii we can also require $\pi_e\phi^+\psi = id$ and $A_{\phi^e} = A$ for all $\psi \in [q](p)_r \setminus \overrightarrow{q}$, $e = Im(\phi^+\psi)$, which gives $\phi^+ \in X^*_{E(\phi),H(\phi)}$.

For the modified Splitting Phase, recalling that F_i is the set of used $e \in \Phi^\circ_r$, we let $D_i = \cup_{e\in F_i}M^*(e)$, and choose $\phi^*_i \in X^*_{E(\phi_i),H(\phi_i)}$ uniformly at random subject to $\phi^*_i(\Omega') \cap (M^*(S) \cup D_i) = \emptyset$. Note that each $\phi^*_i(\Omega') \subseteq G^*$ is rainbow, so $M^*(\phi^*_i(\Omega'))$ is a set.

The modified form of Lemma 4.4 is to show whp $D_{|\Psi^0|}$ is c_3-bounded. Accordingly, the bad event $\mathcal{B}_i$ is that D_i is not c_3-bounded. To see that at most half of the choices of $\phi^*_i \in X^*_{E(\phi_i),H(\phi_i)}$ are forbidden we use (1), which gives $X^*_{E(\phi_i),H(\phi_i)} > \omega n^{pq-q}(\omega^2 z\rho/2)^{Qp^r} > 4(pq)^q c_3 n^{pq-q}$.

For $e \in G^*$ we define $r_e = \sum_{i<i_0}\mathbb{P}'(e \in M^*(\phi^*_i(\Omega'))) = \sum_{i<i_0}\sum_{e'\in M^*(e)}\mathbb{P}'(e' \in \phi^*_i(\Omega'))$. As Ψ^0 is c'_2-bounded, there are at most $c'_2\binom{r}{r'}n^{r'}$ choices of i with $|e' \setminus Im(\phi_i)| = r'$, each $\mathbb{P}'(e' \in \phi^*_i(\Omega')) < 2r!|\Omega'|\omega^{-1}(\omega^2 z\rho/2)^{-Qp^r}n^{-r'}$, so $r_e < (pq)^q\omega^{-1}(\omega^2 z\rho/2)^{-Qp^r}2^{r+1}c'_2$. We conclude that whp no $\mathcal{B}_i$ occurs, so whp $D_{|\Psi^0|}$ is c_3-bounded.

Defining Ψ^1 as before, we have $\partial^\gamma\Psi^1 = \partial^\gamma\Psi^0 = J$ and Ψ^1 is supported on maps added during Splitting Phase, which are now rainbow in G^* and M^*-compatible bar at most one edge.

As before, we classify maps ϕ' and pairs (O, ϕ') added in Splitting Phase as near/far and positive/negative, and assign types to pairs. As γ is now elementary, the next part of the algorithm

[10] The '+1' is only needed to account for cancelling pairs in the case $C_0 = 1$.

becomes simpler. Indeed, for each $O \in \Phi_r/\Sigma$ we can group the near pairs on O into cancelling groups of size one (near pairs of type zero) or size two (of the same type and opposite sign), and at most one additional positive near pair (O, ϕ^O), which we call 'solo', where if $Im(O) = e \in S$ with $\gamma(e) \neq 0$ then $\phi^O(Q) \subseteq G^*$, ϕ^O is M^*-compatible bar e, and $\gamma(e) = \gamma(\psi)$ with $\psi \subseteq \phi^O$.

In Grouping Phase we only need to consider the solo near pairs, which we denote by $\{(O^i, \phi^{O^i}) : i \in [s']\}$. We let $e_i = Im(O^i) \in G^*$, $A^i = A_{\phi^{e_i}}$, and $\theta^i \in A^i_{B^i}$ be such that $\psi'_i := \psi_{O^i}(\theta^i)^{-1} = \pi^{-1}_{e_i} \in A^i$, so $\gamma(\psi'_i) = \gamma(e_i)$. Writing $D'_i = \cup_{i'<i} M^*(\phi_i(Q)))$, we choose $\phi_i \in X_{E_i}(\Phi)$ with $E_i = (\overrightarrow{q}, Im(\theta^i), \psi_{O^i}(\theta^i)^{-1})$ uniformly at random subject to $(\phi^*_i(Q) \setminus \{e_i\}) \cap (M^*(S) \cup D_{|\Psi^0|} \cup D'_i) = \emptyset$ and $\phi_i \in \mathcal{Q}^*$ being cascading. Similarly to Lemma 4.15 (see below), there are at least $0.9(\omega/z)^{q-r}$ choices for each ϕ_i, and similarly to Lemma 4.4 whp $D'_{s'}$ is c_3-bounded.

The next definition and accompanying lemma set up the notation for the Elimination Phase and show that there are whp many choices for each step.

Definition 4.14. Given $A \in \mathcal{A}$, $B \in Q$, $\psi \in A(\Phi)^{\leq}_B = \Phi_B$ we let $E(\psi) = (B(2), B, \psi)$ and $H(\psi) = B(2)_B \setminus \{id_B\}$.

Suppose $\psi \subseteq \phi \in A(\Phi)$ with $\phi(Q) \setminus Im(\psi)$ rainbow in G^* and ϕ is M^*-compatible bar $Im(\psi)$. Let $X^{\phi}_{E(\psi),H(\psi)}$ be the set or number of $\psi^* \in X_{E(\psi),H(\psi)}(\Phi, \gamma[\partial^\gamma M^*]^A)$ such that

i. $Im(\psi^*) \cap Im(\phi) = Im(\psi)$,
ii. $Q^* := (\phi(Q) \cup \{Im(\psi^*\psi') : \psi' \in B(2)_B\}) \setminus \{Im(\psi)\}$ is rainbow in G^*, and
iii. for all $e = Im(\psi^*\psi')$ with $\psi' \in B(2)_B \setminus \{id_B\}$ we have $A_{\phi^e} = A$ and $\pi_e\psi^*\psi' = id$.

For $\psi^* \in X^{\phi}_{E(\psi),H(\psi)}$ we let $E^{\phi}_{\psi^*} = (w^B, F, \psi^* \cup \phi)$, where $F = [q] \cup (B \times [2])$.

We let $H^{\phi}_{\psi^*} = w^B \setminus w^B[F]$ and $v_c := v_{E^{\phi}_{\psi^*}}$.

Let $X^c(E^{\phi}_{\psi^*})^{\pm}$ be the set or number of $\phi^+ \in X_{E^{\phi}_{\psi^*},H^{\phi}_{\psi^*}}(\Phi, \gamma[\partial^\gamma M^*]^A)$ that are 'rainbow $\Upsilon^{\pm}$ cascading', i.e. $\phi^+x \in \mathcal{Q}^*$ is cascading for all $x \in \Upsilon^{\pm} := \{x \in [s]^q \setminus \{1\} : w^B_x = \pm 1\}$, and $j \neq j'$ whenever $\{x, x'\} \subseteq \Upsilon^{\pm}$ with $\phi^+x \in \mathcal{Q}_j$, $\phi^+x' \in \mathcal{Q}_{j'}$.

Lemma 4.15. *For ψ, ϕ, ψ^* as in Definition 4.14 whp $X^c(E^{\phi}_{\psi^*})^{\pm} > (\omega/z)^{3Q^2 s^r} n^{v_c}$.*

The proof of Lemma 4.15 requires the following analogue of Lemma 3.10.

Lemma 4.16. *Let $S \subseteq \Phi^\circ_r$ with $|S| < h = z$ and $\mathcal{E} = \cap_{f\in S}\mathcal{E}^f$. Suppose $\phi \in A(\Phi)$ with $\gamma(\phi) \leq_\gamma G$ such that $|\phi(Q) \cap S| \leq 1$ and each $e' \in \phi(Q) \setminus S$ is not touched by $\mathcal{E}$. Let $j \in [z]$ be such that $T_{e'} \neq j$ for all $e' \in S \setminus \phi(Q)$. If $\phi(Q) \cap S = \{e\}$ suppose also that $\pi_e\phi = id$, $e \in G^*_j$, $\phi^e \in A(\Phi)$. Then $\mathbb{P}(\phi \in \mathcal{Q}_j \mid \mathcal{E}) > (\omega/z)^{3Q^2}$.*

Proof. Let 1_e be 1 if $\phi(Q) \cap S = \{e\}$ or 0 otherwise. For $e' \in \phi(Q) \setminus S$ let $\pi^0_{e'} : e' \to [q]$ be such that $\pi^0_{e'}\phi = id$. For each $e' \in \phi(Q) \setminus S$ we fix $\phi^{e'}_0 \in A(\Phi)$ with $\pi^0_{e'}\phi^{e'}_0 = id$ and estimate the probability that all such $e' \in G^*_j$ with $\phi^{e'} = \phi^{e'}_0$. Let U be the set of vertices touched by $\mathcal{E}$. As $(\Phi, \gamma[G]^A)$ is (ω, h)-extendable, there are at least $(1 - O(n^{-1}))\omega n^{(Q-1_e)(q-r)}$ choices for all $\phi^{e'}_0$ such that the sets $Im(\phi^{e'}_0) \setminus e'$ are pairwise disjoint and disjoint from $Im(\phi) \cup U$, and for each $e' \in \phi(Q) \setminus S$ and $\psi \subseteq \phi^{e'}_0$ with $Im(\psi) \in \phi^{e'}_0(Q) \setminus \{e'\}$ we have $\gamma(\psi) \leq_\gamma G$. The probability that $\phi^{e'}_0$ is activated, $A_{\phi^{e'}_0} = A$, $T_f = j$ and $\pi_f\phi^{e'}_0 = id$ for all such e' and $f \in \phi^{e'}_0$ is at least $((z(q)_r)^{-Q}|\mathcal{A}|^{-1}\omega^2)^{Q-1_e}$.

We condition on $f_j \,|_{Im(\phi)}$ such that $\dim(f_j\phi) = q$; this occurs with probability $1 - O(n^{-1})$. For each $e' \in \phi(Q) \setminus S$ there is a unique $y^{e'} \in \mathbb{F}^r_{p^a}$ such that $(My^{e'})_i = f_j\pi^{-1}_{e'}(i)$ for all $i \in Im(\pi_{e'})$. With probability $(1 + O(n^{-1}))(p^{-a})^{(q-r)(1_e-Q)}$ we have $f_j(\phi^{e'}(i)) = (My^{e'})_i$ for all such e' and

$i \in [q] \setminus Im(\pi_{e'})$. Therefore $\mathbb{P}(\cap_{e'}\{\phi^{e'} = \phi_0^{e'}\} \mid \mathcal{E}) > (1 + O(n^{-1}))[(z(q)_r|\mathcal{A}|^{-1}\omega^{-2})^Q p^{a(q-r)}]^{1_e - Q}$. Summing over all choices for $\phi_0^{e'}$ gives $\mathbb{P}(\phi \in \mathcal{Q}_j \mid \mathcal{E}) > (\omega/z)^{3Q^2}$. □

Proof of Lemma 4.15. As $(\Phi, \gamma[G]^A)$ is (ω, h)-extendable, there are at least ωn^{v_c} choices of $\phi^+ \in X_{E^\phi_{\psi^*}, H^\phi_{\psi^*}}(\Phi, \gamma[G]^A)$. We fix any such ϕ^+ and estimate $\mathbb{P}(\phi^+ \in X^c(E^\phi_{\psi^*})^+)$ by repeated application of Lemma 4.16. (The same estimates will apply to $X^c(E^\phi_{\psi^*})^-$.) We consider sequentially each $\phi' \in \Upsilon^+$, and fix $j \in [z]$ distinct from all previous choices such that (recall Q^* from Definition 4.14) if $Im(\phi') \cap Q^* = e$ then $e \in G^*_j$.

We let $\mathcal{E}$ be the intersection of all local events $\mathcal{E}^e$ where $e \in Q^*$ or $e \subseteq Im(\phi^+\phi'')$ for some previously considered $\phi'' \in \Upsilon^+$. We discard $O(n^{v_c - 1})$ choices of ϕ^+ such that any $e' \in \phi^+\phi'(Q)$ is touched by $\mathcal{E}$. Then $\mathbb{P}(\phi^+\phi' \in \mathcal{Q}_j \mid \mathcal{E}) > (\omega/z)^{3Q^2}$ by Lemma 4.16. Multiplying all conditional probabilities and summing over ϕ^+ gives $\mathbb{E}X^c(E^\phi_{\psi^*})^+ > (1 - O(n^{-1}))((\omega/z)^{3Q^2})^{s^r - 1} n^{v_c}$; concentration follows from Lemma 3.5. □

In the modified Elimination Phase, we recall that the cancelling groups have size one (zero near pairs) or two (cancelling pairs), say $C^i = \{(O^i, \phi^i)\}$ or $C^i = \{(O^i, \phi^i_+), (O^i, \phi^i_-)\}$. We adopt the notation of Definition 4.14 and fix representatives $\psi_{O^i} \in O^i$ as in Algorithm 4.10.

If $C^i = \{(O^i, \phi^i)\}$ we write[11] $\phi^i_+ = \phi^i$, $\psi_{O^i} = \phi^i_+\theta^i_+$ and $\psi^i_+ = \psi_{O^i}(\theta^i_+)^{-1} \subseteq \phi^i_+ \in A^i(\Phi)$, where $\gamma(\psi^i_+) = 0$, we choose $\psi^+_i \in X^{\phi^i_+}_{E(\psi^i_+), H(\psi^i_+)}$ and then $\phi^+_i \in X^c(E^{\phi^i_+}_{\psi^+_i})^+$.

If $C^i = \{(O^i, \phi^i_+), (O^i, \phi^i_-)\}$, we write $\psi_{O^i} = \phi^i_\pm\theta^i_\pm$ and $\psi^i_\pm = \psi_{O^i}(\theta^i_\pm)^{-1} \subseteq \phi^i_\pm \in A^i_\pm(\Phi)$, where $\gamma(\psi^i_+) = \gamma(\psi^i_-)$. We note that $\gamma(\phi^i_\pm) - \gamma(\psi^i_\pm) \leq_\gamma \partial^\gamma M^*$, let $B^i_\pm = \theta^i_\pm(B^i)$ and choose $\psi^+_i \in X^{\phi^i_+}_{E(\psi^i_+), H(\psi^i_+)}$ such that

$$Q^\pm_i = (\phi^i_\pm(Q) \cup \{Im(\psi^+_i\psi') : \psi' \in B^i_\pm(2)_r\}) \setminus Im(O^i)$$

is rainbow in G^* (this holds by definition for Q^+_i but is an extra requirement for Q^-_i). We define $\psi^-_i \in X^{\phi^i_-}_{E(\psi^i_-), H(\psi^i_-)}$ by $\psi^-_i(\theta^i_-(x), y) = \psi^+_i(\theta^i_+(x), y)$ and choose $\phi^\pm_i \in X^c(E^{\phi^i_\pm}_{\psi^\pm_i})^\pm$.

We define type 1 and 2 uses similarly to before and let $D^\alpha_i = \cup_{\psi \in F^\alpha_i} M^*(Im(\psi))$. We make the above choices uniformly at random such that $M^*(Q^\pm_i)$ both avoid $M^*(S) \cup D_{|\Psi^0|} \cup D'_{|S^J|} \cup D^1_i \cup D^2_i$.

To modify Lemma 4.12, we let $\mathcal{B}_i$ be the event that $D^1_i \cup D^2_i$ is not c_4-bounded. To see that at most half of the choices for any ψ^+_i or $\phi^\pm_i$ are forbidden, we use $X^{\phi^i_+}_{E(\psi^i_+), H(\psi^i_+)} > \omega n^{q-r}(\omega^2 z\rho/2)^Q > q^q c_4 n^{q-r}$ (this bound is similar to that in (1)) and $X^c(E^{\phi^i_\pm}_{\psi^\pm_i})^\pm > (\omega/z)^{3Q^2 s^r} n^{v_c} > (qs)^q c_4 n^{v_c}$ (by Lemma 4.15).

Then for $e \in G^*$ we have

$$r^1_e := \sum_{i<i_0} \mathbb{P}'(e \in M^*(\psi^+_i(\Omega'_i))) = \sum_{i<i_0} \sum_{e' \in M^*(e)} \mathbb{P}'(e' \in \psi^+_i(\Omega'_i)) < 2r2^{2r}\omega^{-1}(\omega^2 z\rho/2)^{-Q} c_3,$$

recalling $\Omega'_i = B^i(2)_r \setminus \{id_{B^i}\}$, and

$$r^2_e := \sum_{i<i_0} \sum_{\psi \in \Gamma_i} \mathbb{P}'(Im(\psi) \in M^*(e)) < (qs)^q(\omega/z)^{-3Q^2 s^r} c_3,$$

[11] The sign of a zero pair is irrelevant; we fix + for convenient notation.

where, writing $\Omega^i_\pm = w_r^{B^i_\pm} \setminus (B^i_\pm(2) \cup \overrightarrow{q}_r)$, we let $\Gamma_i = \phi_i^+\Omega^i_+$ if $|C^i| = 1$ or $\Gamma_i = \phi_i^+\Omega^i_+ \cup \phi^i_-\Omega^i_-$ if $|C^i| = 2$. As before, we deduce whp no $\mathcal{B}_i$ occurs, so $D^1_P \cup D^2_P$ is c_4-bounded.

To conclude, we obtain Ψ from Ψ^1 where for each $i \in [P]$ we add $\sum_x w_x^{B^i_+} A^i(\Phi[\phi_i^+ x])$ with the opposite sign to (O^i, ϕ^i) if $|C^i| = 1$, or $\sum_x w_x^{B^i_+} A^i_+(\Phi[\phi_i^+ x]) - \sum_x w_x^{B^i_-} A^i_-(\Phi[\phi_+^- x])$ if $|C^2| = 1$; this cancels all cancelling pairs and preserves $\partial^\gamma \Psi = \partial^\gamma \Psi^0$. Also, for all positive maps ϕ added in Grouping Phase and not cancelled, or added during Elimination Phase, ϕ is cascading, $M^*(\phi(Q))$ is a set, all such $M^*(\phi(Q))$ are disjoint, their union is contained in $D^1_P \cup D^2_P$, which is c_4-bounded and disjoint from $M^*(S)$. This completes the proof of Lemma 3.27. □

5 Integral decomposition

In this section we give a characterisation of the decomposition lattice $\langle\gamma(\Phi)\rangle$, which generalises the degree-type conditions for K^r_q-divisibility to the labelled setting. The characterisation is given in the second subsection, using a characterisation of the simpler auxiliary problem of octahedral decomposition, which is given in the first subsection.

5.1 Octahedral decomposition

A key ingredient in the results of Graver and Jurkat [12] and Wilson [34] (generalised in [15]) is the decomposition of null vectors by octahedra. In this subsection we establish an analogous result for adapted complexes. We start by defining null vectors. Throughout, Φ is a Σ-adapted R-complex and Γ is a finite abelian group.

Definition 5.1. (null) For $J \in \Gamma^\Phi$ and $\psi \in \Phi$ we write $\partial J_\psi = \sum J\,|_\psi = \sum\{J_\phi : \psi \subseteq \phi \in \Phi\}$.
We define $\partial_i J \in \Gamma^{\Phi_i}$ by $(\partial_i J)_\psi = \partial J_\psi$ for $\psi \in \Phi_i$. We say J is i-null if $\partial_i J = 0$.
For $J \in \Gamma^{\Phi_j}$ we write $\partial J = \partial_{j-1} J$; we say J is null if $\partial J = 0$, i.e. J is $(j-1)$-null.

Next we introduce the symmetric analogues of octahedra and their associated signed characteristic vectors (recall Definitions 2.33 and 4.7).

Definition 5.2. Given $\psi^* \in \mathcal{O}^B(\Phi)$ and $v \in \Gamma^{\Sigma^B}$ let $\chi(v, \psi^*)$ denote the 'symmetric characteristic vector' in $(\Gamma^{\Sigma^B})^{\Phi_B}$ where $\chi(v,\psi^*)_{\psi^*\psi\tau} = s(\psi)v\tau$ whenever $\psi \in O^B_B$, $\tau \in \Sigma^B_B$, and all other entries of $\chi(v,\psi^*)$ are zero. For $\Psi \in (\Gamma^{\Sigma^B})^{\mathcal{O}^B(\Phi)}$ we write $\partial\Psi = \sum_{\psi^*} \chi(\Psi_{\psi^*}, \psi^*)$.

We note the linearity $\chi(v+v', \psi^*) = \chi(v,\psi^*) + \chi(v',\psi^*)$, which follows from $(v+v')\tau = v\tau + v'\tau$.

Lemma 5.3. *If $\psi^* \in \mathcal{O}^B(\Phi)$ and $v \in \Gamma^{\Sigma^B}$ then $\chi(v,\psi^*)$ is symmetric and null.*

Proof. For $\psi \in O^B_B$ and $\tau, \tau' \in \Sigma^B_B$ we have $\chi(v,\psi^*)_{\psi^*\psi\tau'\tau} = s(\psi)v\tau'\tau = \chi(v,\psi^*)_{\psi^*\psi\tau'\tau}$, so $\chi(v,\psi^*)$ is symmetric. Also, for any $\psi^\pm \in O^B_B$ that agree on $\psi' \in O^{B'}_{B'}$ with $|B'| = |B| - 1$ we have $\partial\chi(v,\psi^*)_{\psi^*\psi'\tau} = \chi(v,\psi^*)_{\psi^*\psi^+\tau} + \chi(v,\psi^*)_{\psi^*\psi^-\tau} = s(\psi^+)v\tau + s(\psi^-)v\tau = 0$, so $\chi(v,\psi^*)$ is null. □

The following main lemma of this subsection shows that groups of symmetric null vectors are generated by symmetric characteristic vectors of octahedra when Φ is extendable.

Lemma 5.4. *Let Φ be an (ω, s)-extendable Σ-adapted R-complex and $B \subseteq R$ with $|B| = r$, where $s = 3r^2$, $n = |V(\Phi)| > n_0(r,\Gamma)$ is large and $\omega > n^{-1/2}$. Suppose H is a symmetric subgroup of Γ^{Σ^B} and $J \in H^{\Phi_B}$ is symmetric and null. Then $J = \partial\Psi$ for some $\Psi \in H^{\mathcal{O}^B(\Phi)}$.*

It is convenient to first reduce the proof of Lemma 5.4 to the case $B = R$.

Lemma 5.5. *It suffices to prove Lemma 5.4 when $B = R$.*

Proof. We reduce the general case of Lemma 5.4 to the case $B = R$ as follows. Let Φ' be the B-complex with $\Phi'_{B'} = \Phi_{B'}$ for all $B' \subseteq B$. Then Φ' is Σ^B_B-adapted and (ω', s)-extendable, with $\omega' = \omega n/n' > n^{1/2}/n' > n'^{-1/2}$, where $n' = |V(\Phi')|$.

Suppose $B \in C \in \mathcal{P}^\Sigma$. Let $X = \{x^{B'} : B' \in C\}$ where for each $B' \in C$ we fix any representative $x^{B'} \in \Sigma^B_{B'}$. Note that any $\sigma \in \Sigma^B$ has a unique expression $\sigma = \tau x$ with $\tau \in \Sigma^B_B$, $x \in X$.

We define $\pi : (\Gamma^X)^{\Sigma^B_B} \to \Gamma^{\Sigma^B}$ by $\pi(v)_{\tau x} = (v_\tau)_x$. For any set Y we define $\pi : ((\Gamma^X)^{\Sigma^B_B})^Y \to (\Gamma^{\Sigma^B})^Y$ by $\pi(w)_y = \pi(w_y)$ for all $y \in Y$. Note that for any $v \in (\Gamma^X)^{\Sigma^B_B}$ and $\tau' \in \Sigma^B_B$ we have $\pi(v\tau') = \pi(v)\tau'$; indeed, for any $\tau \in \Sigma^B_B$ and $x \in X$ we have $\pi(v\tau')_{\tau x} = ((v\tau')_\tau)_x = (v_{\tau'\tau})_x = \pi(v)_{\tau'\tau x} = (\pi(v)\tau')_{\tau x}$. We let $H' = \{h' : \pi(h') \in H\}$ and note that H' is a symmetric subgroup of $(\Gamma^X)^{\Sigma^B_B}$.

Suppose $J \in H^{\Phi_B}$ is symmetric and null. Define $J' \in (H')^{\Phi'_B}$ by $((J'_\psi)_\tau)_x = (J_\psi)_{\tau x}$. Note that $\pi(J') = J$.

We claim that J' is symmetric and null. To see this, note that for any $\tau, \tau' \in \Sigma^B_B$ and $x \in X$ we have $((J'_\psi\tau')_\tau)_x = ((J'_\psi)_{\tau'\tau})_x = (J_\psi)_{\tau'\tau x} = (J_{\psi\tau'})_{\tau x} = (J_{\psi\tau'})_{\tau x} = ((J'_{\psi\tau'})_\tau)_x$, i.e. $J'_\psi\tau' = J'_{\psi\tau'}$, i.e. J' is symmetric. Also, for any $\psi' \in \Phi'_{B'}$ with $B' \subseteq B$, $|B'| = r - 1$ and $\tau \in \Sigma^B_B$, $x \in X$ we have $((\partial J'_{\psi'})_\tau)_x = \sum\{((J'_\psi)_\tau)_x : \psi' \subseteq \psi\} = \sum\{(J_\psi)_{\tau x} : \psi' \subseteq \psi\} = (\partial J_\psi)_{\tau x} = 0$, so J' is null, as claimed.

Now by the case $B = R$ of Lemma 5.4 we have $J' = \partial\Psi'$ for some $\Psi' \in (H')^{\mathcal{O}^B(\Phi)}$. Let $\Psi = \pi(\Psi') \in H^{\mathcal{O}^B(\Phi)}$. It remains to show that $\partial\Psi = J$, i.e. for any $\psi \in \Phi_B$ that $\partial\Psi_\psi = \pi(\partial\Psi'_\psi)$. It suffices to show for any $\psi^* \in \mathcal{O}^B(\Phi)$ that $\chi(\Psi_{\psi^*}, \psi^*)_\psi = \pi\chi(\Psi'_{\psi^*}, \psi^*)_\psi$. Let $\tau_0 \in \Sigma^B_B$ be such that $\psi\tau_0^{-1} \in \psi^* O^B_B$. Then $\chi(\Psi'_{\psi^*}, \psi^*)_\psi = \Psi'_{\psi^*}\tau_0$ and $\chi(\pi(\Psi'_{\psi^*}), \psi^*)_\psi = \pi(\Psi'_{\psi^*})\tau_0 = \pi(\Psi'_{\psi^*}\tau_0)$, as required. □

We will prove Lemma 5.4 (with $B = R$) by induction on r; the proof of the following lemma uses the induction hypothesis.

Lemma 5.6. *Let Φ be an (ω, s')-extendable Σ-adapted B-complex and $B' \subseteq B$, with $|B| = r$, $|B'| = r'$, $s' = 3(r - r')^2$ and $n = |V(\Phi)| > n_0(r, \Gamma)$ large and $\omega > n^{-1/2}$. Suppose $B' \in C' \in \mathcal{P}^\Sigma$ and $\Phi' \subseteq \Phi_{C'}$ is such that (Φ, Φ') is $(\omega, 2)$-extendable and $\Phi[\Phi']$ is Σ-adapted. Suppose H is a symmetric subgroup of Γ^Σ and $J \in H^{\Phi_B}$ is null and symmetric. Then there is $\Psi \in H^{\mathcal{O}^B(\Phi)}$ with $J - \partial\Psi \in H^{\Phi[\Phi']_B}$.*

Proof. We can assume $B' \neq \emptyset$, otherwise the lemma holds trivially with $\Psi = 0$. Let $B^* = B \setminus B'$, $\Sigma' = \{\sigma \in \Sigma : \sigma\,|_{B'} = id_{B'}\}$ and $\Sigma^* = \Sigma/B' = \{\sigma\,|_{B^*} : \sigma \in \Sigma'\}$. Let $X = \{x^C : C \in \Sigma' \setminus \Sigma\}$ be a set of representatives of the right cosets of Σ' in Σ. Then any $\sigma \in \Sigma$ has a unique representation $\sigma = \sigma' x$ with $\sigma' \in \Sigma'$, $x \in X$.

