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Disjoint representability of sets and their complements

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Abstract

For a hypergraph $\mathcal{H}$ and a set S, the *trace* of $\mathcal{H}$ on S is the set of all intersections of edges of $\mathcal{H}$ with S. We will consider forbidden trace problems, in which we want to find the largest hypergraph $\mathcal{H}$ that does not contain some list of forbidden configurations as traces, possibly with some restriction on the number of vertices or the size of the edges in $\mathcal{H}$. In this paper we will focus on combinations of three forbidden configurations: the k-singleton $[k]^{(1)}$, the k-co-singleton $[k]^{(k-1)}$ and the k-chain $\mathcal{C}_k = \{\emptyset, \{1\}, [1, 2], \ldots, [1, k-1]\}$, where we write $[k]^{(\ell)}$ for the set of all ℓ-subsets of $[k] = \{1, \ldots, k\}$. Our main topic is hypergraphs with no k-singleton or k-co-singleton trace. We obtain an exact result in the case $k = 3$, both for uniform and non-uniform hypergraphs, and classify the extremal examples. In the general case, we show that the number of edges in the largest r-uniform hypergraph with no k-singleton or k-co-singleton trace is of order r^{k-2}. By contrast, Frankl and Pach showed that the number of edges in the largest r-uniform hypergraph with no k-singleton trace is of order r^{k-1}. We also give a very short proof of the recent result of Balogh and Bollobás that there is a finite bound on the number of sets in any hypergraph without a k-singleton, k-co-singleton or k-chain trace, independently of the number of vertices or the size of the edges.

Keywords: Set systems; Trace; Extremal problems

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1. Introduction

Many problems in combinatorics ask for the largest structure satisfying some local condition. Frequently, the local condition is that we have some list of forbidden configurations $\{\mathcal{F}_i : i \in I\}$ and the problem is to find the largest set system that does not contain any forbidden configuration. Perhaps the most famous is the Turán problem, which, in full generality, asks for the largest r-uniform hypergraph $\mathcal{H}$ on n vertices that does not contain some fixed r-uniform hypergraph $\mathcal{F}$ as a subhypergraph.

A natural variation on these problems arises when we modify the notion of containment to allow restrictions, in the following sense. For a hypergraph $\mathcal{H}$ and a subset of its vertex set $S \subset V(\mathcal{H})$, the *trace* of $\mathcal{H}$ on S is the hypergraph $\mathcal{H}|_S = \{E \cap S : E \in E(\mathcal{H})\}$. Given a fixed hypergraph $\mathcal{F}$, we say that $\mathcal{H}$ has $\mathcal{F}$ as a trace if there is a set $S \subset V(\mathcal{H})$ so that $\mathcal{H}|_S$ has a subhypergraph isomorphic to $\mathcal{F}$. Thus we arise at the forbidden trace problem of finding the largest hypergraph $\mathcal{H}$ which does not have $\mathcal{F}$ as a trace. For a survey of these problems and their applications see [6].

There is a variety of notation used for these problems, so we offer the following attempt at standardisation. Given a list of forbidden traces $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$ we write $Tr(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ for the maximum number of edges in a hypergraph $\mathcal{H}$ which does not have any $\mathcal{F}_i$ as a trace. For some forbidden traces this will be infinite, and in those cases we impose other restrictions on $\mathcal{H}$, such as fixing the vertex set or the sizes of the edges. Our notation reflects this by including the number of vertices in the brackets and the uniformity as a superscript. For the restriction that $|V(\mathcal{H})| = n$ we use the notation $Tr(n, \mathcal{F}_1, \ldots, \mathcal{F}_m)$, for the restriction that $\mathcal{H}$ is r-uniform we use $Tr^{(r)}(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ and for both restrictions we use $Tr^{(r)}(n, \mathcal{F}_1, \ldots, \mathcal{F}_m)$.