We define $\pi : (\Gamma^X)^{\Sigma^*} \to \Gamma^\Sigma$ by $\pi(v)_{\sigma' x} = (v_{\sigma^*})_x$ whenever $x \in X$, $\sigma' \in \Sigma'$, $\sigma^* = \sigma'\,|_{B^*}$. Note that for any $v \in (\Gamma^X)^{\Sigma^*}$, $\tau' \in \Sigma'$, $\tau^* = \tau'\,|_{B^*}$ we have $\pi(v\tau^*) = \pi(v)\tau'$; indeed, $\pi(v\tau^*)_{\sigma' x} = ((v\tau^*)_{\sigma^*})_x = (v_{\tau^*\sigma^*})_x = \pi(v)_{\tau'\sigma' x} = (\pi(v)\tau')_{\sigma' x}$. We let $H^* = \{h^* : \pi(h^*) \in H\}$ and note that H^* is a symmetric subgroup of $(\Gamma^X)^{\Sigma^*}$.

Consider any $\psi^* \in \Phi_{B'} \setminus \Phi'$. Recall (Lemma 2.18) that $\Phi^* = \Phi/\psi^*$ is Σ^*-adapted. Define $J^* \in (H^*)^{\Phi^*_{B^*}}$ by $((J^*_{\psi/\psi^*})_{\sigma^*})_x = (J_\psi)_{\sigma' x}$ whenever $\psi^* \subseteq \psi \in \Phi_B$, $x \in X$ and $\sigma^* = \sigma'\,|_{B^*}$ with $\sigma' \in \Sigma'$. Note that $\pi(J^*_{\psi/\psi^*}) = J_\psi$.

We claim that J^* is symmetric and null. To see this, consider any $\psi^* \subseteq \psi \in \Phi_B$, $x \in X$ and $\sigma^* = \sigma' \mid_{B^*}$, $\tau^* = \tau' \mid_{B^*}$ with $\sigma', \tau' \in \Sigma'$. Then $((J^*_{\psi/\psi^*}\tau^*)_{\sigma^*})_x = ((J^*_{\psi/\psi^*})_{\tau^*\sigma^*})_x = (J_\psi)_{\tau'\sigma' x} = (J_\psi \tau')_{\sigma' x} = (J_{\psi\tau'})_{\sigma' x} = ((J^*_{\psi\tau'/\psi^*})_{\sigma^*})_x = ((J^*_{(\psi/\psi^*)\tau^*})_{\sigma^*})_x$, so J^* is symmetric. Also, for any $\psi'/\psi^* \in \Phi^*_{r-r'-1}$ we have $((\partial J^*_{\psi'/\psi^*})_{\sigma^*})_x = \sum\{((J^*_{\psi/\psi^*})_{\sigma^*})_x : \psi/\psi^* \in \Phi^*_{B^*} \mid_{\psi'/\psi^*}\} = \sum\{(J_\psi)_{\sigma' x} : \psi \in \Phi \mid_{\psi'}\} = 0$, so J^* is null, as claimed.

By the inductive hypothesis of Lemma 5.4 we have $J^* = \partial\Psi^{\psi^*}$ for some $\Psi^{\psi^*} \in (H^*)^{\mathcal{O}^{B^*}(\Phi^*)}$. For each $\phi^* \in \mathcal{O}^{B^*}(\Phi^*)$ we consider the Φ-extension $E^{\psi^*}_{\phi^*} = (O^B, F, \psi^* \cup \phi^*)$, where $F = B \cup V(O^{B^*})$. We construct $\Psi \in H^{\mathcal{O}^B(\Phi)}$ by letting ψ^* range over a set of orbit representatives for $(\Phi_{B'} \setminus \Phi')/\Sigma$, letting ϕ^* range over $\mathcal{O}^{B^*}(\Phi^*)$, and adding $\pi(\Psi^{\psi^*}_{\phi^*})\{\phi\}$ to Ψ for some $\phi \in X_{E^{\psi^*}_{\phi^*}}(\Phi, \Phi')$.

To complete the proof, it suffices to show that $(\partial\Psi)_\psi = J_\psi$ for any $\psi \in \Phi_B$ with $\psi^*\Sigma \subseteq \psi\Sigma$ for some representative ψ^* used in the construction. As $\partial\Psi$ and J are both symmetric, it suffices to prove this when $\psi^* \subseteq \psi$. As $J_\psi = \pi(J^*_{\psi/\psi^*})$ and $J^*_{\psi/\psi^*} = \partial\Psi^*_{\psi/\psi^*}$ it suffices to show that $\pi(\chi(\Psi^{\psi^*}_{\phi^*}, \phi^*)_{\psi/\psi^*}) = \chi(\pi(\Psi^{\psi^*}_{\phi^*}), \phi)_\psi$ for any ϕ^* and ϕ as above. Both sides are zero unless $\psi(\tau')^{-1} \in \phi O^B_B$ for some $\tau' \in \Sigma$. Then $\tau' \in \Sigma'$ as $\psi^* \subseteq \psi$. Let $\tau^* = \tau' \mid_{B^*}$. Then $(\psi/\psi^*)(\tau^*)^{-1} \in \phi^* O^{B^*}_{B^*}$. We have $\chi(\Psi^{\psi^*}_{\phi^*}, \phi^*)_{\psi/\psi^*} = \Psi^{\psi^*}_{\phi^*}\tau^*$ and $\chi(\pi(\Psi^{\psi^*}_{\phi^*}), \phi)_\psi = \pi(\Psi^{\psi^*}_{\phi^*})\tau' = \pi(\Psi^{\psi^*}_{\phi^*}\tau^*)$, as required. □

Proof of Lemma 5.4. We can assume $B = R$ by Lemma 5.5. For the purposes of induction we prove a slightly stronger statement, in which we replace the assumption $\omega > n^{-1/2}$ by the weaker assumption $\omega > (2r)^{r^5} n^{-0.6}$. Fix any Φ-embedding ψ_0 of O^B. Let $V_0 = Im(\psi_0)$ and $\tau : V(\Phi)\backslash V_0 \to B$ be uniformly random. For $(i, x_i) \in V(O^B)$ let $\tau(\psi_0(i, x_i)) = i$. Let Φ^τ be the set of $\phi \in \Phi$ with $\tau\phi = id$. Consider the $\Phi[\Phi^\tau]$-extension $E_0 = (B(3), V(O^B), \psi_0)$. For $j \in [r]$ we let

$$L^j = \bigcup\{\psi^* e\Sigma : \psi^* \in X_{E_0}(\Phi[\Phi^\tau]), e \in B(3)_{B'}, B' \in \tbinom{B}{j}\}.$$

The main part of the proof lies in showing that we can reduce the support of J to $\Phi[L^r]_B$.

Before doing so, we start by making the support disjoint from V_0. We identify B with $[r]$ and for each $j \in B$ we let $L'_j = \bigcup\{\psi\Sigma : \psi \in \Phi_j, \psi(j) \notin V_0\}$. We define Φ'^j for $0 \le j \le r$ by $\Phi'^0 = \Phi$ and $\Phi'^j = \Phi'^{j-1}[L'_j]$ for $j \in [r]$. Then $\Phi' := \Phi'^r = \Phi[V(\Phi) \setminus V_0]$.

We claim that each (Φ'^{j-1}, L'_j) is $(\omega - O(n^{-1}), s)$-extendable. To see this, consider any (Φ'^{j-1}, L'_j)-extension (E, H') where $E = (H, F, \phi)$ is a Φ'^{j-1}-extension and $H' \subseteq H \setminus H[F]$. Note that if $\phi^* \in X_E(\Phi'^{j-1})$ with $Im(\phi^*) \cap V_0 = Im(\phi) \cap V_0$ then $\phi^* \in X_E(\Phi'^{j-1}, L'_j)$. Thus $X_E(\Phi'^{j-1}, L'_j) > X_E(\Phi'^{j-1}) - O(n^{v_E - 1})$. As Φ is (ω, s)-extendable, the claim follows. Now by Lemma 5.6 applied to each (Φ'^{j-1}, Φ'^j) successively, there is $\Psi' \in H^{\mathcal{O}^B(\Phi)}$ with $J' = J - \partial\Psi' \in H^{\Phi'_B}$.

Next we will reduce the support of J' to $\Phi[L^r]_B$. We define $\Phi^0 = \Phi'$ and $\Phi^j = \Phi^{j-1}[L^j]$ for $j \in [r]$, and show that whp each (Φ^{j-1}, L^j) is $((2r)^{-jrs}\omega, s - 3j)$-extendable. We show by induction on $j \in [r]$ that any Φ^{j-1}-extension $E = (H', F, \phi)$ of rank $s - 3j$ is $(2r)^{-jrs}\omega$-dense in (Φ^{j-1}, L^j). Note that $\Phi^0 = \Phi'$ is $(\omega - O(n^{-1}), s)$-extendable, and the induction statement for j implies that Φ^j is $((2r)^{-jrs}\omega, s - 3j)$-extendable. Thus for the induction step we can assume that Φ^{j-1} is $((2r)^{-(j-1)rs}\omega - O(n^{-1}), s - 3j + 3)$-extendable.

We can assume that $H' \subseteq B([s-3j+3]\setminus[3])$. For each $e \in H'_{B'}$ with $B' \in \binom{B}{j}$ and $e \setminus F \neq \emptyset$ we fix a $B(s-3j+3)$-embedding ψ^e of $B(3)$ such that ψ^e is the identity on O^B, $e = \psi^e e'$ where $e' \in B(3)_{B'}$ with $e'(x) = (x, 3)$ for all $x \in B'$, and ψ^e is otherwise disjoint from H', i.e. $\psi^e(V(B(3))) \cap V(H') = Im(e)$.

Let $E^+ = (H^+, F^+, \phi^+)$, where $H^+ = H' \cup \bigcup\{\psi^e e' : e \in H'_{B'}, B' \in \binom{B}{j}, e' \in B(3)\}$, $F^+ = F \cup V(O^B)$ and ϕ^+ restricts to ϕ on F and ψ_0 on $V(O^B)$. We claim that ϕ^+ is a Φ^{j-1}-embedding

of $H^+[F^+]$. To see this, consider any $f \in H^+[F^+]$ with $|Im(f)| = i < j$, and write $f = f^1 \cup f^2$, where $Im(f^1) \subseteq F$ and $f^2 \in O^B$. As ϕ is a Φ^{j-1}-embedding of $H'[F]$, we have $\phi f^1 = \psi^1 e^1 \sigma$ for some $\psi^1 \in X_{E_0}(\Phi[\Phi^\tau])$, $\sigma \in \Sigma$, $e^1 \in B(3)$. Then $\phi^+ f = \psi^1(e^1 \cup f^2\sigma^{-1})\sigma$, so $\phi^+ f \in L^i$, which proves the claim. Thus E^+ is a Φ^{j-1}-extension.

Let X be the number of $\phi^* \in X_{E^+}(\Phi^{j-1})$ such that $\tau(\phi^*(i,x_i)) = i$ for all $(i,x_i) \in V(H^+) \setminus F^+$. Then $X_E(\Phi^{j-1}, L^j) \geq Xn^{v_E - v_{E^+}}$ by construction of E^+. As Φ^{j-1} is $((2r)^{-(j-1)rs}\omega - O(n^{-1}), s - 3j + 3)$-extendable, we have $X_{E^+}(\Phi^{j-1}) \geq (2r)^{-(j-1)rs}\omega n^{v_{E^+}} - O(n^{v_E - 1})$, so $\mathbb{E}X_E(\Phi^{j-1}, L^j) \geq r^{-v_{E^+}}(2r)^{-(j-1)rs}\omega n^{v_E} - O(n^{v_E-1})$. Changing any value of τ affects $X_E(\Phi^{j-1}, L^j)$ by $O(n^{v_E-1})$, so by Lemma 3.5 whp $X_E(\Phi^{j-1}, L^j) \geq (2r)^{-jrs}\omega n^{v_E}$. This completes the induction step, so each (Φ^{j-1}, L^j) is $((2r)^{-jrs}\omega, s - 3j)$-extendable.

Now by Lemma 5.6 repeatedly applied to each (Φ^{j-1}, Φ^j) successively, always with $s' \geq s - 3r \geq 3(r-1)^2$ and extendability parameter $\omega' > (2r)^{-3r^4}(2r)^{r^5}n^{-0.6} > (2(r-1))^{(r-1)^5}n^{-0.6}$, there is $\Psi^0 \in H^{\mathcal{O}^B(\Phi)}$ with $J^0 = J' - \partial\Psi^0 \in H^{\Phi[L^r]_B}$. We will define null $J^1, \dots, J^r \in H^{\Phi[L^r]_B}$ such that $J^j_\psi = 0$ whenever $|Im(\psi) \cap V_0| < j$, via the following construction of 'reducing octahedra'.

Let L^* be the set of $e \in L^r$ with $Im(e) \setminus V_0 \neq \emptyset$ such that $e = \psi^e_0 f^e_0$ for some $\psi^e_0 \in X_{E_0}(\Phi[\Phi^\tau])$ and $f^e_0 \in B(3)_B$ (i.e. we can take $\sigma = id$ in the definition of L^r). For each $e \in L^*$ we fix some Φ-embedding ψ^e of O^B of the form $\psi^e = \psi^e_0 \mid_{F^e} \pi^e$, where $f^e_1 \in O^B$ and $\pi^e O^B$ is the copy of O^B in $B(3)$ spanned by $F^e := Im(f^e_0) \cup Im(f^e_1)$, identified so that f^e_0 has sign 1. Note that for any $e' \in \psi^e O^B_B$ with $e' \neq e$ we have $|Im(e') \cap V_0| > |Im(e) \cap V_0|$ and $e' \in L^r$. We define $J^1 = J^0 - \partial\Psi^1$, where $\Psi^1 = \sum_{e \in L^*} J^0_e\{\psi^e\}$. As J^0 and each $\chi(J^0_e, \psi^e)$ are symmetric, we have $J^1_\psi = 0$ whenever $Im(\psi) \cap V_0 = \emptyset$.

Given J^j with $0 < j < r$, we define J^{j+1} as follows. Consider any $f \in \Phi^\circ_{r-j}$ disjoint from V_0, write $B' = \tau(f) \subseteq B$, and suppose $f = Im(\psi^*)$, where $\psi^* \in \Phi_{B'}$ with $\tau\psi^* = id$. For any $\psi \in \Phi_B$ with $J^j_\psi \neq 0$ and $\psi^*\Sigma \subseteq \psi\Sigma$, by definition of L^r we can pick a representative ψ of $\psi\Sigma$ with $\psi^* \subseteq \psi$ and $\tau\psi = id$. Furthermore, for any $x \in B \setminus B'$ and $\psi' \in \Phi_{B-x}$ with $\psi^* \subseteq \psi'$ and $\tau\psi' = id$, if there is $\psi' \subseteq \psi$ with $J^j_\psi \neq 0$ then there are exactly two such ψ, say $\psi^\pm$, obtained from each other by interchanging $\psi_0((x,0))$ and $\psi_0((x,1))$, where as J^j is null we have $J^j_{\psi^-} = -J^j_{\psi^+}$. Thus there is $a_f \in H$ such that $J^j_\psi = \pm a_f$ whenever $\psi^* \subseteq \psi$ and $\tau\psi = id$, where the sign is that of $\psi_0^{-1}\psi \mid_{B \setminus B'}$ in $O^{B \setminus B'}$. Fix e with $\psi^* \subseteq e$, $\tau e = id$, $J^j_e = a_f$. By symmetry, we have $J^j_\psi = \chi(a_f, \psi^e)_\psi$ whenever $J^j_\psi \neq 0$ with $\psi^*\Sigma \subseteq \psi\Sigma$. We add $a_f\{\psi^e\}$ to Ψ^{j+1} for each such f, e and let $J^{j+1} = J^j - \partial\Psi^{j+1}$.

We conclude with J^r such that J^r_e is zero unless $e \in \psi_0 O^B_B \Sigma$. As J^r is symmetric and null, we have $J^r = \chi(a, \psi_0)$ for some $a \in H$. Then $\Psi := \Psi' + a\{\psi_0\} + \sum_{j=0}^r \Psi^j$ has $\partial\Psi = J$. □

Next we give two quantitative versions of Lemma 5.4. These will be used in the next subsection to prove two quantitative versions of the main lemma of this section, which will in turn both be used in the proof of Lemma 3.18 in the next section. We make the following definitions.

Definition 5.7. (G-use) Suppose H is a symmetric subgroup of Γ^Σ and G is a symmetric generating set of H. For $v \in \Gamma$ we write $|v|_G$ for the minimum possible $\sum_{g \in G} |c_g|$ where $v = \sum_{g \in G} c_g g$ with all $c_g \in \mathbb{Z}$. For $\Psi \in H^{\mathcal{O}^B(\Phi)}$ we write $|\Psi|_G = \sum |\Psi_\phi|_G$. If $J \in H^{\Phi_B}$ is symmetric we write $|J|_G = \sum'_\psi |J_\psi|_G$, where the sum is over any choice of orbit representatives for Φ_B/Σ.

The following lemma quantifies the total 'G-use' of the octahedral decomposition Ψ in terms of that of J. We define $C(i) = 2^{(9i+2)^{i+5}}$.

Lemma 5.8. *Let Φ be an (ω, s)-extendable Σ-adapted B-complex with $|B| = r$, $s = 3r^2$, $n = |V(\Phi)| > n_0(r, \Gamma)$ large and $\omega > n^{-1/2}$. Suppose H is a symmetric subgroup of Γ^Σ and G is a*

symmetric generating set of H. Suppose $J \in H^{\Phi_B}$ is symmetric and null. Then there is $\Psi \in H^{\mathcal{O}^B(\Phi)}$ with $\partial\Psi = J$ such that $|\Psi|_G \leq C(r)|J|_G$.

Following the proof of Lemma 5.4, we quantify the total G-use in Lemma 5.6.

Lemma 5.9. *In Lemma 5.6, we can choose Ψ with $|\Psi|_G \leq C(r-r')|J|_G$ and $|J - \partial\Psi|_G \leq 2^r C(r-r')|J|_G$.*

The proof of Lemma 5.9 is the same as that of Lemma 5.6, noting also that when we apply the inductive hypothesis each $|\Psi^{\psi^*}|_G \leq C(r-r')|J^*|_G$, so $|\Psi|_G \leq C(r-r')|J|_G$, and this also gives $|J - \partial\Psi|_G \leq 2^r C(r-r')|J|_G$.

Proof of Lemma 5.8. We will estimate the total G-use during the proof of Lemma 5.4. We write $\Psi' = \sum_{j=1}^r \Psi'^j$, $J'^0 = J$ and $J'^j = J'^{j-1} - \partial\Psi'^j$ for $j > 0$, so $J'^r = J'$. By Lemma 5.9 each $|\Psi'^j|_G \leq C(r-1)|J'^{j-1}|_G$ and $|J'^j|_G \leq 2^r C(r-1)|J'^{j-1}|_G$, so $|J'|_G \leq 2^{r^2} C(r-1)^r |J|_G$.

Similarly, we write $\Psi^0 = \sum_{j=1}^r \Psi^{0,j}$, $J^{00} = J'$ and $J^{0,j} = J^{0,j-1} - \partial\Psi^{0,j}$ for $j > 0$. Then $J^{0r} = J^0$, and each $\Phi^j = \Phi^{j-1}[L^j]$ is obtained by repeated restriction to each $L^j_{B'}$ with $B' \in \binom{B}{j}$, so writing $r_j = \binom{r}{j}$ we have $|\Psi^{0,j}|_G \leq C(r-j)^{r_j}|J^{0,j-1}|_G$, and $|J^{0,j}|_G \leq (2^r C(r-j))^{r_j}|J^{0,j-1}|_G$, so $|J^0|_G \leq 2^{r2^r}|J'|_G \prod_{i=0}^{r-1} C(i)^{r_i}$.

Next we have $|\Psi^1|_G \leq |J^0|_G$ and $|J^1|_G \leq 2^r|J^0|_G$. For $j > 0$ we have $|\Psi^{j+1}|_G \leq |J^j|_G$ and $|J^{j+1}|_G \leq 2^r|J^j|_G$, so $|\Psi^r|_G \leq 2^{r^2}|J^0|_G \leq 2^{r2^r+2r^2}|J|_G C(r-1)^{2r}\prod_{i=0}^{r-2} C(i)^{r_i}$. Recalling $C(i) = 2^{(9i+2)^{i+5}}$, we see that $\Psi := \Psi' + a\{\psi_0\} + \sum_{j=0}^r \Psi^j$ has $|\Psi|_G < C(r)|J|_G$. □

In our second quantitative version, we suppose $\Gamma = \mathbb{Z}^D$ is free, and consider rational decompositions, where we now bound G-uses on every function in Φ_r (as opposed to the total bound in the previous version).

Definition 5.10. Suppose H is a symmetric subgroup of $(\mathbb{Z}^D)^\Sigma$ and G is a symmetric generating set of H. For $v \in \mathbb{Q}H$ we write $|v|_G$ for the minimum possible $\sum_{g\in G}|c_g|$ where $v = \sum_{g\in G} c_g g$ with all $c_g \in \mathbb{Q}$. For $\Psi \in (\mathbb{Q}H)^{\mathcal{O}^B(\Phi)}$ and $\psi \in \Phi_B$ we write $U_G(\Psi)_\psi = \sum\{|\Psi_\phi|_G : \psi = \phi\psi', \psi' \in O_B^B\}$.

Lemma 5.11. *Let Φ be an (ω, s)-extendable Σ-adapted B-complex where $n = |V(\Phi)| > n_0(r,\Gamma)$ is large, $|B| = r$, $n^{-1/2} < \omega < \omega_0(r)$ and $s = 3r^2$. Suppose H is a symmetric subgroup of $(\mathbb{Z}^D)^\Sigma$ and G is a symmetric generating set of H. Suppose $J \in (\mathbb{Q}H)^{\Phi_B}$ is symmetric and null with $|J_\psi|_G \leq \theta$ for all $\psi \in \Phi_B$. Then there is $\Psi \in (\mathbb{Q}H)^{\mathcal{O}^B(\Phi)}$ with $\partial\Psi = J$ such that $U_G(\Psi)_\psi \leq C(r,\omega)\theta$ for all $\psi \in \Phi_B$, where $C(i,\omega) = 2^{C(i)}\omega^{-(9i)^{i+4}}$.*

Again we require the corresponding quantitative version of Lemma 5.6.

Lemma 5.12. *Let Φ be an (ω, s')-extendable Σ-adapted B-complex and $B' \subseteq B$, with $|B| = r$, $|B'| = r'$, $s' = 3(r')^2$, $n = |V(\Phi)| > n_0(r,\Gamma)$ large and $n^{-1/2} < \omega < \omega_0(r)$. Suppose $B' \in X \in \mathcal{P}^\Sigma$ and $\Phi' \subseteq \Phi_X$ is such that (Φ, Φ') is $(\omega, 2)$-extendable and $\Phi[\Phi']$ is Σ-adapted. Suppose H is a symmetric subgroup of $(\mathbb{Z}^D)^\Sigma$ and G is a symmetric generating set of H. Suppose $J \in (\mathbb{Q}H)^{\Phi_B}$ is symmetric and null with all $|J_\psi|_G \leq \theta$. Then there is $\Psi \in (\mathbb{Q}H)^{\mathcal{O}^B(\Phi)}$ with $J - \partial\Psi \in (\mathbb{Q}H)^{\Phi[\Phi']_B}$ such that for all $\psi \in \Phi_B$ both $U_G(\Psi)_\psi$ and $|(J-\partial\Psi)_\psi|_G$ are at most $2^r\omega^{-1}C'\theta$, where $C' = C(r-r',\omega)$.*

Proof. We follow the proof of Lemma 5.6. For each $\psi^* \in \Phi_{B'} \setminus \Phi'$, we define Φ^*, J^*, Ψ^{ψ^*} as before, where by the inductive hypothesis of Lemma 5.11 there is $\Psi^{\psi^*} \in (\mathbb{Q}H^*)^{\mathcal{O}^{B^*}(\Phi^*)}$ with $\partial\Psi^{\psi^*} = J^*$

and all $U_{G^*}(\Psi^{\psi^*})_{\psi/\psi^*} \leq C'\theta$, where $G^* = \{h^* \in H^* : \pi(h^*) \in G\}$. For each orbit representative ψ^* and $\phi^* \in \mathcal{O}^{B^*}(\Phi^*)$ we add $\pi(\Psi^{\psi^*}_{\phi^*})\mathbb{E}\{\phi\}$ to Ψ, where ϕ is uniformly random in $X_{E^{\psi^*}_{\phi^*}}(\Phi, \Phi')$. Then $J - \partial\Psi \in (\mathbb{Q}H)^{\Phi[\Phi']_B}$ as in the proof of Lemma 5.6.

For any $\psi \in \Phi_B$ we have $U_G(\Psi)_\psi \leq \sum_{\psi^* \in \Phi_{B'} \setminus \Phi'} \sum_{\phi^* \in \mathcal{O}^{B^*}(\Phi^*)} |\Psi^{\psi^*}_{\phi^*}|_{G^*} \sum_{\psi^B \in O^B_B} \mathbb{P}(\psi = \phi\psi^B)$, where ϕ is as above. Note that any given ψ^B can only contribute to $U_G(\Psi)_\psi$ if $\psi(x) = \phi^*\psi^B(x)$ for all $x \in B^*$ and $\psi(x) = \psi^*(x)$ for all $x \in B' \setminus D$, where $D = D(\psi^B) = \{x \in B' : \psi^B(x) = (x, 2)\}$.

For each $\psi' \in S_D := \{\psi' \in \Phi_B : \psi' \mid_{B\setminus D} = \psi \mid_{B\setminus D}\}$ and $\psi^* = \psi' \mid_{B'}$ we have $\mathbb{P}(\psi = \phi\psi^B) < \omega^{-1}n^{-|D|}$, as (Φ, Φ') is $(\omega, 2)$-extendable. We have $|S_D| < n^{|D|}$ and all $|\Psi^{\psi^*}_{\phi^*}|_{G^*} \leq C'\theta$, so summing over ψ^B and $\psi' \in S_{D(\psi^B)}$ gives $U_G(\Psi)_\psi \leq 2^r\omega^{-1}C'\theta$. The same bound applies to $|(J - \partial\Psi)_\psi|_G$. □

Proof of Lemma 5.11. We follow the proof of Lemma 5.4, estimating uses of any fixed $\psi \in \Phi_B$, and then average over all choices of the initial Φ-embedding ψ_0 of O^B. We use the same notation as in the proofs of Lemmas 5.4 and 5.8.

Letting $\theta'_0 = \theta$, and θ'_j for $j > 0$ be such that all $|J'^j_\psi|_G \leq \theta'_j$, by Lemma 5.12, for each $j > 0$ both $U_G(\Psi'^j)_\psi$ and $|J'^j_\psi|_G$ are at most $2^{r-1}(\omega - O(n^{-1}))^{-1}C(r-1, \omega - O(n^{-1}))\theta'_{j-1}$, so we can define $\theta'_j = 2^r\omega^{-1}C(r-1, \omega)\theta'_{j-1}$. Then both $U_G(\Psi')_\psi$ and $|J'_\psi|_G$ are at most $2\theta'_r$.

Similarly, letting $\theta_0 = 2\theta'_r$ and θ_j for $j > 0$ be such that all $|J^{0,j}_\psi|_G \leq \theta_j$, by Lemma 5.12, for each $j > 0$ both $U_G(\Psi^{0,j})_\psi$ and $|J^{0,j}_\psi|_G$ are at most $2\theta_{j-1}[2^{r-j}((2r)^{-jrs}\omega)^{-1}C(r-j, (2r)^{-jrs}\omega)]^{r_j}$, so we can define $\theta_j = \theta_{j-1}[2^r((2r)^{-jrs}\omega)^{-1}C(r-j, (2r)^{-jrs}\omega)]^{r_j}$. Then $U_G(\Psi^0)_\psi$ and $|J^0_\psi|_G$ are at most $\theta^* := 2\theta'_r(2^r\omega^{-1}(2r)^{sr^2})^{2^r}\prod_{i=0}^{r-1} C(i, (2r)^{-sr^2}\omega)^{r_i}$.

Now write $B' = \{x : \psi(x) \notin V_0\}$ and $\psi' = \psi \mid_{B'}$. For $0 \leq j < r - |B'|$ we have $U_G(\Psi^{j+1})_{\psi'} \leq 2^r|(J^j \mid_{\psi'})|_G$, so $\sum_{j=1}^{r-|B'|} U_G(\Psi^j)_\psi \leq 2^{r^2}|(J^0 \mid_{\psi'})|_G \leq 2^{r^2}\theta^* n^{r-|B'|}$.

Letting $\overline{\Psi}$ be the average of Ψ (from the proof of Lemma 5.4) over all Φ-embeddings ψ_0 of O^B, as Φ is $(\omega, 2)$-extendable, $U_G(\overline{\Psi})_\psi \leq 2^{r^2+1}\theta^* \sum_{B' \subseteq B} n^{2r-(r-|B'|)}(\omega n^{2r})^{-1}n^{r-|B'|} < 2^{(r+1)^2}\omega^{-1}\theta^* < C(r, \omega)\theta$, for $\omega < \omega_0(r)$, using $s = 3r^2$, $C(i, \omega) = 2^{\overline{C}(i)}\omega^{-(9i)^{i+4}}$ and $C(i) = 2^{(9i+2)^{i+5}}$. □

5.2 Lattices

In this subsection we use the octahedral decompositions from the previous subsection to characterise the decomposition lattice $\langle\gamma(\Phi)\rangle$. For the following lemma, we recall (Lemma 2.32) that $\langle\gamma(\Phi)\rangle \subseteq \mathcal{L}^-_\gamma(\Phi)$. We will show that if Φ is extendable then this inclusion becomes an equality when we restrict to null vectors.

Lemma 5.13. *Let $\Sigma \leq S_q$, $\mathcal{A}$ be a $\Sigma^{\leq}$-family and $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be a Σ-adapted (ω, s)-extendable $[q]$-complex with $s = 3r^2$, $n = |V(\Phi)| > n_0(q, D)$ large and $\omega > n^{-1/2}$. Suppose $B \in C \in \mathcal{P}^\Sigma_r$ and $J \in \mathcal{L}^-_\gamma(\Phi) \cap (\mathbb{Z}^D)^{\Phi_C}$ is null. Then $J \in \langle\gamma(\Phi)\rangle$.*

Proof. Recall (Lemma 2.34) that $f_B(J) \in (\gamma^B)^{\Phi_B}$ is symmetric, and (Lemma 2.30) that γ^B is symmetric. We also note that $f_B(J)$ is null, as for any $\psi' \in \Phi_{B'}$ with $B' \subseteq B$, $|B'| = r - 1$ and $\sigma \in \Sigma^B$ we have $(\partial f_B(J)_{\psi'})_\sigma = \sum\{(f_B(J)_\psi)_\sigma : \psi' \subseteq \psi\} = \sum\{J_{\psi\sigma} : \psi' \subseteq \psi\} = \partial J_{\psi'\sigma} = 0$. By Lemma 5.4 we have $\Psi \in (\gamma^B)^{\mathcal{O}^B(\Phi)}$ with $f_B(J) = \partial\Psi = \sum_{\psi^*} \chi(\Psi_{\psi^*}, \psi^*)$. By definition of γ^B we can fix integers $m^\theta_{\psi^*}$ so that each $\Psi_{\psi^*} = \sum_{\theta \in \mathcal{A}_B} m^\theta_{\psi^*}\gamma^\theta$.

We will show for any $\psi^* \in \mathcal{O}^B(\Phi)$ and $\theta \in A_B$ that there is $\Psi^{\psi^*\theta} \in \mathbb{Z}^{A(\Phi)}$ with $f_B(\partial^\gamma\Psi^{\psi^*\theta}) = \chi(\gamma^\theta, \psi^*)$. This will suffice to prove the lemma. Indeed, letting $\Psi^* = \sum_{\psi^*,\theta} m^\theta_{\psi^*}\Psi^{\psi^*\theta}$, we will have

$\partial^\gamma \Psi^* = \sum_{\psi^*,\theta} m^\theta_{\psi^*} \chi(\gamma^\theta, \psi^*) = \sum_{\psi^*} \chi(\Psi_{\psi^*}, \psi^*) = \partial\Psi = J$.

Consider $\psi^* \in \mathcal{O}^B(\Phi)$, $A \in \mathcal{A}$, $\theta \in A_B$, let $F = \theta(B) \times [2]$ and define $\psi_0 : F \to V(\Phi)$ by $\psi_0(\theta(i), x) = \psi^*(i, x)$ for $i \in B$, $x \in [2]$. We claim that $\psi_0 \in \mathcal{O}^{\theta(B)}(\Phi)$, and so $E = ([q](2), F, \psi_0)$ is a Φ-extension. To see this, note that for any $\upsilon \in O^{\theta(B)}_{\theta(B)}$, defining $\upsilon' \in O^B_B$ such that $\upsilon'(i) = (i, x)$ when $\upsilon(\theta(i)) = (\theta(i), x)$, as $\psi^* \in \mathcal{O}^B(\Phi)$ we have $\psi_0\upsilon\theta = \psi^*\upsilon' \in \Phi_B$. As Φ is Σ-adapted, we deduce $\psi_0\upsilon \in \Phi_{\theta(B)}$, which proves the claim.

As Φ is (ω, s)-extendable, we can choose $\psi^+ \in X_E(\Phi)$. We let $\Psi^{\psi^*\theta}$ be the sum of all $\pm\{\psi^+ \circ \phi\}$ over all $A(2)$-embeddings ϕ of A such that $\phi(i) = (i, 1)$ when $i \notin \theta(B)$, where the sign is that of $\phi \restriction_{\theta(B)}$ in $O^{\theta(B)}$, i.e. that of $\psi' \in O^B$ defined by $\psi' = (\psi^*)^{-1}\psi$, where $\psi = \psi_0\phi\theta$ with ψ_0 as above. Then all entries in $\partial^\gamma \Psi^{\psi^*\theta}$ cancel in $\pm$ pairs, except that for each $\pm\{\psi^+ \circ \phi\}$ in $\Psi^{\psi^*\theta}$ there is a non-cancelling term $\gamma(\psi^+\phi)^{\psi\Sigma} = \gamma[\psi]^\theta$ (by Lemma 2.24.i) as $\psi = \psi_0\phi\theta = \psi^+\phi\theta$. Now $f_B(\gamma[\psi]^\theta)_\psi = \pm\gamma^\theta = \chi(\gamma^\theta, \psi^*)_\psi$, where the sign is that of $\psi' = (\psi^*)^{-1}\psi \in O^B$, so by symmetry $f_B(\partial^\gamma \Psi^{\psi^*\theta}) = \chi(\gamma^\theta, \psi^*)$, as required to prove the lemma. □

To motivate the characterisation of decomposition lattices in general, it may be helpful to consider the decomposition lattice of triangles in a complete tripartite graph (a very special case of Theorem 1.7). Say T is a complete tripartite graph with parts (A, B, C). It is not hard to show that $J \in \mathbb{Z}^T$ is in the decomposition lattice iff J is 'balanced', in that each $a \in A$ has $\sum_{b\in B} J_{ab} = \sum_{c\in C} J_{ac}$ and similarly for each $b \in B$ and $c \in C$. At first sight this seems a rather different condition to the tridivisibility condition that arises for nonpartite triangle decomposition (even degrees and $3 \mid \sum J$). However, we can unify the conditions by lifting J to a vector $J^+ \in (\mathbb{Z}^3)^T$ in which we assign different basis vectors to the three bipartite subgraphs, say $(1, 0, 0)$ to (B, C), $(0, 1, 0)$ to (A, C) and $(0, 0, 1)$ to (A, B). We want to characterise when J^+ is in the lattice generated by all vectors $v(abc) \in (\mathbb{Z}^3)^T$, where for $a \in A$, $b \in B$, $c \in C$ we let $v(abc)_{bc} = (1, 0, 0)$, $v(abc)_{ac} = (0, 1, 0)$, $v(abc)_{ab} = (0, 0, 1)$ and $v(abc)_{xy} = 0$ otherwise. The lifted vertex degree condition is that for any $a \in A$ we have $\sum_{x\in V(T)} J^+_{ax}$ in the lattice generated by $(0, 1, 0)$ and $(0, 0, 1)$, and similarly for any $b \in B$ and $c \in C$; this is equivalent to J being balanced. This example suggests the form of the degree conditions in the following definition.

Definition 5.14. Let Φ be a $[q]$-complex, $\mathcal{A}$ be a $[q]$-complex family and $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. For $J \in (\mathbb{Z}^D)^{\Phi_r}$ we define $J^\sharp \in ((\mathbb{Z}^D)^Q)^\Phi$ by $(J^\sharp_{\psi'})_B = \sum\{J_\psi : \psi' \subseteq \psi \in \Phi_B\}$ for $B \in Q$, $\psi' \in \Phi$. Similarly, we define $\gamma^\sharp \in ((\mathbb{Z}^D)^Q)^{\cup\mathcal{A}}$ by $(\gamma^\sharp_{\theta'})_B = \sum\{\gamma_\theta : \theta' \subseteq \theta \in A_B\}$ for $B \in Q$, $\theta' \in A \in \mathcal{A}$. We let $\mathcal{L}_\gamma(\Phi)$ be the set of all $J \in (\mathbb{Z}^D)^{\Phi_r}$ such that $(J^\sharp)^O \in \langle\gamma^\sharp[O]\rangle$ for any $O \in \Phi/\Sigma$.

Note that $\gamma^\sharp = (\gamma^\sharp_i : 0 \le i \le r)$ where each $\gamma^\sharp_i$ is a vector system for $\mathcal{A}_i$. For convenient use in Lemma 5.19 we will reformulate Definition 5.14 in terms of iterated independent shadows, in the sense of the following definition.

Definition 5.15. (independent shadows)

For each $B \in [q]_i$ and any set S we define $\pi_B : \mathbb{Z}^S \to \mathbb{Z}^{[q]_i \times S}$ by $\pi_B(v) = e_B \otimes v$, i.e. $\pi_B(v)$ is v in the block of S coordinates belonging to B and 0 otherwise.

Let Φ be a $[q]$-complex. We define $\pi_i : (\mathbb{Z}^S)^{\Phi_i} \to (\mathbb{Z}^{[q]_i \times S})^{\Phi_i}$ by $\pi_i(J)_\psi = \pi_B(J_\psi)$ for $\psi \in \Phi_B$, $B \in [q]_i$. We also write $J^+ = \pi_i(J)$ for $J \in (\mathbb{Z}^S)^{\Phi_i}$.

For $i = r, \dots, 0$ we define D_i by $D_r = [q]_r \times [D]$ and $D_i = [q]_i \times D_{i+1}$ for $0 \le i < r$. For $J \in (\mathbb{Z}^{D_{i+1}})^{\Phi_{i+1}}$ we define $\partial^* J \in (\mathbb{Z}^{D_i})^{\Phi_i}$ by $\partial^* J = \pi_i(\partial_i J) = (\partial J)^+$.

For $J \in (\mathbb{Z}^D)^{\Phi_r}$ and $0 \le i \le r$ we define $\partial^*_i J = (\partial^*)^{r-i}(J^+)$.

Similarly, given a $[q]$-complex family $\mathcal{A}$ and $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$ we define $\partial_i^*\gamma = (\partial^*)^{r-i}(\pi_r(\gamma))$, where for $\gamma' \in (\mathbb{Z}^{D_{i+1}})^{\mathcal{A}_{i+1}}$ and $\theta' \in A \in \mathcal{A}$ we define $(\partial\gamma')_{\theta'} = \sum\{\gamma'_\theta : \theta \in A \mid_{\theta'}\}$ and $(\partial^*\gamma')_{\theta'} = \pi_i((\partial\gamma')_{\theta'})$.

We let $\mathcal{L}^i_\gamma(\Phi)$ be the set of all $J \in (\mathbb{Z}^D)^{\Phi_r}$ such that $\partial_i^* J \in \mathcal{L}^-_{\partial_i^*\gamma}(\Phi)$, i.e. $(\partial_i^* J)^O \in \langle(\partial_i^*\gamma)[O]\rangle$ for any $O \in \Phi_i/\Sigma$.

Remark 5.16. Unravelling the iterative definitions in Definition 5.15, we see that each $D_i = [D] \times \prod_{j=i}^r [q]_j$, and for each $B^i \in [q]_i$, $\psi^i \in \Phi_{B^i}$, we have $(\partial_i^* J)_{\psi^i}$ supported on 'full chains from B^i', i.e. if $((\partial_i^* J)_{\psi^i})_C \neq 0$ then $C = (B^i, \dots, B^r)$ with each $B^j \in [q]_j$ and $B^j \subseteq B^{j+1}$ for $i \leq j < r$.

We obtain $((\partial_i^* J)_{\psi^i})_C \in \mathbb{Z}^D$ by summing J_{ψ^r} over all choices of $(\psi^i, \dots, \psi^r)$ with each $\psi^j \in \Phi_{B^j}$ and $\psi^j \subseteq \psi^{j+1}$ for $i \leq j < r$.

Similarly, for $A \in \mathcal{A}$, $\theta^i \in A_{B^i}$ we obtain $((\partial_i^*\gamma)_{\theta^i})_C \in \mathbb{Z}^D$ by summing γ_{θ^r} over all choices of $(\theta^i, \dots, \theta^r)$ with each $\theta^j \in A_{B^j}$ and $\theta^j \subseteq \theta^{j+1}$ for $i \leq j < r$.

Now we reformulate Definition 5.14 using Definition 5.15.

Lemma 5.17. $\mathcal{L}_\gamma(\Phi) = \cap_{i=0}^r \mathcal{L}^i_\gamma(\Phi)$.

Proof. Fix $\psi^O \in O \in \Phi_i/\Sigma$, say with $\psi^O \in \Phi_{B'}$. We need to show $(J^\sharp)^O \in \langle\gamma^\sharp[O]\rangle$ iff $(\partial_i^* J)^O \in \langle(\partial_i^*\gamma)[O]\rangle$.

We have $(J^\sharp)^O \in \langle\gamma^\sharp[O]\rangle$ iff there is $n \in \mathbb{Z}^{\mathcal{A}_{B'}}$ with $(J^\sharp)^O = \sum_\sigma\{n_\sigma\gamma^\sharp(\psi^O\sigma^{-1})\}$, i.e. all $J^\sharp_{\psi^O\sigma'} = \sum_\sigma\{n_\sigma\gamma^\sharp_{\sigma\sigma'}\}$, i.e. for each $A \in \mathcal{A}$, $\sigma' \in A[B']$, $B \in Q$ we have $\sum\{J_\psi : \psi^O\sigma' \subseteq \psi \in \Phi_B\} = \sum_{\sigma,\theta}\{n_\sigma\gamma_\theta : \sigma\sigma' \subseteq \theta \in A_B\}$.

We have $(\partial_i^* J)^O \in \langle(\partial_i^*\gamma)[O]\rangle$ iff there is $n' \in \mathbb{Z}^{\mathcal{A}_{B'}}$ such that for each $A \in \mathcal{A}$, $\sigma' \in A[B']$ and chain $C = (B^i, \dots, B^r)$ from $B^i = B'$ to $B^r = B \in Q$, we have $\sum_{\psi^i,\dots,\psi^r} J_{\psi^r} = \sum_{\sigma,\theta^i,\dots,\theta^r} n'_\sigma\gamma_{\theta^r}$, where we sum over $(\theta^i, \dots, \theta^r)$ and $(\psi^i, \dots, \psi^r)$ with $\theta^i = \sigma\sigma'$, $\psi^i = \psi^O\sigma'$, each $\theta^j \in A_{B^j}$, $\psi^j \in \Phi_{B^j}$, and each $\theta^j \subseteq \theta^{j+1}$, $\psi^j \subseteq \psi^{j+1}$.

Setting $n' = n$ we see that the two conditions are identical, as any such chain C and σ', σ uniquely specifies all $\psi^j = \psi^r \mid_{B^j}$ and $\theta^j = \theta^r \mid_{B^j}$. □

Our next lemma shows that the two definitions of ∂_i^* (for vectors and for vector systems) are compatible with each other.

Lemma 5.18. *If $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ then $\partial^{\partial_i^*\gamma}\Psi = \partial_i^*(\partial^\gamma\Psi)$.*

Proof. By linearity we can assume $\Psi = \{\phi\}$ for some $\phi \in A(\Phi)$. We prove the identity by induction on $i = r, \dots, 0$. In the base case $i = r$ we have $\partial^{\partial_r^*\gamma}\Psi = (\partial_r^*\gamma)(\phi) = (\pi_r\gamma)(\phi) = \pi_r(\gamma(\phi)) = \partial_r^*(\partial^\gamma\Psi)$, where in the third equality we used $(\pi_r\gamma)(\phi)_{\phi\theta} = (\pi_r\gamma)_\theta = \pi_r(\gamma_\theta) = \pi_r(\gamma(\phi)_{\phi\theta})$. For the induction step with $i < r$ we have $\partial^{\partial_i^*\gamma}\Psi = (\partial_i^*\gamma)(\phi) = (\partial^*\partial_{i+1}^*\gamma)(\phi) = \partial^*(\partial_{i+1}^*\gamma)(\phi) = \partial^*\partial^{\partial_{i+1}^*\gamma}\Psi = \partial^*\partial_{i+1}^*(\partial^\gamma\Psi) = \partial_i^*(\partial^\gamma\Psi)$, where in the third equality, writing $\gamma' = \partial_{i+1}^*\gamma$, for $\theta' \in A \in \mathcal{A}$ we used $(\partial^*\gamma')(\phi)_{\phi\theta'} = (\partial^*\gamma')_\theta = \pi_i\sum\{\gamma'_\theta : \theta \in A \mid_{\theta'}\} = \pi_i\sum\{\gamma'_\theta : \phi\theta \in \Phi \mid_{\phi\theta'}\} = \partial^*(\gamma'(\phi)_{\phi\theta'})$. □

Now we come to the main lemma of this section, which characterises the decomposition lattice.

Lemma 5.19. *Let $\Sigma \leq S_q$, $\mathcal{A}$ be a $\Sigma^{\leq}$-family and $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be a Σ-adapted (ω, s)-extendable $[q]$-complex with $s = 3r^2$, $n = |V(\Phi)| > n_0(q, D)$ large and $\omega > n^{-1/2}$. Then $\langle\gamma(\Phi)\rangle = \mathcal{L}_\gamma(\Phi)$.*

Proof. Note for any molecule $\gamma(\phi) \in \gamma(\Phi)$ that $(\partial_i^*\gamma)(\phi) \in (\partial_i^*\gamma)(\Phi) \subseteq \mathcal{L}^-_{\partial_i^*\gamma}(\Phi)$, so $\langle\gamma(\Phi)\rangle \subseteq \mathcal{L}^i_\gamma(\Phi)$. We now show that the reverse inclusions hold. Suppose $J \in \mathcal{L}_\gamma(\Phi)$. We will define $\Psi^0, \dots, \Psi^r \in \mathbb{Z}^{\mathcal{A}(\Phi)}$

so that, letting $J^0 = J - \partial^\gamma \Psi^0$ and $J^i = J^{i-1} - \partial^\gamma \Psi^i$ for $i \in [r]$, each $\partial_i^* J^i = 0$. We will then have $J^r = 0$, so $J = \sum_{i=0}^r \partial^\gamma \Psi^i \in \langle \gamma(\Phi) \rangle$, as required.

We start by noting that[12] $\partial_0^* J_\emptyset \in (\partial_0^* \gamma)_\emptyset$ as $J \in \mathcal{L}_0(\Phi)$, so we have integers k_A with $\partial_0^* J_\emptyset = \sum_{A \in \mathcal{A}} k_A (\partial_0^* \gamma)_{\emptyset^A}$. We can take $\Psi^0 = \sum_{A \in \mathcal{A}} k_A \{\phi^A\}$ for any choices of $\phi^A \in A(\Phi)$. Then $J^0 = J - \partial^\gamma \Psi^0$ has $\partial_0^* J^0 = 0$. It remains to define Ψ^i given J^{i-1} for some $i \in [r]$.

Note that $J^{i-1} \in \mathcal{L}_\gamma^i(\Phi)$ as $J \in \mathcal{L}_\gamma^i(\Phi)$ and $\gamma(\Phi) \subseteq \mathcal{L}_\gamma^i(\Phi)$, so $\partial_i^* J^{i-1} \in \mathcal{L}^-_{\partial_i^* \gamma}(\Phi)$. We write $\partial_i^* J^{i-1} = \sum_{C \in \mathcal{P}_i^\Sigma} J^C$, where each J^C is uniquely defined by $J^C_\psi = \partial_i^* J^{i-1}_\psi$ for $\psi \in \Phi_C$.

We claim that each J^C is null. Indeed, for any $\psi \in \Phi_{i-1}$ we have $0 = \partial_{i-1}^* J^{i-1}_\psi = (\partial \partial_i^* J^{i-1})^+_\psi = \sum_C (\partial J^C)^+_\psi$, so by linear independence each $(\partial J^C)_\psi = 0$, as claimed.

By Lemma 5.13 each $J^C \in \langle (\partial_i^* \gamma)(\Phi) \rangle$, so we have $\Psi^i \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial^{\partial_i^* \gamma} \Psi^i = \partial_i^* J^{i-1}$. Letting $J^i = J^{i-1} - \partial^\gamma \Psi^i$, by Lemma 5.18 we have $\partial_i^* J^i = 0$, as required. □

To illustrate the use of Lemma 5.19, we give the following characterisation of the decomposition lattice for nonpartite hypergraph decomposition, thus giving an independent proof of (a generalisation of) a result of Wilson [35] (a similar generalisation is implicit in [9]).

Lemma 5.20. *Let H be an r-graph on $[q]$ and Φ be an (ω, s)-extendable S_q-adapted $[q]$-complex where $n = |V(\Phi)| > n_0(q)$ is large, $s = 3r^2$ and $\omega > n^{-1/2}$. Let $H(\Phi) = \{\phi(H) : \phi \in \Phi_q\}$. Suppose $G \in \mathbb{N}^{\Phi_r^\circ}$. Then $G \in \langle H(\Phi) \rangle$ iff G is H-divisible.*

Proof. As in example i in subsection 2.4, we have $G \in \langle H(\Phi) \rangle$ iff $G^* \in \langle \gamma(\Phi) \rangle$, with $G^* \in \mathbb{N}^{\Phi_r}$ defined by $G^*_\psi = G_{Im(\psi)}$ for $\psi \in \Phi_r$, and $\gamma \in \{0,1\}^{A_r}$ with $A = S_q^{\leq}$ and $\gamma_\theta = 1_{Im(\theta) \in H}$. By Lemma 5.19 we have $\langle \gamma(\Phi) \rangle = \mathcal{L}_\gamma(\Phi)$. By Definition 5.14 we need to show that G is H-divisible iff $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O] \rangle$ for any $O \in \Phi/S_q$.

Fix any $O \in \Phi/S_q$, write $e = Im(O) \in \Phi^\circ$ and $i = |e|$. Then $((G^*)^\sharp)^O \in (\mathbb{Z}^Q)^O = \mathbb{Z}^{Q \times O}$ is a vector supported on the coordinates (B, ψ') with $B' \subseteq B \in Q$ and $\psi' \in O \cap \Phi_{B'}$ in which every nonzero coordinate is equal: we have $((G^*)^\sharp_{\psi'})_B) = \sum \{G^*_\psi : \psi' \subseteq \psi \in \Phi_B\} = (r-i)!|G(e)|$. Similarly, $\langle \gamma^\sharp[O] \rangle$ is generated by vectors with the same support that are constant on the support, where the constant can be $(r-i)!|H(f)|$ for any $f \in [q]_i$, and so any multiple of $(r-i)!gcd_i(H)$. Therefore $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O] \rangle$ iff $gcd_i(H)$ divides $|G(e)|$, as required. □

For the proof of Lemma 3.18 in the next section, we also require two quantitative versions of Lemma 5.19, analogous to those given above for Lemma 5.4. We recall the notation for G-use from Definition 5.7, and for $B \in Q$ fix the symmetric generating set $G = G^B = \{\gamma^\theta : \theta \in \mathcal{A}_B\}$ for γ^B. Then we have the following relationship between G-use and use in the sense of Definition 3.13.

Lemma 5.21. *Suppose $B \in C \in \mathcal{P}_r^\Sigma$ and $J \in \mathcal{L}_\gamma^-(\Phi) \cap (\mathbb{Z}^D)^{\Phi_C}$. Then $U(J^O) = |f_B(J^O)|_G$ for each $O \in \Phi_r/\Sigma$, so $U(J)_\emptyset = |f_B(J)|_G$.*

Proof. We fix an orbit representative $\psi \in O$ and write $U(J^O)$ as the minimum possible value of $\sum\{|x^\theta_\psi|\}$ over all expressions of $J^O = \sum\{x^\theta_\psi \gamma[\psi]^\theta\}$ as a $\mathbb{Z}$-linear combination of γ-atoms at O. We can write any such expression using some fixed representative $\psi \in \Phi_B$; then $f_B(J^O)_\psi = \sum_\theta x^\theta_\psi \gamma^\theta$. As $|f_B(J^O)|_G$ is the minimum value of $\sum\{|x^\theta_\psi|\}$ over all such expressions the lemma follows. □

Our first quantitative version of Lemma 5.19 will bound the total use $U(\Psi)$ of atoms by Ψ in terms of that by J, where $U(\Psi) = \sum_\phi |\Psi_\phi| U(\gamma(\phi)) \leq q^{2r} |\Psi|$, where $|\Psi| = \sum_\phi |\Psi_\phi|$. We start by stating the analogous statement for Lemma 5.13.

[12] Recall that $\emptyset$ may denote the empty set or the function with empty domain. We write $\emptyset^A$ for the copy of $\emptyset$ in A.

Lemma 5.22. *Let $\Sigma \leq S_q$, $\mathcal{A}$ be a $\Sigma^{\leq}$-family and $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be a Σ-adapted (ω, s)-extendable $[q]$-complex with $s = 3r^2$, $n = |V(\Phi)| > n_0(q, D)$ large and $\omega > n^{-1/2}$. Suppose $B \in C \in \mathcal{P}_r^{\Sigma}$ and $J \in \mathcal{L}_\gamma^-(\Phi) \cap (\mathbb{Z}^D)^{\Phi_C}$ is null. Then there is $\Psi^* \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial^\gamma \Psi^* = J$ and $U(\Psi^*) \leq 2^r q^{2r} C(r) U(J)$.*

The proof of Lemma 5.22 is the same as that of Lemma 5.13. When we apply Lemma 5.8 to $f_B(J) \in (\gamma^B)^{\Phi_B}$ we obtain $\Psi \in (\gamma^B)^{\mathcal{O}^B(\Phi)}$ with $f_B(J) = \partial\Psi$ and $|\Psi|_G \leq C(r)|f_B(J)|_G$. We write each $\Psi_{\psi^*} = \sum_{\theta \in \mathcal{A}_B} m^\theta_{\psi^*} \gamma^\theta$ with $\sum_{\theta \in \mathcal{A}_B} |m^\theta_{\psi^*}| = |\Psi_{\psi^*}|_G$. Defining Ψ^* as in the proof of Lemma 5.13, we have $|\Psi^*| \leq 2^r |\Psi|_G \leq 2^r C(r)|f_B(J)|_G$, so $U(\Psi^*) \leq 2^r q^{2r} C(r) U(J)$ by Lemma 5.21.

Lemma 5.23. *Let $\Sigma \leq S_q$, $\mathcal{A}$ be a $\Sigma^{\leq}$-family with $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be a Σ-adapted (ω, s)-extendable $[q]$-complex with $s = 3r^2$, $n = |V(\Phi)| > n_0(q, D)$ large and $\omega > n^{-1/2}$. Suppose $J \in \mathcal{L}_\gamma(\Phi)$. Then there is $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial^\gamma \Psi = J$ and $U(\Psi) \leq 2^{(9q)^{q+2}} U(J)$.*

Proof. We follow the proof of Lemma 5.19. We can choose the k_A so that $U(\Psi^0) = \sum |k_A| \leq U(J)$ and $U(J^0) \leq q^{2r} U(J)$. For each $i \in [r]$, by Lemma 5.22 we can take $U(\Psi^i) \leq 2^i q^{2i} C(i) U(\partial_i^* J^{i-1}) \leq (2rq^2)^i C(i) U(J^{i-1})$, and so $U(J^i) \leq q^{2r}(2r)^i C(i) U(J^{i-1})$, where $U(\partial_i^* J^{i-1})$ is defined with respect to $(\partial_i^* \gamma)$-atoms. Then $U(\Psi) \leq 2U(J^0) \prod_i q^{2r} 2^{r+i} C(i) \leq 2^{(9q)^{q+2}} U(J)$, as $C(i) = 2^{(9i+2)^{i+5}}$. □

Our second quantitative version of Lemma 5.19 is analogous to Lemma 5.11: we seek a rational decomposition Ψ for which we bound the usage of every orbit in terms of that in J. We start with the analogous statement for Lemma 5.13.

Lemma 5.24. *Let $\Sigma \leq S_q$, $\mathcal{A}$ be a $\Sigma^{\leq}$-family and $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be a Σ-adapted (ω, s)-extendable $[q]$-complex with $s = 3r^2$, $n = |V(\Phi)| > n_0(q, D)$ large and $\omega > n^{-1/2}$. Suppose $C \in \mathcal{P}_r^{\Sigma}$ and $J \in \mathbb{Q}\mathcal{L}_\gamma^-(\Phi) \cap (\mathbb{Q}^D)^{\Phi_C}$ is null with $U(J)_\psi \leq \varepsilon$ for all $\psi \in \Phi_B$. Then there is $\Psi^* \in \mathbb{Q}^{\mathcal{A}(\Phi)}$ with $\partial^\gamma \Psi^* = J$ and $U(\Psi^*)_O \leq q^{2r} C(r, \omega) \omega^{-1} \varepsilon$ for each orbit $O \in \Phi_r/\Sigma$.*

Proof. We follow the proof of Lemma 5.13. Applying Lemma 5.11 to $f_B(J) \in (\mathbb{Q}\gamma^B)^{\Phi_B}$ gives $\Psi \in (\mathbb{Q}\gamma^B)^{\mathcal{O}^B(\Phi)}$ with $f_B(J) = \partial\Psi$ and $U_G(\Psi)_\psi \leq C(r, \omega)\varepsilon$ for all $\psi \in \Phi_B$. We write each $\Psi_{\psi^*} = \sum_{\theta \in \mathcal{A}_B} m^\theta_{\psi^*} \gamma^\theta$ with $\sum_{\theta \in \mathcal{A}_B} |m^\theta_{\psi^*}| = |\Psi_{\psi^*}|_G$.

We let $\Psi^* = \sum_{\psi^*, \theta} m^\theta_{\psi^*} \Psi^{\psi^* \theta}$, where we modify the definition of each $\Psi^{\psi^*\theta}$ by averaging over the choice of $\psi^+ \in X_E(\Phi)$. To estimate $U(\Psi^*)_O$ for some $O \in \Phi_r/\Sigma$ we fix any representative $\psi' \in O$, say with $\psi' \in \Phi_{B'}$. For each $A \in \mathcal{A}$, $\theta \in A_B$, writing $r' = |\theta(B) \setminus B'|$, there are at most $n^{r'}$ choices of $\psi \in \Phi_B$ such that $\psi\theta^{-1}$ agrees with ψ' on $\theta(B) \cap B'$ and each has $U_G(\Psi)_\psi \leq C(r, \omega)\varepsilon$. For each $\psi^* \in \mathcal{O}^B(\Phi)$ with $\psi = \psi^* \upsilon$ for some $\upsilon \in O_B^B$, letting ϕ be the $A(2)$-embedding of A such that $\phi(i) = (i, 1)$ for $i \notin \theta(B)$ and $\phi(i) = (i, x)$ when $i = \theta(j)$ with $j \in B$ and $\upsilon(j) = (j, x)$, for uniformly random $\psi^+ \in X_E(\Phi)$ we have $\mathbb{P}(\psi' \subseteq \psi^+\phi) < \omega^{-1} n^{-r'}$, as Φ is (ω, s)-extendable. Summing over ψ' and θ gives $U(\Psi^*)_O \leq q^{2r} C(r, \omega) \omega^{-1} \varepsilon$. □

We conclude this section by proving the second quantitative version of Lemma 5.19, which can be viewed as a rational version of Lemma 3.18, and will form the basis of the 'randomised rounding' aspect of the proof referred to in the introduction.

Lemma 5.25. *Let $\Sigma \leq S_q$, $\mathcal{A}$ be a $\Sigma^{\leq}$-family with $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be a Σ-adapted (ω, s)-extendable $[q]$-complex with $s = 3r^2$, $n = |V(\Phi)| > n_0(q, D)$ large and $n^{-1/2} < \omega < \omega_0(r)$. Suppose $J \in \mathbb{Q}\mathcal{L}_\gamma(\Phi)$ with $U(J)_O \leq \varepsilon$ for all $O \in \Phi_r/\Sigma$. Then there is $\Psi \in \mathbb{Q}^{\mathcal{A}(\Phi)}$ with $\partial^\gamma \Psi = J$ and all $U(\Psi)_O \leq C(q, \omega)\varepsilon$.*

Proof. We follow the proof of Lemma 5.19. We can choose the k_A so that $\sum_A |k_A| \leq U(J) < \varepsilon n^r$. We define Ψ^0 by averaging over each choice of $\phi^A \in A(\Phi)$. Then for any orbit O we have $\mathbb{P}(O \subseteq$

$\phi^A\Sigma) < q^r\omega^{-1}n^{-r}$, as Φ is (ω,s)-extendable, so $U(\Psi^0)_O < \sum_A |k_A|q^r n^{-r} < q^r\omega^{-1}\varepsilon$ and similarly $U(J^0)_O \le q^{2r}\omega^{-1}\varepsilon$.

By Lemma 5.24 we can construct Ψ^i and $J^i = J^{i-1} - \partial^\gamma\Psi^i$ as in the proof of Lemma 5.19 so that all $U(\Psi^i)_O \le \varepsilon_i$ and $U(J^i)_O \le q^r\varepsilon_i$, where $\varepsilon_0 = q^r\omega^{-1}\varepsilon$ and $\varepsilon_i = q^{r+2i}C(i,\omega)\omega^{-1}\varepsilon_{i-1}$. Then all $U(\Psi)_O \le 2q^{2r}\omega^{-1}\varepsilon\prod_{i\in[r]}(q^{r+2i}C(i,\omega)\omega^{-1}) < C(q,\omega)\varepsilon$, recalling that $C(i,\omega) = 2^{C(i)}\omega^{-(9i)^{i+4}}$ and $C(i) = 2^{(9i+2)^{i+5}}$. □

6 Bounded integral decomposition

To complete the proof of Theorem 3.1, it remains to prove Lemma 3.18. The high-level strategy is similar to the randomised rounding and focussing argument from [15] (version 1), although there are some additional complications in the general setting. The proof is by induction on q. In the inductive step we can assume Lemma 4.2 for smaller values of q (this will be used in the proof of Lemma 6.2). Note that we do not assume that γ is elementary, as this property is not preserved by the inductive step.

6.1 Proof modulo lemmas

We start by stating two key lemmas and using them to deduce Lemma 3.18; the remainder of the section will then be devoted to proving the key lemmas. The first lemma is an approximate version of Lemma 3.18; the second will allow us to focus the support in a smaller set of vertices. The proof of Lemma 3.18 is then to alternate applications of these lemmas until the support is sufficiently small that it suffices to use the total use quantitative version of the decomposition lattice lemma. In the statements of the lemmas we denote the labelled complex by Φ, but note that they will be applied to restrictions of Φ as in the statement of Lemma 3.18, so we allow for weaker lower bounds on the extendability and number of vertices. Throughout we fix a $\Sigma^{\le}$-family $\mathcal{A}$ with $\Sigma \le S_q$ and $|\mathcal{A}| \le K$, suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$ and let $\omega_0 := \omega_0(q,D,K)$, $n_0 := n_0(q,D,K)$ be as in Lemma 3.18.

Lemma 6.1. *Let Φ be an (ω',h)-extendable Σ-adapted $[q]$-complex on $[n]$, where $n^{-h^{-2q}} < \omega' < \omega_0$ and $n > n_0^{1/2r}$. Suppose $J \in \langle\gamma(\Phi)\rangle$ is θ-bounded, with $n^{-(4hq)^{-r}} < \theta < 1$. Then there is some $(\omega')_q^{3qh}\theta$-bounded $J' \in (\mathbb{Z}^D)^{\Phi_r}$ and $(\omega')_q^{-1}\theta$-bounded $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial^\gamma\Psi = J - J'$.*

Lemma 6.2. *Let Φ be an (ω',h)-extendable Σ-adapted $[q]$-complex on $[n]$, where $n^{-h^{-2q}} < \omega' < \omega_0$ and $n > n_0^{1/2r}$. Let $V' \subseteq V(\Phi)$ with $|V'| = n/2$ be such that (Φ,V') is (ω',h)-extendable wrt V'. Suppose $J \in \langle\gamma(\Phi)\rangle$ is θ-bounded, with $n^{-(3hq)^{-r}} < \theta < (\omega')_q^{3qh}$. Then there is some $(\omega')_q^{-2qh}\theta$-bounded $J' \in (\mathbb{Z}^D)^{\Phi[V']_r}$ and $(\omega')_q^{-2qh}\theta$-bounded $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial^\gamma\Psi = J - J'$.*

Proof of Lemma 3.18. Let $\mathcal{A}$ be a $\Sigma^{\le}$-family with $\Sigma \le S_q$ and $|\mathcal{A}| \le K$ and suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be an (ω,h)-extendable Σ-adapted $[q]$-complex on $[n]$, where $n^{-h^{-3q}} < \omega < \omega_0$ and $n > n_0$. Suppose $J \in \langle\gamma(\Phi)\rangle$ is θ-bounded, with $n^{-(5hq)^{-r}} < \theta < 1$. We need to show that there is some $\omega_q^{-2h}\theta$-bounded $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial^\gamma\Psi = J$.

Let t be such that $n^{1/2r}/2 < 2^{-t}n \le n^{1/2r}$. Choose $V_t \subseteq \ldots \subseteq V_1 \subseteq V_0 = V(\Phi)$ with $|V_i| = 2^{-i}n$ uniformly at random. By Lemma 6.5 (a simple concentration argument given in the next subsection) whp all $(\Phi[V_i],V_{i+1})$ are (ω',h)-extendable wrt V_{i+1}, where $\omega' = (\omega/2)^h > n^{-h^{-2q}}$. We define θ-bounded $J_i \in \langle\gamma(\Phi[V_i])\rangle$ as follows.

Let $J_0 = J$. Given J_i with $0 \le i < t$, we apply Lemma 6.1 to obtain some $(\omega')_q^{3qh}\theta$-bounded $J_i' \in (\mathbb{Z}^D)^{\Phi[V_i]_r}$ and $(\omega')_q^{-1}\theta$-bounded $\Psi_i \in \mathbb{Z}^{\mathcal{A}(\Phi[V_i])}$ with $\partial^\gamma \Psi_i = J_i - J_i'$. Note that $J_i' \in \langle \gamma(\Phi[V_i])\rangle$.

Next we apply Lemma 6.2 to J_i' (with $(\omega')_q^{3qh}\theta$ in place of θ) to obtain some θ-bounded $J_{i+1} \in (\mathbb{Z}^D)^{\Phi[V_{i+1}]_r}$ and θ-bounded $\Psi_i' \in \mathbb{Z}^{\mathcal{A}(\Phi[V_i])}$ with $\partial^\gamma \Psi_i' = J_i' - J_{i+1}$.

To continue the process, we need to show $J_{i+1} \in \langle \gamma(\Phi[V_{i+1}])\rangle$. To see this, first note $J_{i+1} \in \langle \gamma(\Phi)\rangle = \mathcal{L}_\gamma(\Phi)$ by Lemma 5.19. Now for any $O \in \Phi[V_{i+1}]/\Sigma$, by definition of $\mathcal{L}_\gamma(\Phi)$ we have $(J_{i+1}^\sharp)^O \in \langle \gamma^\sharp[O]\rangle$, so $J_{i+1} \in \mathcal{L}_\gamma(\Phi[V_{i+1}]) = \langle \gamma(\Phi[V_{i+1}])\rangle$, again by Lemma 5.19.

We conclude with some θ-bounded $J_t \in \langle \gamma(\Phi[V_t])\rangle$, where $|V_t| \le n^{1/2r}$. By Lemma 5.23 there is $\Psi_t \in \mathbb{Z}^{\mathcal{A}(\Phi[V_t])}$ such that $\partial^\gamma \Psi_t = J_t$ and $U(\Psi_t) \le 2^{(9q)^{q+2}} U(J_t)$. Let $\Psi = \Psi_t + \sum_{i=0}^{t-1}(\Psi_i + \Psi_i')$. Then $\partial^\gamma \Psi = J_t + \sum_{i=0}^{t-1}(J_i - J_i' + J_i' - J_{i+1}) = J$.

Also, for any $\psi \in \Phi_{r-1}$ we have $U(\Psi)_\psi \le U(\Psi_t)_\psi + \sum_{i=0}^{t-1}(U(\Psi_i)_\psi + U(\Psi_i')_\psi) < 2^{(9q)^{q+2}}\theta(n^{1/2r})^r + \sum_{i=0}^{t-1}((\omega')_q^{-1}\theta 2^{-i}n + \theta 2^{-i}n) < 2(\omega')_q^{-1}\theta n$, so Ψ is (say) $\omega_q^{-2h}\theta$-bounded. □

6.2 Random subgraphs

In the next subsection we will extend the rational decomposition lemma (Lemma 5.25) to a version relative to a sparse random subgraph L. We establish some preliminary properties of L in this subsection. First we show that whp L is 'typical' in Φ, in that specifying that certain edges of an extension should belong to L scales the number of extensions in the expected way.

Definition 6.3. Let Φ be an $[q]$-complex and $L \subseteq \Phi_r^\circ$. Let $d_\Phi(L) = |L||\Phi_r^\circ|^{-1}$. We say L is (c,s)-typical in Φ if for any Φ-extension $E = (H, F, \phi)$ of rank s and $H' \subseteq H_r^\circ \setminus H_r^\circ[F]$ we have $X_{E,H'}(\Phi, L) = (1 \pm c)d_\Phi(L)^{|H'|}X_E(\Phi)$.

Lemma 6.4. *Let Φ be an (ω, s)-extendable $[q]$-complex. Suppose L is ν-random in Φ_r°, where $\nu > n^{-(3sq)^{-r}}$, $n = |V(\Phi)|$. Then whp L is $(n^{-1/3}, s)$-typical in Φ. In particular, $\Phi[L]$ is (ω', s)-extendable, where $\omega' = 0.9\nu^{Qs^r}\omega$.*

Proof. First note by the Chernoff bound that whp $d_\Phi(L) = (1 \pm n^{-0.4})\nu$. Let $E = (H, F, \phi)$ be any Φ-extension of rank s, $H' \subseteq H_r^\circ \setminus H_r^\circ[F]$ and $X = X_{E,H'}(\Phi, L)$. Note that $\mathbb{E}X = \nu^{|H'|}X_E(\Phi)$, where $X_E(\Phi) > \omega n^{v_E}$. Also, for any $k \in [r]$ there are $O(n^k)$ choices of $f \in \Phi_r^\circ$ with $f \setminus \phi(F)| = k$, and for each such f, changing whether $f \in L$ affects X by $O(n^{v_E - k})$. Thus X is $O(n^{2v_E - 1})$-varying, so by Lemma 3.5 whp $X = (1 \pm n^{-1/3})\nu^{|H'|}X_E(\Phi)$. In particular, $X > \omega' n^{v_E}$. □

Similarly, we obtain the following variant form of the previous lemma that was used in the previous subsection.

Lemma 6.5. *Let Φ be an (ω, s)-extendable $[q]$-complex on $[n]$. Suppose S is uniformly random in $\binom{[n]}{m}$, where $m > \omega^{-1}\log n$ and n is large. Then (Φ, S) is $((\omega/2)^h, s)$-extendable wrt S with probability at least $1 - e^{-(\omega m)^2/20}$.*

Proof. It suffices to estimate the probability that any simple Φ-extension of rank s is $\omega/2$-dense in (Φ, S). Let $E = (H, F, \phi)$ be any Φ-extension of rank s with $F = V(H)\setminus\{x\}$ for some $x \in V(H)$. Note that $X_E(\Phi, S) = \sum_{\phi^+ \in X_E(\Phi)} 1_{\phi^+(x) \in S}$ and $X_E(\Phi) > \omega n$ as Φ is (ω, s)-extendable. Then $X_E(\Phi, S)$ is hypergeometric with $\mathbb{E}X_E(\Phi, S) > \omega m$, so $\mathbb{P}(X_E(\Phi, S) < \omega m/2) < e^{-(\omega m)^2/12}$. The lemma follows by taking a union bound over at most $qsn^{qs} < e^{(\omega m)^2/48}$ choices of E. □

We also require the following refined notion of boundedness that operates with respect to all small extensions in L. The following lemma is analogous to [15, Lemma 2.21].

Definition 6.6. Let Φ be an $[q]$-complex, $L \subseteq \Phi_r^\circ$, and $J \in (\mathbb{Z}^D)^{\Phi_r}$. Let $E = (H, F, \phi)$ with $H \subseteq [q](s)$ be a Φ-extension, $G \subseteq H_r^\circ \setminus H_r^\circ[F]$, $\psi \in [q](s)_r$ and $e = Im(\psi)$. We write $X_{E,G}^{e,J}(\Phi, L) = \sum_{\phi^* \in X_{E,G}(\Phi,L)} U(J)_{\phi^*\psi}$. We say that J is (θ, s)-bounded wrt (Φ, L) if $X_{E,G}^{e,J}(\Phi, L) < \theta d_\Phi(L)^{|G|}|V(\Phi)|^{v_E}$ for any such E, G and e with $e \notin G$ and $e \setminus F \neq \emptyset$.

Lemma 6.7. *Let Φ be an $[q]$-complex with $|V(\Phi)| = n$. Suppose $J \in (\mathbb{Z}^D)^{\Phi_r}$ is θ-bounded, with $\theta > n^{-0.01}$, and all $U(J)_\psi < n^{0.1}$. Let L be ν-random in Φ_r°, where $\nu > n^{-(3sq)^{-r}}$. Then whp J is $(1.1\theta, s)$-bounded wrt (Φ, L).*

Proof. Let $E = (H, F, \phi)$ with $H \subseteq [q](s)$ be a Φ-extension, $G \subseteq H_r^\circ \setminus H_r^\circ[F]$ and $\psi \in [q](s)_r$ with $e := Im(\psi) \notin G$ and $e \setminus F \neq \emptyset$. Write $X = X_{E,G}^{e,J}(\Phi, L) = \sum_{\phi^* \in X_E(\Phi)} 1_{\phi^* \in X_{E,G}(\Phi,L)} U(J)_{\phi^*\psi}$. As J is θ-bounded we have $\sum_{\phi^* \in X_E(\Phi)} U(J)_{\phi^*\psi} < \theta n^{v_E}$. For each $\phi^* \in X_E(\Phi)$ we have $\mathbb{P}(\phi^* \in X_{E,G}(\Phi, L)) = \nu^{|G|}$, so $\mathbb{E}X < \theta\nu^{|G|}n^{v_E}$. For any $k \in [r]$ there are $O(n^k)$ choices of $f \in \Phi_r^\circ$ with $|f \setminus \phi(F)| = k$, and for each such f, changing whether $f \in L$ affects X by $O(n^{v_E - k + 0.1})$. Thus X is $O(n^{2v_E - 0.8})$-varying, so by Lemma 3.5 whp $X < 1.1\theta d_\Phi(L)^{|G|}n^{v_E}$. □

6.3 Rational decompositions

In this subsection we prove the following result, which is a version of Lemma 5.25 relative to a sparse random subgraph L; note the key point that we incur a loss in boundedness that depends only on q, not on the density of L. Throughout, as in the hypotheses of Lemma 6.1, we let $\mathcal{A}$ be a $\Sigma^{\leq}$-family with $\Sigma \leq S_q$ and $|\mathcal{A}| \leq K$, suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$, and let Φ be an (ω, h)-extendable Σ-adapted $[q]$-complex on $[n]$, where $n^{-h^{-2q}} < \omega < \omega_0$ and $n > n_0^{-1/2}$. (For convenient notation we rename ω' as ω.)

Lemma 6.8. *Suppose $L \subseteq \Phi_r^\circ$ is (c, h)-typical in Φ with $d_\Phi(L) = \nu \geq n^{-(3hq)^{-r}}$. Let $J \in \langle \gamma(\Phi) \rangle_{\mathbb{Q}} \cap \mathbb{Q}^{\Phi[L]_r}$. Suppose J is θ-bounded with all $|J_\psi| < n^{0.1}$ and (θ, h)-bounded wrt (Φ, L). Then there is some $\omega_q^{-0.9}\theta$-bounded $\Psi \in \mathbb{Q}^{\mathcal{A}(\Phi[L])}$ with $\partial^\gamma \Psi = J$.*

The proof of Lemma 6.8 uses the following result, which reduces to the case when we bound the use of every orbit.

Lemma 6.9. *For any θ-bounded $J \in (\mathbb{Q}^D)^{\Phi_r}$ there is some $J' \in (\mathbb{Q}^D)^{\Phi_r}$ and $\Psi \in \mathbb{Q}^{\mathcal{A}(\Phi)}$ such that $\partial\Psi = J - J'$, and for any $O \in \Phi_r/\Sigma$ both $U(\Psi)_O - U(J)_O$ and $U(J')_O$ are at most $q^q\omega^{-1}\theta$.*

Proof. For each $O \in \Phi_r/\Sigma$ we fix a representative $\psi^O \in \Phi_{B^O}$ and $n^O \in \mathbb{Q}^{\mathcal{A}_{B^O}}$ with $|n^O| = U(J)_O$ and $f_{B^O}(J)_{\psi^O} = \sum_\theta n_\theta^O \gamma^\theta$, so $J^O = \sum_\theta n_\theta^O \gamma(\psi^O\theta^{-1})$. For each such (O, θ) we let $E_\theta^O = (\overrightarrow{q}, Im(\theta), \psi^O\theta^{-1})$. We let $\Psi = \sum_{O,\theta} n_\theta^O |X_{E_\theta^O}(\Phi)|^{-1} X_{E_\theta^O}(\Phi)$, where if $\theta \in A \in \mathcal{A}$ we choose the copy of $\phi \in A(\Phi)$ for each $\phi \in X_{E_\theta^O}(\Phi)$. Then J is exactly cancelled by the γ-atoms of $\partial^\gamma\Psi$ corresponding to $\gamma(\phi)^O$ for $\phi \in A(\Phi)$. To estimate the remaining contributions of $\partial^\gamma\Psi$, note that for any $O \in \Phi_r/\Sigma$ and $r' \in [r]$ we have $\sum\{|n_\theta^{O'}| : |Im(O') \setminus Im(O)| = r'\} \leq \binom{r}{r'}\theta n^{r'}$. For each such (O', θ), there are at least ωn^{q-r} choices of ϕ, of which at most $(q-r)!n^{q-r-r'}$ contain $Im(O)$, so for random $\phi \in X_{E_\theta^{O'}}(\Phi)$ we have $\mathbb{P}(O \subseteq \phi\Sigma) \leq (q-r)!\omega^{-1}n^{-r'}$. Summing over r' gives the stated bounds on $U(\Psi)_O - U(J)_O$ and $U(J')_O$. □

Proof of Lemma 6.8. We start by defining $\Psi^0 \in \mathbb{Q}^{\mathcal{A}(\Phi)}$ such that $\partial^\gamma\Psi^0 = J$ and $U(\Psi^0)_O < U(J)_O + (C^* - 1)\theta$ for all $O \in \Phi_r/\Sigma$, where $C^* = 2C(q, \omega)q^q\omega^{-1}$ (so Ψ^0 is $C^*\theta$-bounded). Then we will modify Ψ^0 to obtain Ψ using a version of the Clique Exchange Algorithm.

First we apply Lemma 6.9 to obtain $\Psi' \in \mathbb{Q}^{\mathcal{A}(\Phi)}$ and $J' = J - \partial^\gamma \Psi$ so that all $U(\Psi)_O - U(J)_O$ and $U(J')_O$ are at most $q^q\omega^{-1}\theta$. Then by Lemma 5.25 there is $\Psi'' \in \mathbb{Q}^{\mathcal{A}(\Phi)}$ such that $\partial^\gamma \Psi'' = J'$ and all $U(\Psi'')_O < C(q,\omega)q^q\omega^{-1}\theta$. We take $\Psi^0 = \Psi' + \Psi''$.

We apply two Splitting Phases, the first in Φ and the second in $\Phi[L]$. For the first, we fix $N_0 \in \mathbb{N}$ such that $N_0\Psi^0 \in \mathbb{Z}^{\mathcal{A}(\Phi)}$, and list the signed elements of $N_0\Psi^0$ as $(s_i\phi_i : i \in [|N_0\Psi^0|])$, where each $s_i \in \pm 1$. For each i, say with $\phi_i \in A^i(\Phi)$, we consider the Φ-extension $E_i = ([q](p), [q], \phi_i)$, and define $\Psi^1 \in \mathbb{Q}^{\mathcal{A}(\Phi)}$ by $\Psi^1 = \Psi^0 + N_0^{-1}\sum_{i\in[|N_0\Psi^0|]} s_i\mathbb{E}_{\phi_i^*\in X_{E_i}(\Phi)}(A^i(\Phi[\phi_i^*\Upsilon']) - A^i(\Phi[\phi_i^*\Upsilon]))$. Then $\partial^\gamma\Psi^1 = \partial^\gamma\Psi^0 = J$, and all signed elements in Ψ^0 are cancelled.

We claim for any $O \in \Phi_r/\Sigma$ that $\Gamma_O := U(\Psi^1)_O - U(\Psi^0)_O \le r!\omega^{-1}(2pq)^r C^*\theta$. To see this, we estimate $\Gamma_O \le \mathbb{E}_{i\in[|N_0\Psi^0|]}\mathbb{P}(Im(O) \in \phi_i^*(\Omega'))$ (recall $\Omega' = K_q^r(p) \setminus Q$). For any $r' \in [r]$, as Ψ^0 is $C^*\theta$-bounded there are at most $N_0\binom{r}{r'}C^*\theta n^{r'}$ choices of i such that $|Im(O) \setminus Im(\phi_i))| = r'$. For each such i, as Φ is (ω,h)-extendable there are at least ωn^{pq-q} choices of $\phi_i^* \in X_{E_i}(\Phi)$, of which at most $r!|\Omega'|n^{v_{E_i}-r'}$ have $Im(O) \in \phi_i^*(\Omega')$, so $\mathbb{P}(Im(O) \in \phi_i^*(\Omega')) < r!\omega^{-1}|\Omega'|n^{-r'}$. Therefore $\Gamma_O \le N_0^{-1}\sum_{r'\in[r]} N_0\binom{r}{r'}C^*\theta n^{r'}\cdot r!\omega^{-1}|\Omega'|n^{-r'} < r!\omega^{-1}(2pq)^rC^*\theta$, as claimed.

Also, for any $\phi \in \mathcal{A}(\Phi)$ we claim that $|\Psi^1_\phi| < \omega^{-1}p^qn^{r-q}\sum_{O\subseteq\phi\Sigma}(qC^*\theta + U(J)_O)$. To see this, we consider separately the contributions from ϕ_i according to $r' = |Im(\phi) \cap Im(\phi_i)|$.

First we consider $0 \le r' < r$. As Ψ^0 is $C^*\theta$-bounded there are at most $N_0QC^*\theta n^{r-r'}$ such choices of ϕ_i. For each such i, there are at least ωn^{pq-q} choices of $\phi_i^* \in X_{E_i}(\Phi)$, of which at most $p^qn^{pq-q-(q-r')}$ have $\phi = \phi_i^*\phi'$ for some $\phi' \in [q](p)$, so the total such contribution to $|\Psi^1_\phi|$ is at most $N_0^{-1}\sum_{r'} N_0QC^*\theta n^{r-r'}\cdot\omega^{-1}p^qn^{r'-q} = rp^qQC^*\omega^{-1}\theta n^{r-q}$.

It remains to consider $r' = r$. For each $O \subseteq \phi\Sigma$ there are at most $N_0(U(J)_O + (C^*-1)\theta)$ choices of ϕ_i containing $Im(O)$. For each of these with $|Im(\phi_i) \cap Im(\phi)| = r$ we have $\phi = \phi_i^*\phi'$ for some $\phi' \in [q](p)$ with probability at most $\omega^{-1}p^qn^{r-q}$, so the total such contribution to $|\Psi^1_\phi|$ is at most $(U(J)_O + (C^*-1)\theta)\omega^{-1}p^qn^{r-q}$. The claim follows.

In the second Splitting Phase, we fix $N_1 \in \mathbb{N}$ such that $N_1\Psi^1 \in \mathbb{Z}^{\mathcal{A}(\Phi)}$, and list the signed elements of $N_1\Psi^1$ as $(s_i\phi_i : i \in [|N_1\Psi^1|])$, where each $s_i \in \pm 1$. For each i, say with $\phi_i \in A^i(\Phi)$, we consider the Φ-extension $E_i = ([q](p), [q], \phi_i)$, and define $\Psi^2 \in \mathbb{Q}^{\mathcal{A}(\Phi)}$ by $\Psi^2 = \Psi^1 + N_1^{-1}\sum_{i\in[|N_1\Psi^1|]} s_i\mathbb{E}_{\phi_i^*\in X_{E_i,L}(\Phi)}(A^i(\Phi[\phi_i^*\Upsilon']) - A^i(\Phi[\phi_i^*\Upsilon]))$. Then $\partial\Psi^2 = \partial\Psi^1 = J$, and all signed elements in Φ^1 are cancelled, so Ψ^2 is supported on maps ϕ such that there is at most one $e \in Q$ with $\phi(e) \notin L$. (This was the same as the first Splitting Phase except that we changed $X_{E_i}(\Phi)$ to $X_{E_i}(\Phi, L)$.)

We claim for any $O \in \Phi_r/\Sigma$ that $\Gamma'_O := U(\Psi^2)_O - U(\Psi^1)_O \le (pq)^{2q}\omega^{-2}C^*\theta\nu^{-1}$, where $\nu := d_\Phi(L)$. To see this, we estimate $\Gamma'_O \le \mathbb{E}_{i\in[|N_1\Psi^1|]}\mathbb{P}(Im(O) \in \phi_i^*(\Omega'))$. We fix $f' \in \Omega'$ and consider the contribution from i with $O = \phi_i^*f'\Sigma$. Consider any $\phi' \in \Phi$ with $O = \phi'f'\Sigma$ and the Φ-extension $E_{f'} = (H, f', \phi')$, where $H = [q](p)[[q] \cup f']$ is the restriction of $[q](p)$ to $[q] \cup f'$. Let $H' = H_r^\circ \setminus ([q]_r \cup \{f'\})$.

The number of i with $\phi_i = \phi^*|_{[q]}$ for some $\phi^* \in X_{E_{f'},H'}(\Phi, L)$ is at most

$$\begin{aligned}
N_1 \sum_{\phi^*\in X_{E_{f'},H'}(\Phi,L)} |\Psi^1_{\phi^*|_{[q]}}| &< \sum_{\phi^*\in X_{E_{f'},H'}(\Phi,L)} N_1\omega^{-1}p^qn^{r-q} \sum_{\psi\subseteq\phi^*|_{[q]}} (qC^*\theta + U(J)_\psi) \\
&= N_1\omega^{-1}p^qn^{r-q}\left[qQC^*\theta X_{E_{f'},H'}(\Phi,L) + \sum_{e'\in Q} X^{e',J}_{E_{f'},H'}(\Phi,L)\right] \\
&< 2N_1\omega^{-1}p^qn^{r-q}qQC^*\theta\nu^{|H'|}n^{q-|f'\cap[q]|},
\end{aligned}$$

as L is (c,h)-typical and J is (θ,h)-bounded wrt L.

For each such i, there are at least $0.9\omega n^{pq-q}\nu^{|\Omega'|}$ choices of $\phi_i^* \in X_{E_i}(\Phi, L)$, of which the number containing ϕ' is at most $1.1\nu^{|\Omega'|-|H'\cup\{f'\}|}n^{pq-q-r+|f'\cap[q]|}$, so $\mathbb{P}(\phi' \subseteq \phi_i^*) < 1.3\omega^{-1}\nu^{-|H'|-1}n^{|f'\cap[q]-r}$. Thus

$$\begin{aligned}\Gamma'_O &< q^r N_1^{-1} \sum_{f'\in\Omega'} 2N_1\omega^{-1}p^q n^{r-q} qQC^*\theta\nu^{|H'|}n^{q-|f'\cap[q]|} \cdot 1.3\omega^{-1}\nu^{-|H'|-1}n^{|f'\cap[q]|-r} \\ &< (pq)^{2q}\omega^{-2}C^*\theta\nu^{-1}, \quad \text{using } p > 2^{8q}, \text{ as claimed.}\end{aligned}$$

We fix $N_2 \in \mathbb{N}$ such that $N_2\Psi^2 \in \mathbb{Z}^{\mathcal{A}(\Phi)}$, and classify signed elements of $N_2\Psi^2$ as before: recall that a pair (O, ϕ') is near or far, has the same sign as that of ϕ' in $N_2\Psi^2$, and has a type θ determined by an orbit representative $\psi^O \in O$. For $\phi \in \mathcal{A}(\Phi)$ let B_ϕ be the number of pairs (O, ϕ) in $N_2\Psi^2$ such that $Im(O) \notin L$; note that all such pairs are near and $Im(O)$ is uniquely determined by ϕ.

We claim that each $B_\phi < 3N_2\omega^{-2}p^q\nu^{-Q+1}n^{r-q}\sum_{\psi\subseteq\phi}(qQC^*\theta + U(J)_\psi)$, To see this, we fix $\psi' \in \Upsilon'\cup\Upsilon\setminus\{[q]\}$ and consider the contributions from i with $\phi_i^*\psi' = \phi$. We consider the Φ-extension $E_{\psi'} = (H, F', \phi')$, where $F' = Im(\psi')$ (so $|F'\cap[q]| = r$), $H = [q](p)[[q]\cup F']$ and $\phi = \phi'\psi'$. Let $H' = H_r^\circ \setminus ([q]_r \cup F'_r)$.

The number of i with $\phi_i = \phi^* \mid_{[q]}$ for some $\phi^* \in X_{E_{\psi'},H'}(\Phi, L)$ is at most

$$\begin{aligned}N_1 \sum_{\phi^*\in X_{E_{\psi'},H'}(\Phi,L)} |\Psi^1_{\phi^*|_{[q]}}| &\le \sum_{\phi^*\in X_{E_{\psi'},H'}(\Phi,L)} N_1\omega^{-1}p^q n^{r-q} \sum_{\psi\subseteq\phi^*|_{[q]}} (qC^*\theta + U(J)_\psi) \\ &= N_1\omega^{-1}p^q n^{r-q}\left[qQC^*\theta X_{E_{\psi'},H'}(\Phi, L) + \sum_{e'\in Q} X^{e',J}_{E_{\psi'},H'}(\Phi, L)\right] \\ &< 2N_1\omega^{-1}p^q n^{r-q}\nu^{|H'|}n^{q-|F'\cap[q]|}(qQC^*\theta + U(J)_O) \\ &= 2N_1\omega^{-1}p^q\nu^{|H'|}(qQC^*\theta + U(J)_O),\end{aligned}$$

where for $e' = F'\cap[q]$ we let $O \subseteq \phi\Sigma$ be such that $Im(O) = \phi(e')$ and use $X^{e',J}_{E_{\psi'},H'}(\Phi, L) \le X_{E_{\psi'},H'}(\Phi, L)U(J)_O$.

For each such i, there are at least $0.9\omega n^{pq-q}\nu^{|\Omega'|}$ choices of $\phi_i^* \in X_{E_i}(\Phi, L)$, of which the number containing ψ' is at most $1.1\nu^{|\Omega'|-|H'|-(Q-1)}n^{pq-q-q+|F'\cap[q]|}$ (as $|F'_r\setminus[q]_r| = Q-1$), so $\mathbb{P}(\psi' \subseteq \phi_i^*) < 1.3\omega^{-1}\nu^{-|H'|-Q+1}n^{r-q}$. Then

$$\begin{aligned}B_\phi &< N_2N_1^{-1} \sum_{\psi':|F'\cap[q]|=r} 2N_1\omega^{-1}p^q\nu^{|H'|}(qQC^*\theta + U(J)_O) \cdot 1.3\omega^{-1}\nu^{-|H'|-Q+1}n^{r-q} \\ &< 3N_2\omega^{-2}p^q\nu^{-Q+1}n^{r-q}\sum_{\psi\subseteq\phi}(qQC^*\theta + U(J)_\psi), \text{ as claimed.}\end{aligned}$$

In Elimination Phase, we consider each orbit $O \in \Phi_r/\Sigma$ with $Im(O) \notin L$, say with representative $\psi^O \in \Phi_B$, and let $E_O = (B(2), B, \psi^O)$. For each $\psi_O^* \in X_{E_O}(\Phi, L)$ and each signed near pair $\pm(O, \phi)$ in $N_2\Psi^2$, say of type θ, i.e. $\phi\theta = \psi^O$, we let $E^\phi(\psi_O^*) = (w^{B_O^\phi}, F, \phi_0)$, where $B_O^\phi = \theta(B)$, $F = [q]\cup(B_O^\phi\times[2])$, $\phi_0 \mid_{[q]} = \phi$ and $\phi_0(\theta(x), y) = \psi_O^*(x, y)$ for $x \in B$, $y \in [2]$.

We let $\Psi = \Psi^2 + N_2^{-1}\sum_{(O,\phi)} \pm\mathbb{E}_{\psi_O^*\in X_{E_O}(\Phi,L)}\mathbb{E}_{\psi^+\in X_{E^\phi(\psi_O^*)}(\Phi,L)}\sum_{x\in[q](s)} w_x^{B_O^\phi}\psi^* x$, where the sign is that of (O, ϕ), and each $\psi^* x \in A^\phi(\Phi)$ where $\phi \in A^\phi(\Phi)$. Then $\partial^\gamma\Psi = J$, similarly to the previous version of the algorithm, treating all near pairs on O as a single cancelling group, which is valid as $J^O = 0$ when $Im(O) \notin L$. All pairs (O, ϕ) with $Im(O) \notin L$ are cancelled, so $\Psi \in \mathbb{Q}^{\mathcal{A}(\Phi[L])}$.

We claim for any $O' \in \Phi_r/\Sigma$ that $\Gamma''_{O'} := U(\Psi)_{O'} - U(\Psi^2)_{O'} \le \omega^{-3}(pq)^{2q}C^*\theta\nu^{-1}$. To see this, we estimate

$$\Gamma''_{O'} \le N_2^{-1} \sum_{(O,\phi)} \mathbb{E}_{\psi^*_O \in X_{E_O}(\Phi,L)} \mathbb{P}(Im(O') \in \psi^+(\Omega')) = N_2^{-1} \sum_{(O,\phi)} \mathbb{P}(Im(O') \in \psi^\phi_O(\Omega')),$$

with each ψ^ϕ_O uniformly random in $X_{E^\phi_O}(\Phi, L)$ where $E^\phi_O = ([q](s), [q], \phi)$.

We fix $f' \in [q](s)_r \setminus \overrightarrow{q}_r$ and consider the contribution from near pairs (O, ϕ) with $O' = \psi^\phi_O f'\Sigma$. Consider any $\phi' \in \Phi_r$ with $O' = \phi' f'\Sigma$ and the Φ-extension $E_{f'} = (H, f', \phi')$, where $H = [q](p)[[q] \cup f']$. For $B \in [q]_r$ let $H^B = H^\circ_r \setminus \{B, f'\}$.

The number of signed near pairs $\pm(O, \phi)$ in $N_2\Psi^2$ with $\phi = \phi^* \mid_{[q]}$, $Im(O) \notin L$, $O = \phi^* \mid_B \Sigma$ for some $B \in [q]_r$ and $\phi^* \in X_{E_{f'},H^B}(\Phi, L)$ is at most

$$\begin{aligned}
\sum_{\phi^* \in X_{E_{f'},H^B}(\Phi,L)} B_{\phi^*|_{[q]}} &< \sum_{\phi^* \in X_{E_{f'},H^B}(\Phi,L)} 3N_2\omega^{-2}p^q\nu^{-Q+1}n^{r-q} \sum_{\psi \subseteq \phi} (qQC^*\theta + U(J)_\psi) \\
&= 3N_2\omega^{-2}p^q\nu^{-Q+1}n^{r-q} \sum_{e^* \in Q} [qQC^*\theta X_{E_{f'},H^B}(\Phi, L) + X^{e^*,J}_{E_{f'},H^B}(\Phi, L)] \\
&< 3N_2\omega^{-2}p^q\nu^{-Q+1}n^{r-q}2QqQC^*\theta\nu^{|H^B|}n^{q-|f'\cap[q]|},
\end{aligned}$$

as L is (c, h)-typical and J is (θ, h)-bounded wrt L.

For each such (O, ϕ), there are at least $0.9\omega n^{qs-q}\nu^{Q(s^r-1)}$ choices for $\psi^\phi_O \in X_{E^\phi_O}(\Phi, L)$, of which the number with $O' = \psi^\phi_O f'\Sigma$ is at most $1.1n^{qs-q-r+|f'\cap[q]|}\nu^{Qs^r-|H|}$, so $\mathbb{P}(O' = \psi^\phi_O f'\Sigma) < 1.3\omega^{-1}\nu^{-|H^B|+Q-2}n^{-r+|f'\cap[q]|}$, as $Qs^r - |H| = Q(s^r - 1) - |H^B| + Q - 2$. Thus

$$\begin{aligned}
\Gamma''_{O'} &< N_2^{-1} \sum_{f' \in [q](s)_r \setminus \overrightarrow{q}_r} 3N_2\omega^{-2}p^q\nu^{-Q+1}n^{r-q}2QqQC^*\theta\nu^{|H^B|}n^{q-|f'\cap[q]|} \\
&\qquad \cdot 1.3\omega^{-1}\nu^{-|H^B|+Q-2}n^{-r+|f'\cap[q]|} \\
&< \omega^{-3}(pq)^{2q}C^*\theta\nu^{-1}, \quad \text{as claimed.}
\end{aligned}$$

Finally, for any $f \in \Phi_{r-1}$ we have

$$\begin{aligned}
U(\Psi)_f &\le U(\Psi^0)_f + \sum\{\Gamma_O : f\Sigma \subseteq O\} + \sum\{\Gamma'_O + \Gamma''_O : f\Sigma \subseteq O, Im(O) \in L\} \\
&< C^*\theta n + r!\omega^{-1}(2pq)^r C^*\theta n + 1.1q^r\nu n((pq)^{2q}\omega^{-2}C^*\theta\nu^{-1} + \omega^{-3}(pq)^{2q}C^*\theta\nu^{-1}) \\
&< 2\omega^{-3}q^r(pq)^{2q}C^*\theta n.
\end{aligned}$$

Recalling that $C^* = 2C(q, \omega)q^q\omega^{-1}$, $C(i, \omega) = 2^{C(i)}\omega^{-(9i)^{i+4}}$, $C(i) = 2^{(9i+2)^{i+5}}$, $\omega_q := \omega^{(9q)^{q+5}}$, and $\omega < \omega_0$ we see that Ψ is $\omega_q^{-0.9}\theta$-bounded. □

6.4 Approximation

In this subsection we prove Lemma 6.1 (approximate bounded integral decomposition) by randomly rounding Lemma 6.8 (rational decomposition with respect to a sparse random subgraph). Throughout, as in the hypotheses of Lemma 6.1, we let $\mathcal{A}$ be a $\Sigma^{\le}$-family with $\Sigma \le S_q$ and $|\mathcal{A}| \le K$, suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$, and let Φ be an (ω, h)-extendable Σ-adapted $[q]$-complex on $[n]$, where $n^{-h^{-2q}} < \omega < \omega_0$ and $n > n_0$. We start with some preliminary lemmas for flattening and focussing a vector.

Lemma 6.10. *If $J \in (\mathbb{Z}^D)^{\Phi_r}$ is θ-bounded with $\theta > n^{-1/2}$ then there is some $J' \in (\mathbb{Z}^D)^{\Phi_r}$ and $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ such that $\partial\Psi = J - J'$, J' and Ψ are $q^q\omega^{-1}\theta$-bounded, and $U(J')_O < n^{0.1}$ for all $O \in \Phi_r/\Sigma$.*

Proof. Similarly to the proof of Lemma 6.9, for each $O \in \Phi_r/\Sigma$ with representative $\psi^O \in \Phi_{B^O}$, we have $n^O \in \mathbb{Z}^{\mathcal{A}_{B^O}}$ with $|n^O| = U(J)_O$ and $f_{B^O}(J)_{\psi^O} = \sum_\theta n^O_\theta \gamma^\theta$, so $J^O = \sum_\theta n^O_\theta \gamma(\psi^O\theta^{-1})$. Let S be the intset where each (O,θ) appears $|n^O_\theta|$ times with the sign of n^O_θ. For each (O,θ) in S we add to Ψ with the same sign as (O,θ) a uniformly random $\phi \in X_E(\Phi)$ with $E = (\overrightarrow{q}, Im(\theta), \psi^O\theta^{-1})$; then each $\gamma(\psi^O\theta^{-1})$ is cancelled by the corresponding $\gamma(\phi)^O$, where $\theta \in A$, $\phi \in A(\Phi)$.

For any $\psi \in \Phi_{r-1}$ and $k \in [r]$ there are at most $\binom{r-1}{k-1}\theta n^k$ signed elements (O,θ) of S with $|Im(O) \setminus Im(\psi)| = k$. For each such (O,θ), there are at least ωn^{q-r} choices of ϕ, of which at most $(q-r)!n^{q-r-(k-1)}$ contain ϕ, so $\mathbb{P}(\psi \subseteq \phi) \le (q-r)!\omega^{-1}n^{-k+1}$. Then $U(\Psi)_\psi$ is a sum of bounded independent variables with mean at most $0.9q^q\omega^{-1}\theta n$, so whp J' and Φ are $q^q\omega^{-1}\theta$-bounded. Similarly, whp $U(J')_O < n^{0.1}$ for all $O \in \Phi_r/\Sigma$. □

Lemma 6.11. *Suppose $J \in (\mathbb{Z}^D)^{\Phi_r}$ is θ-bounded with $\theta > n^{-1/2}$. Let $L \subseteq \Phi^\circ_r$ be (c,h)-typical in Φ with $d_\Phi(L) > n^{-(3hq)^{-r}}$. Suppose J is (θ,h)-bounded wrt (Φ,L). Then there is some $J' \in (\mathbb{Z}^D)^{\Phi[L]_r}$ and $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ such that $\partial\Psi = J - J'$, J' and Ψ are $2^{2r}\omega^{-1}\theta$-bounded, and J' is $(q!2^{2r}\omega^{-1}\theta, h)$-bounded wrt (Φ,L).*

Proof. We apply the same procedure as in the proof of Lemma 6.10, replacing $X_E(\Phi)$ by $X_E(\Phi,L)$. To spell this out, for each orbit O with representative $\psi^O \in \Phi_{B^O}$, we have $n^O \in \mathbb{Z}^{\mathcal{A}_{B^O}}$ with $|n^O| = U(J)_O$ and $f_{B^O}(J)_{\psi^O} = \sum_\theta n^O_\theta \gamma^\theta$, so $J^O = \sum_\theta n^O_\theta \gamma(\psi^O\theta^{-1})$. Let S be the intset where each (O,θ) appears $|n^O_\theta|$ times with the sign of n^O_θ. For each (O,θ) in S we add to Ψ with the same sign as (O,θ) a uniformly random $\phi^O_\theta \in X_E(\Phi,L)$ with $E = (\overrightarrow{q}, Im(\theta), \psi^O\theta^{-1})$; then each $\gamma(\psi^O\theta^{-1})$ is cancelled by $\gamma(\phi^O_\theta)^O$, where $\theta \in A$, $\phi^O_\theta \in A(\Phi)$.

We claim for any $e' \in L$ that $E_{e'} := \sum_{(O,\theta)} \sum_{B' \in [q]_r \setminus \{Im(\theta)\}} \mathbb{P}(\phi^O_\theta(B') = e') < 1.3q!2^r\omega^{-1}\theta d(L)^{-1}$. To see this, first note that for any $k \in [r]$, as J is (θ,h)-bounded wrt (Φ,L) there are at most $\binom{r}{k}d(L)^{\binom{k+r}{r}-2}\theta n^k$ signed elements (O,θ) of S with $|Im(O) \setminus Im(e')| = k$. As Φ is (ω,h)-extendable and L is (c,h)-typical in Φ, for each such (O,θ), there are at least $0.9d(L)^{Q-1}\omega n^{q-r}$ choices of ϕ^O_θ, of which at most $1.1q!d(L)^{Q-\binom{k+r}{r}}n^{q-r-k}$ have $\phi^O_\theta(B') = e'$ for some $B' \ne Im(\theta)$, so $\mathbb{P}(\phi^O_\theta(B') = e') < 1.3q!\omega^{-1}d(L)^{1-\binom{k+r}{r}}n^{-k}$. The claim follows. Now for any $f \in \Phi_{r-1}$, by typicality $|L(Im(f))| < 1.1d(L)n$, so by the claim $U(\Psi)_f$ is a sum of bounded independent variables with mean at most $1.5q!2^r\omega^{-1}\theta n$, so whp J' and Ψ are $q!2^{r+1}\omega^{-1}\theta$-bounded.

Finally, consider any Φ-extension $E = (H,F,\phi)$ with $H \subseteq [q](h)$, any $G \subseteq H^\circ_r \setminus H^\circ_r[F]$ and $e \in [q](h)^\circ_r \setminus G$ with $e \setminus F \ne \emptyset$. Write $e = Im(\psi)$ with $\psi \in [q](h)$, $E^+ = (H^+,F,\phi)$ with $H^+ = H \cup \psi^{\le}$ and $G^+ = G \cup \{e\}$. Note that $X^{e,J'}_{E,G}(\Phi,L) = \sum_{\phi^+ \in X_{E^+,G^+}(\Phi,L)} U(J')_{\phi^+(e)}$. As L is (c,h)-typical in Φ we have $X_{E^+,G^+}(\Phi,L) < 1.1d(L)^{|G|+1}n^{v_E}$, so by the claim $\mathbb{E}X^{e,J'}_{E,G}(\Phi,L) < 0.9q!2^{2r}\omega^{-1}\theta d(L)^{|G|}n^{v_E}$. Any choice of ϕ^O_θ affects $X^{e,J'}_{E,G}(\Phi,L)$ by $O(n^{v_E-1})$, so by Lemma 3.3 whp $X^{e,J'}_{E,G}(\Phi,L) < q!2^{2r}\omega^{-1}\theta d(L)^{|G|}n^{v_E}$. Thus J' is $(q!2^{2r}\omega^{-1}\theta,h)$-bounded wrt (Φ,L). □

We also need the following estimate for the expected deviation from the mean of a random variable that is a sum of independent indicator variables.

Lemma 6.12. *There is $C > 0$ such that for any sum of independent indicator variables X with mean μ we have $\mathbb{E}|X - \mu| \le C\sqrt{\mu}$.*

Proof. We can assume $\mu > 1$, otherwise we use the bound $\mathbb{E}|X-\mu| \le 2\mu \le 2\sqrt{\mu}$. Write $\mathbb{E}|X-\mu| = \sum_{t\ge 0}|t-\mu|\mathbb{P}(X=t) = E_0 + E_1$, where E_i is the sum of $|t-\mu|\mathbb{P}(X=t)$ over $|t-\mu| > \frac{1}{2}C\sqrt{\mu}$ for $i=0$ or $|t-\mu| \le \frac{1}{2}C\sqrt{\mu}$ for $i=1$. Clearly, $E_1 \le \frac{1}{2}C\sqrt{\mu}$, and by Chernoff bounds, for C large,

$$E_0 \le \sum_{a>\frac{1}{2}C\sqrt{\mu}} a(e^{-a^2/2\mu} + e^{-a^2/2(\mu+a/3)}) \le \tfrac{1}{2}C\sqrt{\mu}. \qquad \square$$

Now we give the proof of our first key lemma, on approximate integral decompositions.

Proof of Lemma 6.1. Suppose $J \in (\mathbb{Z}^D)^{\Phi_r}$ is θ-bounded. By Lemma 6.10 there is some $J^0 \in (\mathbb{Z}^D)^{\Phi_r}$ and $\Psi^0 \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ such that $\partial\Psi^0 = J - J^0$, J^0 and Ψ^0 are $q^q\omega^{-1}\theta$-bounded, and $U(J^0)_O < n^{0.1}$ for all $O \in \Phi_r/\Sigma$.

Let L be ν-random in Φ_r°, where $\nu = n^{-(3hq)^{-r}}$. By Lemma 6.4 whp L is $(n^{-1/3}, h)$-typical in Φ. and by Lemma 6.7 whp J^0 is $(2q^q\omega^{-1}\theta, h)$-bounded wrt (Φ, L).

By Lemma 6.11 there is some $J^1 \in (\mathbb{Z}^D)^{\Phi[L]_r}$ and $\Psi^1 \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ such that $\partial\Psi^1 = J^0 - J^1$, J^1 and Ψ^1 are $(2q)^{2q}\omega^{-2}\theta$-bounded, and J^1 is $((2q)^{2q}\omega^{-2}\theta, h)$-bounded wrt (Φ, L).

By Lemma 6.8 there is some $\omega_q^{-1}\theta/2$-bounded $\Psi^* \in \mathbb{Q}^{\mathcal{A}(\Phi[L])}$ with $\partial\Psi^* = J^1$.

We obtain $\Psi^2 \in \mathbb{Z}^{\mathcal{A}(\Phi[L])}$ from Ψ^* by randomised rounding as follows. For each ϕ with $\Psi^*_\phi \ne 0$, let $s_\phi \in \pm 1$ be the sign of Ψ^*_ϕ, let $m_\phi = \left\lfloor s_\phi \Psi^*_\phi \right\rfloor$, let X_ϕ be independent Bernoulli random variables such that $\Psi^*_\phi = s_\phi(m_\phi + \mathbb{E}X_\phi)$, and let $\Psi^2_\phi = s_\phi(m_\phi + X_\phi)$. Note that each $\mathbb{E}\Psi^2_\phi = \Psi^*_\phi$, so $\mathbb{E}\partial\Psi^2 = \partial\Psi^* = J^1$.

Let $J' = J^1 - \partial\Psi^2$. For any $O \in \Phi[L]_r/\Sigma$ we have $(J')^O = \sum_\phi \gamma(\phi)^O s_\phi(\mathbb{E}X_\phi - X_\phi)$. Separating the positive and negative contributions for each γ-atom a at O we can write $U(J')_O \le \sum_{a\in\pm\gamma[O]}|Y_a - \mu_a|$, where each Y_a is a sum of independent indicator variables with mean $\mu_a \le U(\Psi^*)_O$. By Lemma 6.12 we have $\mathbb{E}U(J')_O < C\sqrt{U(\Psi^*)_O}$, where C depends only on q, D and K. For any $f \in \Phi_{r-1}$, whp $|L(Im(f))| < 1.1\nu n$, so writing $\sum'_O$ for the sum over $O \in \Phi_r/\Sigma$ with $f \subseteq Im(O) \in L$, by Cauchy-Schwartz

$$\begin{aligned}\mathbb{E}U(J')_f &< C\sum_O{}' \sqrt{U(\Psi^*)_O} \le C(1.1\nu n\sum_O{}' U(\Psi^*)_O)^{1/2}\\ &< C(\nu q^r\omega_q^{-1}\theta)^{1/2}n < \omega_q^{3qh}\theta n/2,\end{aligned}$$

as $\nu = n^{-(3hq)^{-r}}$, $\theta > n^{-(4hq)^{-r}}$, $\omega_q = \omega^{(9q)^{q+5}}$, $\omega > n^{-h^{-2q}}$.

Any rounding decision affects $U(J')_f$ by at most 1, so by Lemma 3.3 whp $U(J')_f < \omega_q^{3qh}\theta n$ for all $f \in \Phi_{r-1}$. Similarly, whp $U(\Psi^2)_f < 0.9\omega_q^{-1}\theta n$ for all $f \in \Phi_{r-1}$. Thus J' is $\omega_q^{3qh}\theta$-bounded and $\Psi = \Psi^0 + \Psi^1 + \Psi^2$ is $\omega_q^{-1}\theta$-bounded with $\partial\Psi = J - J'$. $\square$

6.5 Lifts and neighbourhood lattices

This subsection contains some preliminaries for the proof of our second key lemma in the following subsection. Throughout we suppose $\mathcal{A}$ is a $\Sigma^{\le}$-family, $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$ and Φ is a Σ-adapted $[q]$-complex. The construction in the following definition is a technical device for working with neighbourhood lattices.

Definition 6.13. (lifts)

Fix a representative ψ^O for each orbit $O \in \Phi_r/\Sigma$. Given $J \in \Gamma^{\Phi_r}$ we define $J^\uparrow \in (\Gamma^{\Sigma^\leq})^{\Phi_r}$, where for each $O \in \Phi_r/\Sigma$ we let $(J^\uparrow_{\psi^O})_\sigma = J_{\psi^O\sigma}$ where defined, and set all other entries to zero. We call $J^\uparrow$ a lift of J. For $J' \in (\Gamma^{\Sigma^\leq})^{\Phi_r}$ we write $\pi(J') := \sum\{(J'_\psi)_\sigma\{\psi\sigma\} : \psi \in \Phi_r, \sigma \in \Sigma^\leq\}$.

Note that $J = \pi(J^\uparrow)$.

Definition 6.14. (lifted vector systems) Let $\mathcal{A}^\uparrow$ be obtained from $\mathcal{A}$ by including a copy $A^{\overline{\theta}}$ of A for each $A \in \mathcal{A}$ and $\overline{\theta} = (\theta^B : B \in Q)$ with each $\theta^B \in \Sigma^B$.

Suppose $\gamma \in \Gamma^{\mathcal{A}_r}$. Let $\gamma^\uparrow \in (\Gamma^{\Sigma^\leq})^{\mathcal{A}^\uparrow_r}$ where for $\theta \in A^{\overline{\theta}}$ we have $\gamma^\uparrow_\theta = 0$ unless $\theta = \theta^B$ for some $B \in Q$, when $(\gamma^\uparrow_{\theta^B})_\sigma$ is $\gamma_{\theta^B\sigma}$ if $\theta^B \in \Sigma_{B'}$, $\sigma \in \Sigma^{B'}$ or 0 otherwise.

Lemma 6.15.

i. The set of $\gamma^\uparrow$-molecules is the set of lifts of γ-molecules.
ii. The set of $\gamma^\uparrow$-atoms is the set of lifts of γ-atoms.
iii. If $J \in \langle\gamma(\Phi)\rangle$ then any lift $J^\uparrow$ of J is in $\langle\gamma^\uparrow(\Phi)\rangle$, with $U(J^\uparrow)_O = U(J)_O$ for all $O \in \Phi_r/\Sigma$.

Proof. Clearly (ii) and (iii) follow from (i). For (i), consider any $\phi \in A(\Phi)$ and $\phi^\uparrow \in A^{\overline{\theta}}(\Phi)$ with $\phi^\uparrow = \phi$. We claim $\gamma^\uparrow(\phi^\uparrow)$ is a lift of $\gamma(\phi)$. Indeed, for any $O \in \Phi_r/\Sigma$ with $O \subseteq \phi\Sigma$ and $\psi \in O$ we have $\gamma^\uparrow(\phi^\uparrow)_\psi = 0$, except for $\psi = \phi\theta^B \in O$ where $B = \phi^{-1}(Im(O))$, when each $(\gamma^\uparrow(\phi^\uparrow)_\psi)_\sigma = (\gamma^\uparrow(\phi^\uparrow)_{\phi^\uparrow\theta^B})_\sigma = (\gamma^\uparrow_{\theta^B})_\sigma = \gamma_{\theta^B\sigma} = \gamma(\phi)_{\psi\sigma}$. Conversely, any lift of $\gamma(\phi)$ with respect to orbit representatives ψ^O can be expressed as $\gamma^\uparrow(\phi^\uparrow)$ with $\phi^\uparrow \in A^{\overline{\theta}}(\Phi)$ where $\theta^B = \phi^{-1}\psi^O$ for each $B \in Q$, $Im(O) = \phi(B)$. □

We also require some notation and basic properties of neighbourhood lattices (recall the notation of Definitions 2.5 and 2.17).

Definition 6.16. (quotients) Let $\Sigma^* = \Sigma/B^*$ where $B^* \subseteq [q]$ with $r^* = r - |B^*| > 0$. Suppose $\mathcal{A}^*$ is a Σ^*-family that includes a copy A^{θ^*} of $(\Sigma^*)^\leq$ for each $A \in \mathcal{A}$ and $\theta^* \in A_{B^*}$. We call $\gamma^* \in (\mathbb{Z}^D)^{\mathcal{A}^*_{r^*}}$ a B^*-quotient of γ if for each $\theta^* \in A_{B^*}$ there is $\sigma^* = \sigma^*(\theta^*) \in \Sigma$ with $\sigma^*\theta^* = id_{B^*}$ such that $\gamma^*_{\theta'} = \gamma_\theta$ whenever $\theta^* \subseteq \theta \in A_r$, $\theta' = (\sigma^*\theta)/id_{B^*} \in A^{\theta^*}$.

Lemma 6.17. *Let $J \in (\mathbb{Z}^D)^{\Phi_r}$, $\psi^* \in \Phi_{B^*}$, $\Phi^* = \Phi/\psi^*$ and $J^* = J/\psi^*$. Suppose $\gamma^* \in (\mathbb{Z}^D)^{\mathcal{A}^*_{r^*}}$ is a B^*-quotient of $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Then*

i. $\phi' \in \mathcal{A}^*(\Phi^*)$ iff $\phi' = (\phi \circ (\sigma^*)^{-1})/\psi^*$ for some $\phi \in A(\Phi)$, $\theta^* \in A_{B^*}$, $\sigma^* = \sigma^*(\theta^*)$.
ii. for such ϕ', ϕ we have $\gamma(\phi)/\psi^* = \gamma^*(\phi')$,
iii. if $J \in \langle\gamma(\Phi)\rangle$ then $J^* \in \langle\gamma^*(\Phi^*)\rangle$.

Proof. For (i), first consider any $\phi' \in \mathcal{A}^*(\Phi^*)$, say $\phi' \in A^{\theta^*}(\Phi^*)$, and define ϕ such that $\phi' = (\phi \circ (\sigma^*)^{-1})/\psi^*$. As $\phi' \in \Sigma^{*\leq}(\Phi^*) = \Phi^*_{q-|B^*|}$ we have $\phi \circ (\sigma^*)^{-1} \in \Phi$, so $\phi \in \Phi_q = A(\Phi)$. Conversely, consider any $\phi' = (\phi \circ (\sigma^*)^{-1})/\psi^*$ with $\phi \in A(\Phi)$, $\theta^* \in A_{B^*}$, $\sigma^* = \sigma^*(\theta^*)$. For any $\theta' = (\sigma^*\theta)/id_{B^*} \in A^{\theta^*}$ where $\theta^* \subseteq \theta \in A$ we have $\phi\theta \in \Phi$ as $\phi \in A(\Phi)$, so $\phi'\theta' = (\phi\theta)/\psi^* \in \Phi^*$, so $\phi' \in A^{\theta^*}(\Phi^*)$. This proves (i). Furthermore, if $\theta \in A_r$ then $(\gamma(\phi)/\psi^*)_{\phi'\theta'} = \gamma(\phi)_{\phi\theta} = \gamma_\theta = \gamma^*_{\theta'} = \gamma^*(\phi')_{\phi'\theta'}$, so (ii) holds, and (iii) is immediate from (ii). □

6.6 Reducing support

In this section we prove Lemma 6.2, using the inductive hypothesis of Lemma 3.18 if $r > 1$. Our argument will also prove the case $r = 1$, which is the base of the induction. For convenient notation

we rename ω' as ω. Let $V' \subseteq V(\Phi)$ with $|V'| = n/2$ be such that (Φ, V') is (ω, h)-extendable wrt V'. Suppose $J \in \langle \gamma(\Phi) \rangle$ is θ-bounded, where $\theta < \omega_q^{3qh}$. We need to find some $\omega_q^{-2qh}\theta$-bounded $J' \in (\mathbb{Z}^D)^{\Phi[V']_r}$ and $\omega_q^{-2qh}\theta$-bounded $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ with $\partial\Psi = J - J'$.

We will define $J = J^0, \dots, J^r \in \langle \gamma(\Phi) \rangle$ so that $J^j_\psi = 0$ whenever $|Im(\psi) \cap V'| < j$ and J^j is θ_j-bounded, where $\theta_0 = \theta$, $\theta_1 = 2^q\omega^{-1}\theta$ and for $0 < j < r$ we let $\theta_{j+1} = 2^{r^2+2}\eta^{-r+1}M_j^{IH}\theta_j$, where $\eta = (9q)^{-2q}\omega$ and $M_j^{IH} = \omega_{q-r+j}^{-3h} = \omega^{-3h(9(q-r+j))^{q-r+j+5}}$. As $\theta < \omega_q^{3qh}$ we see that $n^{-(5h(q-r+j))^{-j}} < n^{-(5hq)^{-r}} < \theta_j < \omega_{q-r+j}^{3h(q-r+j)}$, i.e. θ_j satisfies the necessary bounds to apply Lemma 4.2 with $(q-r+j, j)$ in place of (q, r), and also that $\theta_r < \omega_q < 2^{-r}\eta$.

We start with $J^0 = J$. To define $J^1 = J - \partial^\gamma\Psi^0$, for each orbit $O \in \Phi_r/\Sigma$ with $Im(O) \cap V' = \emptyset$, we fix a representative, say $\psi^O \in \Phi_{B^O}$, recall $f_{B^O}(J)_{\psi^O} \in \gamma^{B^O}$ by Lemma 2.34, and find $n^O \in \mathbb{Z}^{\mathcal{A}_{B^O}}$ with $|n^O| = U(J)_O$ and $f_{B^O}(J)_{\psi^O} = \sum_\theta n^O_\theta \gamma^\theta$. Then $J^O = \sum_\theta n^O_\theta \gamma[\psi^O]^\theta = \sum_\theta n^O_\theta \gamma(\psi^O\theta^{-1})$.

Let S be the intset where each (O, θ) appears $|n^O_\theta|$ times with the sign of n^O_θ. For each (O, θ) in S we add to Ψ^0 a uniformly random ϕ with $\psi^O\theta^{-1} \subseteq \phi \in A(\Phi)$ and $Im(\phi) \setminus Im(\psi^O) \subseteq V'$, with the same sign as that of (O, θ) in S. Then $\gamma(\psi^O\theta^{-1})$ is cancelled by $\gamma(\phi)$ in $J^1 = J - \partial^\gamma\Psi^0$, and all other ψ with $\gamma(\phi)_\psi \neq 0$ have $Im(\psi) \cap V' \neq \emptyset$. As (Φ, V') is (ω, h)-extendable wrt V' there are at least $\omega(n/2)^{q-r}$ choices for each ϕ, so similarly to Lemma 6.10, whp J^1 and Ψ^0 are θ_1-bounded. If $r = 1$ this completes the construction, so henceforth we suppose $r > 1$.

Given J^j with $0 < j < r$ we will let $J^{j+1} = J^j - \partial^\gamma\Psi^j$, where $|Im(\phi) \cap V'| = j$ whenever $\Psi^j_\phi \neq 0$. To define Ψ^j, we fix any (see Definition 6.13) lift $J^{j\uparrow}$ of J^j with orbit representatives ψ^O for $O \in \Phi_r/\Sigma$, such that writing $B^O = \{i : \psi^O(i) \in V \setminus V'\}$, we have $\psi^{O'} \mid_{B^O} = \psi^O \mid_{B^O}$ for any orbit O' that contains some ψ with $\psi \mid_{B^O} = \psi^O \mid_{B^O}$. Then for each $\psi^* \in \Phi_{r-j}$ with $Im(\psi^*) \cap V' = \emptyset$ such that ψ^* is some $\psi^O \mid_{B^O}$ we consider $\Phi^* = \Phi/\psi^*$, $J^* = J^{j\uparrow}/\psi^*$, $\Sigma^* = \Sigma/B^O$ and some B^O-quotient (see Definition 6.16) $\gamma^* \in ((\mathbb{Z}^D)^{\Sigma^\leq})^{\mathcal{A}^*_{r^*}}$ of $\gamma^\uparrow$. Note that J^* is supported in $\Phi^*[V']_{r^*}$, and $\Phi^*[V']$ is (ω, h)-extendable by Lemma 2.12. Then $J^* \in \langle \gamma^*(\Phi^*) \rangle$ by Lemmas 6.15 and 6.17, so $J^* \in \langle \gamma^*(\Phi^*[V']) \rangle$ by Lemma 5.19 (as in the earlier proof of Lemma 3.18 modulo lemmas).

We will write $J^* = \partial^{\gamma^*}\Psi^*$ for some $\Psi^* = \Psi^{\psi^*} \in \mathbb{Z}^{\mathcal{A}^*(\Phi^*[V'])}$ and define Ψ^j as the sum over all such ψ^* and $\phi' \in \mathcal{A}^*(\Phi^*[V'])$ of $\Psi^{\psi^*}_{\phi'}\{\phi\}$, where $\phi' = (\phi \circ (\sigma^*)^{-1})/\psi^*$ for some $\phi \in A^{\overline{\theta}}(\Phi)$, $\theta^* \in A^{\overline{\theta}}_{B^O}$, $\sigma^* = \sigma^*(\theta^*)$; recall from Lemma 6.17 that any $\phi' \in \mathcal{A}^*(\Phi^*)$ can be thus expressed, and then $\gamma^\uparrow(\phi)/\psi^* = \gamma^*(\phi')$. Then $J^{j+1} = J^j - \partial^\gamma\Psi^j$ will be as required if $\partial^\gamma\Psi^j_\psi = J^j_\psi$ for every $\psi \in \Phi_r$ with $|Im(\psi) \cap V'| = j$.

To see this, consider any $O \in \Phi_r/\Sigma$ with $|B^O| = r - j$ and $\psi = \psi^O\sigma \in O$. Let $\psi^* = \psi^O \mid_{B^O}$, and define J^* and γ^* for ψ^* as above. For each $\psi' \in O$ with $\psi' \mid_{B^O} = \psi^*$ let $\sigma' = \sigma'(\psi')$ be such that $\psi = \psi'\sigma'$. Then

$$\begin{aligned}
\partial^\gamma\Psi^j_\psi &= \pi(\partial^{\gamma^\uparrow}\Psi^j)_\psi = \sum_{\psi'}(\partial^{\gamma^\uparrow}\Psi^j_{\psi'})_{\sigma'} = \sum_{\psi'}\sum_{\phi}\Psi^j_\phi(\gamma^\uparrow(\phi)_{\psi'})_{\sigma'} \\
&= \sum_{\psi'}\sum_{\phi}\Psi^j_\phi((\gamma^\uparrow(\phi)/\psi^*)_{\psi'/\psi^*})_{\sigma'} = \sum_{\psi'}\sum_{\phi' \in \mathcal{A}^*(\Phi^*)}\Psi^{\psi^*}_{\phi'}(\gamma^*(\phi')_{\psi'/\psi^*})_{\sigma'} \\
&= \sum_{\psi'}(\partial^{\gamma^*}\Psi^{\psi^*}_{\psi'/\psi^*})_{\sigma'} = \sum_{\psi'}(J^*_{\psi'/\psi^*})_{\sigma'} = (J^*_{\psi^O/\psi^*})_\sigma = (J^{j\uparrow}_{\psi^O})_\sigma = J^j_\psi, \quad \text{as required.}
\end{aligned}$$

We will construct Ψ^* for each ψ^* as above sequentially using Lemma 4.2 applied to $J^* \in \langle \gamma^*(\Phi^*[V']) \rangle$ which is valid by Lemma 4.1 and the inductive hypothesis of Lemma 3.18, as $\mathcal{A}^*$ is a $(\Sigma^*)^\leq$-family, $\Phi^*[V']$ is Σ^*-adapted (by Lemma 2.18) and (ω, h)-extendable (by Lemma 2.12), and

J^* is θ_j-bounded, as this is true of J^j and so $J^{j\uparrow}$ by Lemma 6.15. Lemma 4.2 will give Ψ^* that is $M_j^{IH}\theta_j$-bounded, and also provides the option to avoid using any η-bounded sets $B_p^* \subseteq \Phi^*[V']_p$ for $j \leq p \leq q-(r-j)$, which we will define below so as to maintain boundedness throughout the algorithm. During the construction of Ψ^j, we say that $\psi \in \Phi$ is full if

i. $|Im(\psi)| = r$ with $U(\Psi^j)_\psi > 0$,
ii. $|Im(\psi)| = r-1$ with $U(\Psi^j)_\psi > \theta_{j+1}n/4 - C_0 - 1$, or
iii. $|Im(\psi)| = i < r-1$ with more than $2^{-r}\eta n - C_0 - 1$ full elements of $\Phi_{i+1}\,|_\psi$.

There will be no uses of full sets by Ψ^j apart from at most $U(J^j)_\psi$ 'forced' uses of ψ for each $\psi \in \Phi_r$, and at most C_0+1 further unforced uses. This implies that for ψ with $|Im(\psi)| = i < r$, if ψ is not full then $\Phi_{i+1}\,|_\psi$ has at most $2^{-r}\eta n$ full elements (if $i = r-1$ we use $\theta_r < 2^{-r}\eta$ here).

We claim that there is no full $\psi' \in \Phi$ with $|Im(\psi') \cap V'| = j-1$. Indeed, suppose we have such ψ'. Let $\psi^a = \psi'[V']$ and $\psi^b = \psi' \setminus \psi^a$. Then

$$\begin{aligned}&(\theta_{j+1}n/4 - C_0 - 1)(2^{-r}\eta n - C_0 - 1)^{r-1-|Im(\psi')|}\\ &< \sum\{U(\Psi^j)_\psi : \psi' \subseteq \psi \in \Phi_{r-1}\} = \sum\{U(\Psi^{\psi^*})_{\psi^a} : \psi^b \subseteq \psi^*\}\\ &< n^{r-j-|\psi^b|} \cdot M_j^{IH}\theta_j n = M_j^{IH}\theta_j n^{r-|Im(\psi')|},\end{aligned}$$

as all Ψ^{ψ^*} are $M_j^{IH}\theta_j$-bounded. This contradicts the definition of θ_{j+1} and so proves the claim.

When applying Lemma 4.2 to $J^* \in \langle \gamma^*(\Phi^*[V'])\rangle$ as above, for $j \leq p \leq q-r+j$ we let B_p^* be the set of $Im(\upsilon)$ with $\upsilon \in \Phi_p^*[V']$ such that $\upsilon \cup \psi^{*\prime}$ is full for some $\psi^{*\prime} \subseteq \psi^*$, and $\upsilon' \cup \psi^{*\prime}$ is not full for any $\upsilon' \subsetneq \upsilon$, $\psi^{*\prime} \subseteq \psi^*$. Then $|B_p^*(Im(\upsilon))| < \eta n$ for all $\upsilon \in \Phi_{p-1}^*[V']$, by definition for $p > j$, and by the claim for $p = j$. Thus we can apply Lemma 4.2 to obtain some $M_j^{IH}\theta_j$-bounded $\Psi^* = \Psi^{\psi^*} \in \mathbb{Z}^{\mathcal{A}^*(\Phi^*[V'])}$ with $\partial^{\gamma^*}\Psi^* = J^*$ such that

i. if $p > j$ then $U(\Psi^*)_\psi \leq 1$ for all $\psi \in \Phi_p[V']$ and $U(\Psi^*)_\psi = 0$ if $Im(\psi) \in B_p^*$,
ii. $U(\Psi^*)_\psi \leq U(J^*)_\psi + C_0 + 1$ for all $\psi \in \Phi_j[V']$, and $U(\Psi^*)_\psi = U(J^*)_\psi$ if $Im(\psi) \in B_j^*$.

As described above, Ψ^j is the sum over all such ψ^* and $\phi' \in \mathcal{A}^*(\Phi^*[V'])$ of $\Psi_{\phi'}^{\psi^*}\{\phi\}$, where $\phi' = (\phi \circ (\sigma^*)^{-1})/\psi^*$ for some $\phi \in A^{\overline{\theta}}(\Phi)$, $\theta^* \in A_{B^O}^{\overline{\theta}}$, $\sigma^* = \sigma^*(\theta^*)$. We claim that Ψ^j is $\theta_{j+1}/2$-bounded. To see this, we fix $\psi \in \Phi_{r-1}$, let $\psi^a = \psi[V']$, $\psi^b = \psi \setminus \psi^a$ and consider cases according to $p = |Im(\psi^a)|$.

i. If $p = j-1$ then $U(\Psi^j)_\psi = U(\Psi^{\psi^b})_{\psi^a} < M_j^{IH}\theta_j n < \theta_{j+1}n/2$, as Ψ^{ψ^b} is $M_j^{IH}\theta_j$-bounded,
ii. If $p > j$ then $U(\Psi^j)_\psi \leq \theta_{j+1}n/4 - C_0 - 1$ before ψ is full, after which $U(\Psi^{\psi^*})_{\psi^a} = 0$ whenever $\psi^b \subseteq \psi^*$, except for at most one such ψ^* with $U(\Psi^{\psi^*})_{\psi^a} \leq C_0 + 1$, so $U(\Psi^j)_\psi < \theta_{j+1}n/4 \leq \theta_{j+1}n/2$.
iii. If $p = j$ then $U(\Psi^j)_\psi \leq \theta_{j+1}n/4 - C_0 - 1$ before ψ is full, after which $U(\Psi^{\psi^*})_{\psi^a} = U(J^j)_{\psi^a \cup \psi^*}$ whenever $\psi^b \subseteq \psi^*$, except for at most one such ψ^* with $U(\Psi^{\psi^*})_{\psi^a} \leq U(J^j)_{\psi^a \cup \psi^*} + C_0 + 1$, so $U(\Psi^j)_\psi \leq \theta_{j+1}n/4 + U(J^j)_\psi \leq \theta_{j+1}n/2$, as $U(J^j)_\psi \leq \theta_j n$.

Thus $U(\Psi^j)_\psi \leq \theta_{j+1}n/2$ in all cases, so the claim holds.

It follows that $J^{j+1} = J^j - \partial\Psi^j$ is θ_{j+1}-bounded, so the construction can proceed to the next step. We conclude with $J' := J^r \in (\mathbb{Z}^D)^{\Phi[V']_r}$ and $\Psi = \sum_j \Psi^j \in \mathbb{Z}^{\mathcal{A}(\Phi)}$, such that J' and Ψ are $\omega_q^{-2qh}\theta$-bounded with $\partial\Psi = J - J'$. □

7 Applications

In this section we give several applications of our main theorem, including the theorems stated in the introduction of the paper. Most of the applications will follow from a decomposition theorem for hypergraphs in various partite settings. We also give some results on coloured hypergraph decomposition, and a simple illustration (the Tryst Table Problem) of other applications that are not equivalent to hypergraph decomposition, but for the sake of brevity we leave a detailed study of these applications for future research.

Our first theorem in this section can be viewed as a simplified form of Theorem 3.1, in which various general definitions are specialised to the setting of hypergraph decompositions. To state it we require two definitions.

Definition 7.1. Let Φ be a $[q]$-complex and H be an r-graph on $[q]$. We say $G \in \mathbb{N}^{\Phi_r^\circ}$ is (H,c,ω)-regular in Φ if there are $y_\phi \in [\omega n^{r-q}, \omega^{-1}n^{r-q}]$ for each $\phi \in \Phi_q$ with $\phi(H) \subseteq G$ so that $\sum_\phi y_\phi \phi(H) = (1 \pm c)G$.

Definition 7.2. We say that an R-complex Φ is exactly Σ-adapted if whenever $\phi \in \Phi_B$ and $\tau \in Bij(B', B)$ we have $\phi \circ \tau \in \Phi_{B'}$ iff $\sigma \in \Sigma^B_{B'}$. We say Φ is exactly adapted if Φ is exactly Σ-adapted for some Σ.

Theorem 7.3. *Let H be an r-graph on $[q]$ and Φ be an (ω,h)-extendable exactly adapted $[q]$-complex where $n = |V(\Phi)| > n_0(q)$ is large, $h = 2^{50q^3}$, $\delta = 2^{-10^3q^5}$, $n^{-\delta} < \omega < \omega_0(q)$ is small and $c = \omega^{h^{20}}$. Suppose $G \in \langle H(\Phi)\rangle$ is (H,c,ω)-regular in Φ and (Φ, G) is (ω,h)-extendable. Then G has an H-decomposition in Φ_q.*

Proof. Suppose Φ is exactly Σ-adapted, let $\mathcal{A} = \{A\}$ with $A = \Sigma^{\leq}$, and $\gamma \in \{0,1\}^{A_r}$ with each $\gamma_\theta = 1_{Im(\theta)\in H}$. Let $G^* \in \mathbb{N}^{\Phi_r}$ with $G^*_\psi = G_{Im(\psi)}$ for $\psi \in \Phi_r$. For any $\phi \in A(\Phi) = \Phi_q$ and $\theta \in A_r$ we have $\gamma(\phi)_{\phi\theta} = \gamma_\theta = 1_{Im(\theta)\in H}$. As Φ is exactly Σ-adapted, we deduce $G \in \langle H(\Phi)\rangle$ iff $G^* \in \langle \gamma(\Phi)\rangle$, and that an H-decomposition of G is equivalent to a $\gamma(\Phi)$-decomposition of G^*.

There are two types in γ for each $B \in [q]_r$: the edge type $\{\theta \in A_B : Im(\theta) \in H\}$ and the nonedge type $\{\theta \in A_B : Im(\theta) \notin H\}$. Each γ^θ is the all-1 vector for θ in an edge type or the all-0 vector for θ in a nonedge type, so γ is elementary. The atom decomposition of G^* is $G^* = \sum_{e\in\Phi_r^\circ} G_e e^*$, where $e^*_\psi = 1$ for all $\psi \in \Phi_r$ with $Im(\psi) = e$, i.e. e^* contains all edge types at e.

As G is (H,c,ω)-regular in Φ, we have $\sum_\phi y_\phi\phi(H) = (1\pm c)G$ for some $y_\phi \in [\omega n^{r-q}, \omega^{-1}n^{r-q}]$ for each $\phi \in \Phi_q$ with $\phi(H) \subseteq G$. For any such ϕ we have $\gamma(\phi) \leq_\gamma G$, so $\phi \in \mathcal{A}(\Phi, G)$. Also, for any $B \in [q]_r$ and $\psi \in \Phi_B$, writing $1_B \in T_B$ for the edge type we have $\partial^{1_B} y_\psi = \sum_{\phi: t_\phi(\psi)=1_B} y_\phi = \sum\{y_\phi : Im(\psi) \in \phi(H)\} = (1\pm c)(G^*)^{1_B}_\psi$, so G^* is (γ, c, ω)-regular.

To apply Theorem 3.1, it remains to show that $(\Phi, \gamma[G])$ is (ω,h)-extendable. We have $\gamma[G] = (\gamma[G]_B : B \in Q)$ where if $B \notin H$ then $\gamma[G]_B = \Phi_B$ and if $B \in H$ then $\gamma[G]_B = \{\psi \in \Phi_B : G_{Im(\psi)} > 0\}$. Let $E = (J, F, \phi)$ be any Φ-extension of rank s and $J' \subseteq J_r \setminus J[F]$. As (Φ, G) is (ω,h)-extendable we have $X_{E,J'}(\Phi, G) > \omega n^{v_E}$. Consider any $\phi^+ \in X_{E,J'}(\Phi, G)$. For any $\psi \in J'$ we have $\phi^+\psi \in \Phi$ and $Im(\phi^+\psi) \in G$, so $\phi^+\psi \in \gamma[G]$. Thus $\phi^+ \in X_{E,J'}(\Phi, \gamma[G])$, so $(\Phi, \gamma[G])$ is (ω,h)-extendable. Now G^* has a $\gamma(\Phi)$-decomposition, so G has an H-decomposition. □

The following theorem solves the H-decomposition problem in the nonpartite setting (so is similar in spirit to [9]); it is an immediate corollary of Theorem 7.3 with $\Sigma = S_q$ and Theorem 5.20.

Theorem 7.4. *Let H be an r-graph on $[q]$ and Φ be an (ω,h)-extendable S_q-adapted $[q]$-complex where $n = |V(\Phi)| > n_0(q)$ is large, $h = 2^{50q^3}$, $\delta = 2^{-10^3q^5}$, $n^{-\delta} < \omega < \omega_0(q)$ is small and $c = \omega^{h^{20}}$.*

Suppose G is H-divisible and (H,c,ω)-regular in Φ and (Φ,G) is (ω,h)-extendable. Then G has an H-decomposition in Φ_q.

In the introduction we stated a simplified form of this result (using typicality rather than extendability and regularity); we now give the deduction. (It will also follow from a later more general result, but we include the proof for the sake of exposition.)

Proof of Theorem 1.5. Let H be an r-graph on $[q]$ and G be an H-divisible (c,h^q)-typical r-graph, where $n=|V(\Phi)|>n_0(q)$, $h=2^{50q^3}$, $\delta=2^{-10^3q^5}$, $d(G)>2n^{-\delta/h^q}$ and $c<c_0d(G)^{h^{30q}}$. We need to show that G has an H-decomposition. Let Φ be the complete $[q]$-complex on $V(G)$, i.e. each $\Phi_B=Inj(B,V(G))$. Then Φ is exactly S_q-adapted.

We claim that (Φ,G) is (ω,h)-extendable with $\omega=\frac{1}{2}d(G)^{h^q}$. To see this, consider any Φ-extension $E=(J,F,\phi)$ with $J\subseteq[q](h)$ and any $J'\subseteq J_r^\circ\setminus J^\circ[F]$. Write $V(J)\setminus F=\{x_1,\dots,x_{v_E}\}$ and suppose for $i\in[v_E]$ that there are m_i edges of J' that use x_i but no x_j with $j>i$. The number of choices for the embedding of each x_i given any previous choices is $(1\pm m_ic)d(G)^{m_i}n$. Thus $X_{E,J'}(\Phi,G)>\frac{1}{2}d(G)^{e_E}n^{v_E}>\omega n^{v_E}$, as claimed.

Furthermore, if $e\in G$, $f\in H$, $J=\overrightarrow{q}$, $F=f$, $\psi\in Bij(f,e)$ we see that there are $(1\pm|H|c)d(G)^{|H|-1}n^{q-r}$ extensions of ψ to $\phi\in\Phi_q$ with $e\in\phi(H)\subseteq G$. Defining $y_\phi=d(G)^{1-|H|}n^{r-q}$ for all $\phi\in\Phi_q$ with $\phi(H)\subseteq G$ we see that G is $(H,|H|c,\omega)$-regular in Φ. The theorem now follows from Theorem 7.4. □

Our next definition sets up notation for hypergraph decompositions in a generalised partite setting that incorporates several earlier examples in the paper. It is followed by a theorem that solves the corresponding hypergraph decomposition problem.

Definition 7.5. Let H be an r-graph on $[q]$ and $\mathcal{P}=(P_1,\dots,P_t)$ be a partition of $[q]$. Let Σ be the group of all $\sigma\in S_q$ with all $\sigma(P_i)=P_i$. Let Φ be an exactly Σ-adapted $[q]$-complex with parts $\mathcal{Q}=(Q_1,\dots,Q_t)$, where each $Q_i=\{\psi(j):j\in P_i,\psi\in\Phi_j\}$. Let $G\in\mathbb{N}^{\Phi_r^\circ}$.

For $S\subseteq[q]$ the $\mathcal{P}$-index of S is $i_{\mathcal{P}}(S)=(|S\cap P_1|,\dots,|S\cap P_t|)$; similarly, we define the $\mathcal{Q}$-index of subsets of $V(\Phi)$, and also refer to both as the 'index'. For $i\in\mathbb{N}^t$ we let H_i and G_i be the (multi)sets of edges in H and G of index i. Let $I=\{i:H_i\neq\emptyset\}$. We call G an $(H,\mathcal{P})$-blowup if $G_i\neq\emptyset\Rightarrow i\in I$.

We say G has a $\mathcal{P}$-partite H-decomposition if it has an H-decomposition using copies $\phi(H)$ of H with all $\phi(P_i)\subseteq Q_i$.

For $e\subseteq V(\Phi)$ we define the degree vector $G_I(e)\in\mathbb{N}^I$ by $G_I(e)_i=|G_i(e)|$ for $i\in I$. Similarly, for $f\subseteq[q]$ we define $H_I(f)$ by $H_I(f)_i=|H_i(f)|$. For $i'\in\mathbb{N}^t$ let $H^I_{i'}$ be the subgroup of $\mathbb{N}^I$ generated by $\{H_I(f):i_{\mathcal{P}}(f)=i'\}$. We say G is $(H,\mathcal{P})$-divisible if $G_I(e)\in H^I_{i'}$ whenever $i_{\mathcal{P}}(e)=i'$.

Let $G^*\in\mathbb{N}^{\Phi_r}$ with $G^*_\psi=G_e$ for $\psi\in\Phi_r$, $e=Im(\psi)$ with $i_{\mathcal{Q}}(e)\in I$, and G^*_ψ is otherwise undefined.

Theorem 7.6. *With notation as in Definition 7.5, suppose $n/h\le|Q_i|\le n$ with $n>n_0(q)$ large, G is an $(H,\mathcal{P})$-divisible $(H,\mathcal{P})$-blowup, G is (H,c,ω)-regular in Φ, and (Φ,G^*) is (ω,h)-extendable, where $h=2^{50q^3}$, $\delta=2^{-10^3q^5}$, $n^{-\delta}<\omega<\omega_0(q)$ is small and $c=\omega^{h^{20}}$. Then G has a $\mathcal{P}$-partite H-decomposition.*

Proof. Let $A=\Sigma^{\le}$, $H^*=\{\theta\in A_r:Im(\theta)\in H\}$ and $\gamma_\theta=1_{\theta\in H^*}$ for $\theta\in A_r$, Then an (integral) $\mathcal{P}$-partite H-decomposition of G is equivalent to an (integral) $\gamma(\Phi)$-decomposition of G^*. By Theorem 7.3, it remains to show $G\in\langle H(\Phi)\rangle$, i.e. $G^*\in\langle\gamma(\Phi)\rangle=\mathcal{L}_\gamma(\Phi)$ (by Lemma 5.19).

Consider any $i\in I=\{i:H_i\neq\emptyset\}$ and $i'\in\mathbb{N}^t$ with all $i'_j\le i_j$. Let $m^i_{i'}=\prod_{j\in[t]}(i_j-i'_j)!$. For any $B'\subseteq B\in Q$ with $i_{\mathcal{P}}(B')=i'$ and $i_{\mathcal{P}}(B)=i$ and $\psi'\in\Phi_{B'}$ with $Im(\psi')=e$ we have

$((G^*)^\sharp_{\psi'})_B) = \sum\{G^*_\psi : \psi' \subseteq \psi \in \Phi_B\} = m^i_{i'}|G_i(e)|$. Writing $O = \psi'\Sigma$, for any $\psi \in O$ we have $((G^*)^\sharp_\psi)_B) = m^i_{i'}|G_i(e)|$. Thus we obtain $(G^*)^\sharp_\psi$ from $G_I(e)$ by copying coordinates and multiplying all copies of each i-coordinate by $m^i_{i'}$.

Similarly, for any $\theta' \in A_{B'}$ with $Im(\theta') = f$ we have $(\gamma^\sharp_{\theta'})_B) = \sum\{\gamma_\theta : \theta' \subseteq \theta \in A_B\} = m^i_{i'}|H_i(f)|$, so $\langle\gamma^\sharp[O]\rangle$ is generated by vectors $v^f \in (\mathbb{Z}^Q)^O$ where $f \subseteq [q]$ with $i_{\mathcal{P}}(f) = i'$ and for each $\psi' \in O$, $B \in Q$ we have $(v^f_{\psi'})_B = m^i_{i'}|H_i(f)|$ where $i = i_{\mathcal{P}}(B)$. Thus all vectors in $\langle\gamma^\sharp[O]\rangle$ are obtained from vectors in $H^I_{i'}$ by the same transformation that maps $G_I(e)$ to $((G^*)^\sharp)^O$. As G is $(H,\mathcal{P})$-divisible we deduce $((G^*)^\sharp)^O \in \langle\gamma^\sharp[O]\rangle$ for any $O \in \Phi/\Sigma$, as required. □

Similarly to the nonpartite setting, which also give a simplified form of the previous theorem in which we replace extendability and regularity by typicality (which we will generalise here to multigraphs).

Definition 7.7. With notation as in Definition 7.5, suppose that Φ is the $[q]$-complex where each Φ_B consists of all maps $\psi : B \to V(G)$ with $\psi(B \cap P_i) \subseteq Q_i$ for all $i \in [t]$; we call Φ the complete $[q]$-complex wrt $(\mathcal{P},\mathcal{Q})$, and note that Φ is exactly Σ-adapted. For $f \in Q$ let $|G^* \cap \Phi_f| = \sum\{G^*_\psi : \psi \in \Phi_f\}$ and $d_f(G^*) = |G^* \cap \Phi_f|/|\Phi_f|$.

For any Φ-extension $E = (J,F,\phi)$ and $J' \subseteq J^\circ_r \setminus J^\circ[F]$ let $X_{E,J'}(\Phi,G) = \sum_{\phi' \in X_E(\Phi)} \prod_{e \in J'} G_{\phi'(e)}$.

We call G a (c,s)-typical $(H,\mathcal{P})$-blowup if for every Φ-extension $E = (J,F,\phi)$ of rank s and $J' \subseteq J^\circ_r \setminus J^\circ[F]$ we have $X_{E,J'}(\Phi,G) = (1 \pm c)X_E(\Phi)\prod_{e \in J'} d_e(G^*)$, where for $e \in J^\circ_f$ we write $d_e(G^*) = d_f(G^*)$.

Now we give our theorem on decompositions of typical multigraphs in the generalised partite setting. We note that the case $\mathcal{P} = ([q])$ implies Theorem 1.5 and the case $\mathcal{P} = (\{1\},\dots,\{q\})$ implies Theorem 1.7.

Theorem 7.8. *Let H be an r-graph on $[q]$ and $\mathcal{P} = (P_1,\dots,P_t)$ be a partition of $[q]$. Suppose each $n/h \le |Q_i| \le n$ with $n > n_0(q)$ and $h = 2^{50q^3}$, that $\delta = 2^{-10^3q^5}$, $d > 2n^{-\delta/h^q}$ and $c < c_0 d^{h^{30q}}$, where $c_0 = c_0(q)$ is small. Let G be an $(H,\mathcal{P})$-divisible (c,h)-typical $(H,\mathcal{P})$-blowup wrt $\mathcal{Q} = (Q_1,\dots,Q_t)$, such that $d_f(G) > d$ for all $f \in H$ and $G_e < d^{-1}$ for all $e \in [n]_r$. Then G has a $\mathcal{P}$-partite H-decomposition.*

Proof. With notation as in Definition 7.5, it follows (as in the proof of Theorem 1.5) from the definition of (c,h)-typical $(H,\mathcal{P})$-blowup that (Φ,G^*) is (ω,h)-extendable with $\omega = \frac{1}{2}d^{h^q} > n^{-\delta}$. By Theorem 7.6 it remains to show that G is $(H,2c,\omega)$-regular in Φ.

To see this, first note that as G is $(H,\mathcal{P})$-divisible we have $G_I(\emptyset) \in H^I_0 = \langle H_I(\emptyset)\rangle$, so there is some integer Y such that $|G_i| = Y|H_i|$ for all $i \in I$. For each $\phi \in \Phi_q$ with $\phi(H) \subseteq G$ we let $y_\phi = YZ^{-1}\prod_{f \in H} G_{\phi(f)}$, where $Z = \prod_{j \in [t]} |Q_j|^{|P_j|}\prod_{f \in H} d_f(G^*)$. Then $y_\phi \in [\omega n^{r-q}, \omega^{-1}n^{r-q}]$ for each such ϕ.

We need to show for any $e \in \Phi^\circ_r$ that $\sum\{y_\phi : e \in \phi(H)\} = (1 \pm 2c)G_e$. We can suppose $G_e \neq 0$, so $i = i(e) \in I$. Let $m_i = \prod_{j \in [t]} i_j!$. For any $B \in H_i$ there are m_i choices of $\psi \in \Phi_B$ with $\psi(B) = e$. It suffices to show for any such B and ψ that $\sum\{y_\phi : \psi \subseteq \phi\} = (1 \pm 2c)G_e/(m_i|H_i|)$.

Let $E = (\overrightarrow{q}, B, \psi)$ and $J' = H\setminus\{B\}$. As G is a (c,h)-typical $(H,\mathcal{P})$-blowup we have $X_{E,J'}(\Phi,G) = (1 \pm c)X_E(\Phi)\prod_{f \in J'} d_f(G^*) = (1 \pm 2c)Z/(m_i|G_i|)$, as $X_E(\Phi) = (1 + O(n^{-1}))\prod_{j \in [t]} |Q_j|^{|P_j|-i_j}$ and $m_i|G_i| = |G^* \cap \Phi_B| = d_B(G^*)|\Phi_B| = (1 + O(n^{-1}))d_B(G^*)\prod_{j \in [t]} |Q_j|^{i_j}$. Therefore

$$\sum\{y_\phi : \psi \subseteq \phi\} = YZ^{-1}\sum_{\phi \in X_E(\Phi)}\prod_{f \in H} G_{\phi(f)} = G_eYZ^{-1}X_{E,J'}(\Phi,G) = (1 \pm 2c)G_e/(m_i|H_i|). \quad \square$$

Now we will prove several other theorems stated in the introduction for which we gave an equivalent reformulation in terms of hypergraph decompositions in partite settings as above. We start with the existence of resolvable designs, or more generally, resolvable hypergraph decompositions of multigraphs.[13] We deduce Theorem 1.1 by applying Theorem 7.9 with $H = K_q^r$ and $G = \lambda K_n^r$.

Theorem 7.9. *Let H be a vertex-regular r-graph on $[q]$ and G be a vertex-regular H-divisible r-multigraph on $[n]$ where $n > n_0(q)$ is large and $q \mid n$. Let Φ be a S_q-adapted $[q]$-complex on $V(G)$. Suppose G is (H, c, ω)-regular in Φ and (Φ, G) is (ω, h)-extendable, where $h = 2^{50q^3}$, $\delta = 2^{-10^3q^5}$, $n^{-\delta/2h} < \omega < \omega_0$, $c = \omega^{h^{22}}$. Then G has a resolvable H-decomposition.*

Proof. We start by recalling the equivalent partite hypergraph decomposition problem. Let Y be a set of m vertices disjoint from X, where m is the least integer with $\binom{m}{r-1} \geq q|G|/|H|n$. Let J be a random $(r-1)$-graph on Y with $|J| = q|G|/|H|n$. Let G' be the r-multigraph obtained from G by adding as edges (with multiplicity one) all r-sets of the form $f \cup \{x\}$ where $f \in J$ and $x \in X$. Let H' be the r-graph whose vertex set is the disjoint union of a q-set $A = [q]$ and an $(r-1)$-set B, and whose edges consist of all r-sets in $A \cup B$ that are contained in H or have exactly one vertex in A. To adopt the notation of Definition 7.5 we let $\mathcal{P} = (P_1, P_2)$ with $P_1 = A$, $P_2 = B$ and $\mathcal{Q} = (Q_1, Q_2)$ with $Q_1 = X$, $Q_2 = Y$. Then G' is an $(H', \mathcal{P})$-blowup and we wish to find a $\mathcal{P}$-partite H'-decomposition of G'.

First we check $(H', \mathcal{P})$-divisibility. The set of edge indices is $I = \{(r, 0), (1, r-1)\}$. We identify $\mathbb{N}^I$ with $\mathbb{N}^2$ by assigning $(r, 0)$ to the first coordinate and $(1, r-1)$ to the second. Let $i' \in \mathbb{N}^2$. Suppose $i'_2 > 0$. Then ${H'}^I_{i'}$ is $\{(0, 0)\}$ unless $i'_1 \leq 1$ and $i'_2 \leq r-1$, in which case ${H'}^I_{i'}$ is generated by $(0, 1)$ if $i'_1 = 1$ or $(0, q)$ if $i'_1 = 0$. The resulting $(H', \mathcal{P})$-divisibility conditions ($G_I(e) \in {H'}^I_{i'}$ whenever $i_{\mathcal{P}}(e) = i'$) are satisfied trivially when $i'_1 = 1$ and as $q \mid n$ when $i'_1 = 0$. Now suppose $i'_2 = 0$. Then ${H'}^I_{i'}$ is generated by all $(|H(f)|, 0)$ with $f \in [q]_{i'_1}$ if $i'_1 > 1$ or by $(|H(f)|, 1)$ with $f \in [q]_{i'_1}$ if $i'_1 \leq 1$. If $i'_1 > 1$ then the $(H', \mathcal{P})$-divisibility condition is equivalent to the H-divisibility condition that $gcd_{i'_1}(H)$ divides $|G(e)|$. For $i'_1 = 0$ we need $(|G|, n|J|)$ to be an integer multiple of $(|H|, q)$, so we require the H-divisibility condition $|H| \mid |G|$ and also $|J| = q|G|/|H|n$. For $i'_1 = 1$ we need $(|G(x)|, |J|)$ to be an integer multiple of $(rq^{-1}|H|, 1)$ for any $x \in X$ (recall that H is vertex-regular), so we require the K_q^r-divisibility condition that $rq^{-1}|H|$ divides $|G(x)|$ and also that G is vertex-regular. Therefore G' is $(H', \mathcal{P})$-divisible.

To apply Theorem 7.6, it remains to check extendability and regularity. We let Φ' be the $(A \cup B)$-complex where each $\Phi'_{A' \cup B'}$ for $A' \subseteq A$, $B' \subseteq B$ consists of all $\phi \in Inj(A' \cup B', X \cup Y)$ with $\phi \mid_{A'} \in \Phi$ and $\phi(B') \subseteq Y$. Consider any Φ'-extension $E = (J, F, \phi)$ with $J \subseteq (A \cup B)(h)$ and any $J' \subseteq J_r \setminus J[F]$ with $J'_{B'} \neq \emptyset \Rightarrow i_{\mathcal{P}}(B') \in I$. Let $E(A)$ and $J'(A)$ be obtained by restricting to $A(h)$, and define $E(B)$ similarly. Then whp $X_{E,J'}(\Phi', G') = (1 + o(1))X_{E(A),J'(A)}(\Phi, G)m^{v_{E(B)}}$, where $X_{E(A),J'(A)}(\Phi, G) > \omega n^{v_{E(A)}}$ as (Φ, G) is (ω, h)-extendable. As $|G| > \omega n^r$ we have $m \geq (q\omega/Q)^{1/(r-1)}n$, so $X_{E,J'}(\Phi', G') > \omega^{2h}(n+m)^{v_E}$ (say), i.e. (Φ', G') is (ω^{2h}, h)-extendable.

For regularity, as G is (H, c, ω)-regular in Φ we can choose $y_\phi \in [\omega n^{r-q}, \omega^{-1}n^{r-q}]$ for each $\phi \in \Phi_A$ with $\phi(H) \subseteq G$ with $\sum\{y_\phi : e \in \phi(H)\} = (1 \pm c)G_e$ for any $e \in X_r$. We define $y'_{\phi'} = y_{\phi'|_A}((r-1)!|J|)^{-1}$ for each $\phi' \in \Phi'_{A \cup B}$ with $\phi'(B) \in J$. Then $y'_{\phi'} \in [\omega^{2h}(n+m)^{1-q}, \omega^{-2h}(n+m)^{1-q}]$ for all $\phi' \in \Phi'_{A \cup B}$ with $\phi'(H') \subseteq G'$. For any $e \in X_r$ we have $\sum\{y'_{\phi'} : e \in \phi'(H')\} = \sum\{y_\phi : e \in \phi(H)\} = (1 \pm c)G'_e$.

[13] For the sake of brevity we just consider the case that H is vertex-regular, and leave the computation of the lattice for general H to the reader (if H is not vertex-regular than we need G the difference between any two vertex degrees in G to be divisible by the gcd of all differences of vertex degrees in H).

Also, for any $e = f \cup \{x\}$ with $x \in X$ and $f \in J$, we have

$$\begin{aligned}\sum\{y'_{\phi'} : e \in \phi'(H')\} &= |J|^{-1}\sum\{y_\phi : x \in Im(\phi)\}\\ &= (|J|rq^{-1}|H|)^{-1}\sum_{x\in e'\in X_r}\sum\{y_\phi : e' \in \phi(H)\}\\ &= (|J|rq^{-1}|H|)^{-1}(1\pm c)|G(x)|.\end{aligned}$$

As G is vertex-regular, $|G(x)| = r|G|/n = r|H||J|/q$, so $\sum\{y'_{\phi'} : e \in \phi'(H')\} = 1 \pm c = (1 \pm c)G'_e$. Therefore G' is (H', c, ω^{2h})-regular in Φ'. □

Next we show the existence of large sets of designs. Again, we first consider the more general setting of decompositions of multidesigns into designs.

Theorem 7.10. *Let Φ be a S_q-adapted $[q]$-complex with $V(\Phi) = [n]$ where $n > n_0(q, \lambda)$ is large. Suppose $G \in \mathbb{N}^{\Phi^\circ_q}$ is an r-multidesign with all $G_e < \omega^{-1}$. For all $0 \le i \le r$ suppose $Z_i := \lambda\binom{q-i}{r-i}^{-1}\binom{n-i}{r-i} \in \mathbb{Z}$ and $Z_i \mid |G(f)|$ for all $f \in [n]_i$. Suppose also that (Φ, G) is (ω, h)-extendable, where $h = 2^{50q^3}$, $\delta = 2^{-10^3q^5}$, $n^{-\delta/2h} < \omega < \omega_0(q, \lambda)$. Then G has a decomposition into (n, q, r, λ)-designs.*

Proof. We start by recalling the equivalent partite hypergraph decomposition problem. Let Y be a set of m vertices disjoint from X, where m is the least integer with $\binom{m}{q-r} \ge |G|/Z_0$. Let J be a random $(q-r)$-graph on Y with $|J| = |G|/Z_0$. Let G' be the q-multigraph obtained from G by adding as edges with multiplicity λ all q-sets of the form $e \cup f$ with $e \subseteq X$ and $f \in J$. Let H be the q-graph whose vertex set is the disjoint union of a q-set A and a $(q-r)$-set B, and whose edges consist of A and all q-sets in $A \cup B$ that contain B. To adopt the notation of Definition 7.5 we let $\mathcal{P} = (P_1, P_2)$ with $P_1 = A$, $P_2 = B$ and $\mathcal{Q} = (Q_1, Q_2)$ with $Q_1 = X$, $Q_2 = Y$. Then G' is an $(H, \mathcal{P})$-blowup and we wish to find a $\mathcal{P}$-partite H-decomposition of G'.

First we check $(H, \mathcal{P})$-divisibility. We identify $\mathbb{N}^I$ with $\mathbb{N}^2$ by assigning $(q, 0)$ to the first coordinate and $(r, q-r)$ to the second. Let $i' \in \mathbb{N}^2$. Suppose $i'_2 > 0$. Then $H^I_{i'}$ is $\{(0,0)\}$ unless $i'_1 \le r$ and $i'_2 \le q-r$, in which case $H^I_{i'}$ is generated by $(0, \binom{q-i'_1}{r-i'_1})$. The corresponding $(H, \mathcal{P})$-divisibility condition is $Z_{i'_1} \in \mathbb{Z}$. Now suppose $i'_2 = 0$. Then $H^I_{i'}$ is generated by $(1, 0)$ if $i'_1 > r$ or by $(1, \binom{q-i'_1}{r-i'_1})$ if $i'_1 \le r$. The corresponding $(H, \mathcal{P})$-divisibility condition is trivial if $i'_1 > r$. If $i'_1 \le r$, for each $f \in [n]_{i'_1}$ we need $(|G(f)|, \lambda|J|\binom{n-i'_1}{r-i'_1})$ to be an integer multiple of $(1, \binom{q-i'_1}{r-i'_1})$, i.e. $|G(f)| = |J|Z_{i'_1}$; this is equivalent to G being an r-multidesign. Therefore G' is $(H, \mathcal{P})$-divisible.

To apply Theorem 7.6, it remains to check extendability and regularity. We let Φ' be the $(A \cup B)$-complex where each $\Phi'_{A'\cup B'}$ for $A' \subseteq A$, $B' \subseteq B$ consists of all $\phi \in Inj(A' \cup B', X \cup Y)$ with $\phi\,|_{A'} \in \Phi$ and $\phi(B') \subseteq Y$. Then (Φ', G') is (ω^{2h}, h)-extendable as in the proof of Theorem 7.9. For regularity, we define $y_\phi = G_{\phi(A)}(q!(q-r)!|J|)^{-1}$ for each $\phi \in \Phi'_{A\cup B}$ with $\phi(B) \in J$. Then $y_\phi \in [\omega^{2h}(n+m)^{r-q}, \omega^{-2h}(n+m)^{r-q}]$ for all $\phi \in \Phi'_{A\cup B}$ with $\phi(H) \subseteq G'$. For any $e \in X_q$ we have $\sum\{y_\phi : e \in \phi(H)\} = G_e = G'_e$. For any $e = f \cup f'$ with $f \in X_r$ and $f' \in J$, as G is an r-multidesign we have $|G(f)| = Q\binom{n}{r}^{-1}|G| = \lambda|G|/Z_0$, so $\sum\{y_\phi : e \in \phi(H)\} = (q!(q-r)!|J|)^{-1}q!|G(f)|(q-r)! = |G(f)|Z_0/|G| = \lambda = G'_e$. □

Now we prove the existence conjecture for large sets of designs.

Proof of Theorem 1.2. First we note that the case that λ is fixed and $n > n_0(q, \lambda)$ is large follows from Theorem 7.10 applied to $G = K^q_n$. Now we can assume $\lambda > \lambda'(q)$ is large. We let $\lambda_0 = \prod_{i=1}^r \binom{q-i}{r-i}$

and write $\lambda = \mu\lambda_0 + \lambda_1$ for some integers μ, λ_1 with $0 \le \lambda_1 < \lambda_0$. Write $Z_i := \lambda\binom{q-i}{r-i}^{-1}\binom{n-i}{r-i}$ and note that by assumption all $Z_i \in \mathbb{Z}$. Let $\ell = \binom{n}{q}/Z_0 = \lambda^{-1}\binom{n-r}{q-r}$. It suffices to decompose K_n^q into $\ell\mu$ designs with parameters (n, q, r, λ_0) and ℓ with parameters (n, q, r, λ_1). Indeed, these can then be combined into ℓ designs with parameters (n, q, r, λ).

We start by choosing the ℓ designs with parameters (n, q, r, λ_1). We do this by a greedy process, where we start with K_n^q and repeatedly delete some (n, q, r, λ_1)-design. Note that the divisibility conditions for the existence of an (n, q, r, λ_1)-design are satisfied, namely all $\binom{q-i}{r-i} \mid \lambda_1\binom{n-i}{r-i}$. At each step of the process we have some q-graph G. We say that a q-set e is full if $e \in K_n^q \setminus G$, and for $i < q$ that an i-set f is full if it is contained in at least $c(q)n$ full $(i+1)$-sets, where we choose $1/\lambda'(q) \ll c(q) \ll 1/h$. Once a set is full we will avoid using it.

There can be no full r-set, as this would belong to at least $(c(q)n)^{q-r}/(q-r)!$ full q-sets, but we are only choosing $\ell\lambda_1 < \lambda_0\lambda'(q)^{-1}\binom{n-r}{q-r}$ such q-sets. Let Φ be the $[q]$-complex on $V(G)$ where each Φ_B consists of all $\phi \in Inj(B, V(G))$ such that all subsets of $Im(\phi)$ are not full. Then Φ is $(1/2, h)$-extendable (say), so by Theorem 7.3 we can find a K_q^r-decomposition of $\lambda_1 K_n^r$ in Φ_q, i.e. an (n, q, r, λ_1)-design avoiding full sets. Thus the algorithm can be completed to choose ℓ designs with parameters (n, q, r, λ_1).

Finally, letting Φ and G be as above after the final step of the algorithm, (Φ, G) is $(1/2, h)$-extendable, G is an r-multidesign, $Z_i^0 := \lambda_0\binom{q-i}{r-i}^{-1}\binom{n-i}{r-i} \in \mathbb{Z}$ and $Z_i^0 \mid |G(f)|$ for all $f \in [n]_i$. By Theorem 7.10 we can decompose G into $\ell\mu$ designs with parameters (n, q, r, λ_0), as required. □

Next we prove the existence of complete resolutions.

Proof of Theorem 1.3. Suppose q is fixed and $n > n_0(q)$ is large with $n = q \bmod lcm([q])$. We start by recalling the reformulation of complete resolution as a partite hypergraph decomposition problem. We consider disjoint sets of vertices X and Y where $|X| = n$ and Y is partitioned into Y_j, $2 \le j \le q+1$ with $|Y_j| = \frac{n-j+2}{q-j+2}$. We let G' be the q-graph whose edges are all q-sets $e \subseteq X \cup Y$ such that $|e \cap Y_j| \le 1$ for all $2 \le j \le q+1$, and if $e \cap Y_j \ne \emptyset$ then $e \cap Y_i \ne \emptyset$ for all $i > j$. Let H be the q-graph whose vertex set is the disjoint union of two q-sets A and $B = \{b_2, \dots, b_{q+1}\}$, whose edges are all q-sets $e \subseteq A \cup B$ such that if $b_j \in e$ then $b_i \in e$ for all $i > j$. To adopt the notation of Definition 7.5 we let $\mathcal{P} = (P_1, \dots, P_{q+1})$ and $\mathcal{Q} = (Q_1, \dots, Q_{q+1})$ with $P_1 = A$, $Q_1 = X$ and $P_j = \{b_j\}$, $Q_j = Y_j$ for $2 \le j \le q+1$. Then G' is an $(H, \mathcal{P})$-blowup and we wish to find a $\mathcal{P}$-partite H-decomposition of G'.

To apply Theorem 7.6, we consider the complete $(A \cup B)$-complex Φ wrt $(\mathcal{P}, \mathcal{Q})$. Then (Φ, G') is clearly $(1/2, h)$-extendable (say). Also, every edge of G' is in exactly $\binom{n}{q}$ copies of H, so G' is (H, c, ω)-regular in Φ for any $c > 0$ and $\omega < \omega_0$. It remains to check $(H, \mathcal{P})$-divisibility.

The set of index vectors of edges is $I = \{i^j : 1 \le j \in q+1\} \subseteq \mathbb{N}^{q+1}$ where $i^j_{j'}$ is 1 for $j+1 \le j' \le q+1$, $i^j_1 = j-1$ and $i^j_{j'} = 0$ otherwise. We identify $\mathbb{N}^I$ with $\mathbb{N}^{q+1}$ by assigning i^j to coordinate j. Consider $i' \in \mathbb{N}^{q+1}$. We can assume $i'_{j'} \le 1$ for $j' > 1$. If there is any $j' > 1$ with $i'_{j'} \ne 0$, we let j^0 be the least such j', otherwise we let $j^0 = q+2$. We can assume $i'_1 \le j^0 - 2$, otherwise $H^I_{i'} = 0$. Then $H^I_{i'}$ is generated by $v^{i'} \in \mathbb{N}^q$ where each $v^{i'}_j = 1_{i'_1+1 \le j \le j^0-1}\binom{q-i'_1}{j-1-i'_1}$.

Let $u^{i'} \in \mathbb{N}^q$ where each $u^{i'}_j$ is $1_{i'_1+1 \le j \le j^0-1}\binom{n-i'_1}{j-1-i'_1}\prod_{j^*=j+1}^{q+1}|Y_{j^*}|^{1-i'_{j^*}}$. Then the corresponding $(H, \mathcal{P})$-divisibility condition is that $u^{i'}$ is an integer multiple of $v^{i'}$. It suffices to consider the case that $i'_j = 1$ for all $j \ge j^0$, and so each $u^{i'}_j$ is $1_{i'_1+1 \le j \le j^0-1}\binom{n-i'_1}{j-1-i'_1}\prod_{j^*=j+1}^{j^0-1}|Y_{j^*}|$. Then for $i'_1+1 \le j \le j^0-1$ we have $u^{i'}_j/v^{i'}_j = \binom{q-i'_1}{j-1-i'_1}^{-1}\binom{n-i'_1}{j-1-i'_1}\prod_{j^*=j+1}^{j^0-1}|Y_{j^*}| = \prod_{j^*=i'_1+2}^{j^0-1}\frac{n-j^*+2}{q-j^*+2}$. This is an integer constant

independent of j, as required. $\square$

Next we solve the Tryst Table Problem.

Proof of Theorem 1.10. Let Φ be the complete [9]-complex on an n-set V where n is large. Let $G^* \in (\mathbb{Z}^2)^{\Phi_3}$ with all $G^*_\phi = (1,1)$. Let $\mathcal{A} = \{A\}$ with $A = S_9^{\leq}$. Let $\gamma \in (\mathbb{Z}^2)^{A_3}$ where

- $\gamma_\theta = (1,0)$ if $Im(\theta) = \{1,4,7\}$,
- $\gamma_\theta = (0,1)$ if $Im(\theta) = \{3i-2, 3i-1, 3i\}$ for some $i \in [3]$ and $\theta(\min(Dom(\theta))) = 3i-2$,
- $\gamma_\theta = (0,0)$ otherwise.

The Tryst Table Problem is equivalent to finding a $\gamma(\Phi)$-decomposition of G^*.

There are three types in γ for each $B \in [9]_3$, where the type of θ is determined by γ_θ as above, so γ is elementary. The atom decomposition of G^* is $G^* = \sum_{e=abc\in[n]_3}(e^1 + e^a + e^b + e^c)$, where e^1_ψ is $(1,0)$ for all ψ with $Im(\psi) = e$ otherwise 0, and each e^x_ψ for $x \in e$ is $(0,1)$ for all ψ with $Im(\psi) = e$ and $\psi(\min(Dom(\psi))) = x$, otherwise 0. (The interpretation of e^1 is that e is the set of captains, and of e^x is that e is a team with captain x.) As all nonzero γ-atoms at e appear in G^* we have $\gamma[G] = (\gamma[G]_B : B \in [9]_3)$ with each $\gamma[G]_B = \Phi_B$, so $(\Phi, \gamma[G])$ is (ω, h)-extendable for any $\omega < 1$ and $n > n_0(\omega, h)$. To show regularity of G^* we let $y_\phi = 1/6(n-3)_6$ for all $\phi \in A(\Phi) = \Phi_9$. Then for any $B \in [9]_3$, $\psi \in \Phi_B$, $t \in T_B$ we have $\partial^t y_\phi = 6(n-3)_6|\{\phi : t_\phi(\psi) = t\}| = 1$, where the factor of 6 either represents all bijections from $\{1,4,7\}$ to $e = Im(\psi)$, or all bijections from $\{3i-2, 3i-1, 3i\}$ to e mapping $3i-2$ to some $x \in e$, where x is fixed and i ranges over [3]. Therefore G^* is $(\gamma, c, 1/7)$-regular in Φ for any $c > 0$.

To apply Theorem 3.1, it remains to show that $G^* \in \langle\gamma(\Phi)\rangle = \mathcal{L}_\gamma(\Phi)$ (by Lemma 5.19). Fix any $O \in \Phi/S_9$, write $e = Im(O) \in \Phi^\circ$ and $i = |e|$. Then $((G^*)^\sharp)^O \in (\mathbb{Z}^2)^{[9]_3 \times O}$ is a vector supported on the coordinates (B, ψ') with $B' \subseteq B \in [9]_3$ and $\psi' \in O \cap \Phi_{B'}$ in which every nonzero coordinate is equal: we have $((G^*)^\sharp_{\psi'})_B) = \sum\{G^*_\psi : \psi' \subseteq \psi \in \Phi_B\} = (n-i)_{3-i}(1,1)$.

We need to show $((G^*)^\sharp)^O \in \langle\gamma^\sharp[O]\rangle$. First consider the case $i = 3$. Then it is clear that $((G^*)^\sharp)^O$ is the sum of the $\gamma^\sharp$-atoms at O, as these are obtained from the γ-atoms e^1, e^a, e^b, e^c described above by identifying each ψ with (B, ψ) where $\psi \in \Phi_B$.

Now suppose $i = 2$, say $Im(O) = e = ab$. There are four $\gamma^\sharp$-atoms at O, which we label $e^{ab} = \gamma^\sharp(1 \to a, 4 \to b)$ (a and b are captains), $e^a = \gamma^\sharp(1 \to a, 2 \to b)$ (a captains a team containing b), $e^b = \gamma^\sharp(1 \to b, 2 \to a)$ (b captains a team containing a), $e^0 = \gamma^\sharp(2 \to a, 3 \to b)$ (a and b are in the same team, neither is the captain).

To calculate e^{ab}, consider any $\theta' \in A_2$ with $\theta'(x) = 1$, $\theta'(y) = 4$ and $\psi' \in \Phi_2$ with $\psi'(x) = a$, $\psi'(y) = b$. Then $e^{ab}_{\psi'} = \gamma^\sharp_{\theta'}$, so each $(e^{ab}_{\psi'})_{xyz} = \sum\{\gamma_\theta : \theta' \subseteq \theta \in A_{xyz}\} = \gamma_{x\to1,y\to4,z\to7} = (1,0)$.

Next, if $\theta' \in A_2$ with $\theta'(x) = 1$, $\theta'(y) = 2$ and $\psi' \in \Phi_2$ with $\psi'(x) = a$, $\psi'(y) = b$ then each $(e^a_{\psi'})_{xyz} = \sum\{\gamma_\theta : \theta' \subseteq \theta \in A_{xyz}\} = \gamma_{x\to1,y\to2,z\to3}$ is $(0,1)$ if $x = \min\{x,y,z\}$ or $(0,0)$ otherwise. Similarly, $(e^b_{\psi'})_{xyz}$ is $(0,1)$ if $y = \min\{x,y,z\}$ or $(0,0)$ otherwise.

Finally, if $\theta' \in A_2$ with $\theta'(x) = 2$, $\theta'(y) = 3$ and $\psi' \in \Phi_2$ with $\psi'(x) = a$, $\psi'(y) = b$ then each $(e^a_{\psi'})_{xyz} = \sum\{\gamma_\theta : \theta' \subseteq \theta \in A_{xyz}\} = \gamma_{z\to1,x\to2,y\to3}$ is $(0,1)$ if $z = \min\{x,y,z\}$ or $(0,0)$ otherwise.

Therefore $((e^{ab}+e^a+e^b+e^0)_{\psi'})_{xyz} = (1,1)$ for every $\psi' \in O$ and $xyz \in [9]_3$, so $((G^*)^\sharp)^O \in \langle\gamma^\sharp[O]\rangle$.

Now suppose $i = 1$, say $Im(O) = a$. There are two $\gamma^\sharp$-atoms at O, which we label $a^1 = \gamma^\sharp(1 \to a)$ (a is a captain), $a^0 = \gamma^\sharp(2 \to a)$ (a is not a captain).

Consider any $\theta' \in A_1$ with $\theta'(x) = 1$ and $\psi' \in \Phi_1$ with $\psi'(x) = a$. Then each $(a^1_{\psi'})_{xyz} = \sum\{\gamma_\theta : \theta' \subseteq \theta \in A_{xyz}\}$ is $(2,2)$ if $x = \min\{x,y,z\}$ or $(2,0)$ otherwise.

Next consider any $\theta' \in A_1$ with $\theta'(x) = 2$ and $\psi' \in \Phi_1$ with $\psi'(x) = a$. Then each $(a^0_{\psi'})_{xyz} = \sum\{\gamma_\theta : \theta' \subseteq \theta \in A_{xyz}\}$ is $(0,0)$ if $x = \min\{x,y,z\}$ or $(0,2)$ otherwise.

Therefore $((a^1 + a^0)_{\psi'})_{xyz} = (2,2)$ for every $\psi' \in O$ and $xyz \in [9]_3$. As each $((G^*)^\sharp_{\psi'})_{xyz} = (n(n-1), n(n-1))$ we have $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O]\rangle$.

Finally, $\gamma^\sharp[\emptyset]$ is generated by a vector v with all $(v_\emptyset)_B = (6,6)$. As $((G^*)^\sharp_\emptyset)_B = (n(n-1)(n-2), n(n-1)(n-2))$ we have $(G^*)^\sharp_\emptyset \in \gamma^\sharp[\emptyset]$. □

Now we consider the more general setting of coloured hypergraph decompositions. We require some definitions.

Definition 7.11. Suppose H is an r-graph on $[q]$, edge-coloured as $H = \cup_{d\in[D]} H^d$. We identify H with a vector $H \in (\mathbb{N}^D)^Q$, where each $(H_f)_d = 1_{f\in H^d}$.

Let Φ be a $[q]$-complex. For $\phi \in \Phi_q$ we define $\phi(H) \in (\mathbb{N}^D)^{\Phi^\circ_r}$ by $\phi(H)_{\phi(f)} = H_f$. Let $\mathcal{H}$ be an family of $[D]$-edge-coloured r-graphs on $[q]$. Let $\mathcal{H}(\Phi) = \{\phi(H) : \phi \in \Phi_q, H \in \mathcal{H}\}$.

We say $G \in (\mathbb{N}^D)^{\Phi^\circ_r}$ is $(\mathcal{H}, c, \omega)$-regular in Φ if there are $y^H_\phi \in [\omega n^{r-q}, \omega^{-1}n^{r-q}]$ for each $H \in \mathcal{H}$, $\phi \in \Phi_q$ with $\phi(H) \le G$ so that $\sum\{y^H_\phi \phi(H)\} = (1 \pm c)G$.

We say that (Φ, G) is (ω, h)-extendable if (Φ, G') is (ω, h)-extendable, where $G' = (G^1, \dots, G^D)$ with each $G^d = \{e \in \Phi^\circ_r : (G_e)_d > 0\}$.

The following generalises Theorem 7.3 by allowing colours and also families of hypergraphs.

Theorem 7.12. *Let $\mathcal{H}$ be an family of $[D]$-edge-coloured r-graphs on $[q]$. Let Φ be an (ω, h)-extendable exactly adapted $[q]$-complex where $n = |V(\Phi)| > n_0(q, D)$ is large, $h = 2^{50q^3}$, $\delta = 2^{-10^3 q^5}$, $n^{-\delta} < \omega < \omega_0(q, D)$ is small and $c = \omega^{h^{20}}$. Suppose $G \in \langle \mathcal{H}(\Phi)\rangle$ is $(\mathcal{H}, c, \omega)$-regular in Φ and (Φ, G) is (ω, h)-extendable. Then G has an H-decomposition in Φ_q.*

Proof. We follow the proof of Theorem 7.3, with appropriate modifications for the more general setting. Suppose Φ is exactly Σ-adapted and let $\mathcal{A} = \{A^H : H \in \mathcal{H}\}$ with each $A^H = \Sigma^{\le}$. Let $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$ with $\gamma_\theta = e_d$ if $\theta \in A^H_r$, $H \in \mathcal{H}$, $d \in [D]$ with $Im(\theta) \in H^d$ or $\gamma_\theta = 0$ otherwise. Let $G^* \in (\mathbb{N}^D)^{\Phi_r}$ with $G^*_\psi = G_{Im(\psi)}$ for $\psi \in \Phi_r$. Then $G \in \langle \mathcal{H}(\Phi)\rangle$ iff $G^* \in \langle \gamma(\Phi)\rangle$, and an $\mathcal{H}$-decomposition of G is equivalent to a $\gamma(\Phi)$-decomposition of G^*.

There are $D+1$ types in γ for each $B \in [q]_r$: the colour d type $\{\theta \in A^H_B : Im(\theta) \in H^d, H \in \mathcal{H}\}$ for each $d \in [D]$, and the nonedge type $\{\theta \in A^H_B : Im(\theta) \notin H \in \mathcal{H}\}$. Each γ^θ is e_d in all coordinates for θ in a colour d type or 0 in all coordinates for θ in a nonedge type, so γ is elementary. The atom decomposition of G^* is $G^* = \sum_{f\in\Phi^\circ_r} \sum_{d\in[D]} (G_f)_d f^d$, where $f^d_\psi = e_d$ for all $\psi \in \Phi_r$ with $Im(\psi) = f$, i.e. f^d contains all colour d types at f.

As G is $(\mathcal{H}, c, \omega)$-regular in Φ we have $\sum\{y^H_\phi \phi(H)\} = (1 \pm c)G$ for some $y^H_\phi \in [\omega n^{r-q}, \omega^{-1}n^{r-q}]$ for each $H \in \mathcal{H}$, $\phi \in \Phi_q$ with $\phi(H) \le G$. For any such $\phi \in H(\Phi)$ we have $\gamma(\phi) \le_\gamma G$, so $\phi \in \mathcal{A}(\Phi, G)$. We define $y_\phi = y^H_\phi$ for $\phi \in A^H(\Phi)$. Then for any $B \in [q]_r$, $\psi \in \Phi_B$ and $d \in [D]$, writing $t_d \in T_B$ for the colour d type we have $\partial^{t_d} y_\psi = \sum_{\phi : t_\phi(\psi) = t_d} y_\phi = \sum\{y^H_\phi : Im(\psi) \in \phi(H^d), H \in \mathcal{H}\} = (1\pm c)(G^*)^{t_d}_\psi$, so G^* is (γ, c, ω)-regular.

To apply Theorem 3.1, it remains to show that each $(\Phi, \gamma[G]^H)$ is (ω, h)-extendable. If $B \notin H$ then $\gamma[G]^H_B = \Phi_B$ and if $B \in H^d$ for $d \in [D]$ then $\gamma[G]^H_B = \{\psi \in \Phi_B : Im(\psi) \in G^d\}$. Let $E = (J, F, \phi)$ be any Φ-extension of rank s and $J' \subseteq J_r \setminus J[F]$. Let $J'' = (J^d : d \in [D])$ where each $J^d = \bigcup\{J'_B : B \in H^d\}$. As (Φ, G) is (ω, h)-extendable we have $X_{E,J''}(\Phi, G) > \omega n^{v_E}$. Consider any $\phi^+ \in X_{E,J''}(\Phi, G)$. For any $\psi \in J^d$ we have $\phi^+\psi \in \Phi$ and $Im(\phi^+\psi) \in G^d$, so $\phi^+\psi \in \gamma[G]^H$. Thus

$\phi^+ \in X_{E,J'}(\Phi, \gamma[G]^H)$, so $(\Phi, \gamma[G]^H)$ is (ω, h)-extendable. Now G^* has a $\gamma(\Phi)$-decomposition, so G has an $\mathcal{H}$-decomposition. □

We conclude by applying Theorem 7.12 to the two results on rainbow clique decompositions stated in the introduction.

Proof of Theorem 1.11. We apply Theorem 7.12 with $G = [\binom{q}{r}]K_n^r$ and $\mathcal{H}$ equal to the set of all rainbow $[\binom{q}{r}]$-colourings of K_q^r. We let Φ be the complete $[q]$-complex on $[n]$ and note that G is $(\mathcal{H}, c, \omega)$-regular in Φ and (Φ, G) is (ω, h)-extendable for any $c > 0$ and some $\omega = \omega(q)$.

It remains to check $G \in \langle \mathcal{H}(\Phi) \rangle$. Let G^* and γ be as in the proof of Theorem 7.12. We need to show $G^* \in \langle \gamma(\Phi) \rangle = \mathcal{L}_\gamma(\Phi)$ (by Lemma 5.19), i.e. $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O] \rangle$ for any $O \in \Phi/S_q$.

Fix any $O \in \Phi/S_q$, write $e = Im(O) \in \Phi^\circ$ and $i = |e|$. Then $((G^*)^\sharp)^O \in ((\mathbb{Z}^Q)^Q)^O = (\mathbb{Z}^Q)^{Q \times O}$ is a vector supported on the coordinates (B, ψ') with $B' \subseteq B \in Q$ and $\psi' \in O \cap \Phi_{B'}$ with each $((G^*)^\sharp_{\psi'})_B) = \sum\{G^*_\psi : \psi' \subseteq \psi \in \Phi_B\} \in \mathbb{Z}^Q$ equal to $(r-i)!\binom{n-i}{r-i}$ in each coordinate. Also, $\langle \gamma^\sharp[O] \rangle$ is generated by $\gamma^\sharp$-atoms $\gamma^\sharp(v)$ at O, each of which is supported on the same coordinates (B, ψ') as $((G^*)^\sharp)^O$, with each $(\gamma^\sharp(v)_{\psi'})_B)$ equal to some $(r-i)!v$ in each coordinate, where $v \in \{0,1\}^Q$ is any vector with $\sum_B v_B = \binom{q-i}{r-i}$.

To see that $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O] \rangle$ we write $((G^*)^\sharp)^O$ as the sum of $\binom{q-i}{r-i}^{-1}\binom{q}{r}\binom{n-i}{r-i}$ atoms at O, where we choose the support of the vectors v cyclically: to be formal, identify Q with $\mathbb{Z}/Q\mathbb{Z}$ and assign the jth atom the vector in $\{0,1\}^{\mathbb{Z}/Q\mathbb{Z}}$ with support $\{j\binom{q-i}{r-i} + x : x \in [\binom{q-i}{r-i}]\}$. □

Proof of Theorem 1.12. The proof is the same as that of Theorem 1.11, except for the verification of $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O] \rangle$ for any $O \in \Phi/S_q$. The generators $\gamma^\sharp[O]$ are vectors of the same form as before except that now v must be a row of the inclusion matrix $M_i^r(q)$ (discussed after the statement of the theorem). To see that $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O] \rangle$, we note that the sum σ^O of all $\gamma^\sharp$-atoms at O is supported (as before) on the coordinates (B, ψ') with $B' \subseteq B \in Q$ and $\psi' \in O \cap \Phi_{B'}$, with each coordinate equal to the vector in $\mathbb{Z}^Q$ that is $(r-i)!\binom{r}{i}$ in each coordinate. Recalling that $((G^*)^\sharp)^O$ has the same description with $\binom{r}{i}$ replaced by $\binom{n-i}{r-i}$, we have $((G^*)^\sharp)^O = \binom{r}{i}^{-1}\binom{n-i}{r-i}\sigma^O$. □

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