One of the earliest results on forbidden traces concerns the case when $\mathcal{F} = 2^{[k]}$ consists of all subsets of the set $[k] = \{1, \ldots, k\}$. A result of Sauer [9], Perles and Shelah [10], Vapnik and Chervonenkis [12] (frequently referred to as the Sauer–Shelah theorem) states that $Tr(n, 2^{[k]}) = \sum_{i=0}^{k-1} \binom{n}{i}$. Equality can be achieved, for example, when $\mathcal{H} = [n]^{(\leqslant k-1)}$ consists of all subsets of $[n]$ of size at most $k-1$.

A uniform version of this question was considered by Frankl and Pach [5], who showed in particular that $\binom{n-1}{k-1} \leqslant Tr^{(k)}(n, 2^{[k]}) \leqslant \binom{n}{k-1}$. They conjectured that the lower bound was tight (which would give a generalisation of the Erdős–Ko–Rado theorem) but a counterexample was constructed by Ahlswede and Khachatrian [1]. The main topic of [5] was the notion of disjointly representable sets, which were introduced by Frankl and Pach as a strengthening of the classical Hall condition. Here one says that the sets $\mathcal{A} = \{A_1, \ldots, A_k\}$ have a system of distinct representatives $\{x_1, \ldots, x_k\}$ if all the x_i's are different and $x_i \in A_i$ for each i. If we can also arrange that $x_i \notin A_j$ if $i \neq j$ then we call the sets *disjointly representable*. This can be rephrased as saying that no set is contained in the union of the others. In terms of traces we say that $\mathcal{A}$ has a k-singleton trace, where a k-singleton is $[k]^{(1)} = \{\{1\}, \{2\}, \ldots, \{k\}\}$.

More generally, one can consider forbidding any $[k]^{(\ell)}$ as a trace, where the ℓ^{th} *level* $[k]^{(\ell)}$ consists all subsets of $[k]$ of size ℓ. (We exclude the trivial cases $\ell = 0$ and k.) Here Füredi and Quinn [7] gave an example to show that no improvement on the Sauer–Shelah bound is possible, i.e. $Tr(n, [k]^{(\ell)}) = \sum_{i=0}^{k-1} \binom{n}{i}$ for any *fixed* ℓ. Moreover, we will give an example in the next section of a hypergraph $\mathcal{H}$ where $E(\mathcal{H}) = \Omega(n^{k-1})$ has the same order of magnitude yet $\mathcal{H}$ does not have *any* non-trivial level $[k]^{(\ell)}$ as a trace.

In the uniform setting Frankl and Pach considered the function $Tr^{(r)}([k]^{(1)})$, which they showed has order of magnitude r^{k-1}. More precisely, they obtained an upper bound of $\binom{r+k-1}{k-1}$ and a lower bound equal to the maximum number of edges in a $(k-1)$-uniform hypergraph on $r+k-1$ vertices not containing a copy of the complete $(k-1)$-uniform hypergraph on k vertices (a hypergraph Turán number). In this paper we consider the general problem of forbidding a number of levels as traces.

The first obvious point is that one must forbid a k-singleton trace to get a finite bound, as one can take any number of mutually disjoint sets without having any trace of the form $[k]^{(\ell)}$ with $\ell > 1$. With the k-singleton forbidden, we show that the order of magnitude depends only on whether the *k-co-singleton* $[k]^{(k-1)}$ is forbidden. The following theorem shows that if the k-singleton and k-co-singleton are forbidden traces then the number of edges is at most of order r^{k-2}, and this is the correct order of magnitude. On the other hand, if we permit a k-co-singleton trace, then forbidding any other levels as traces does not give any improvement in the order of magnitude from the Frankl–Pach bound.

Theorem 1.1. (i) $Tr^{(r)}([k]^{(1)}, [k]^{(k-1)}) < kr^{k-2}$, *i.e. an r-uniform hypergraph with at least* kr^{k-2} *edges has a k-singleton or k-co-singleton trace.*

(ii) $Tr^{(r)}([k]^{(1)}, [k]^{(2)}, \ldots, [k]^{(k-1)}) \geqslant \binom{r+k-2}{k-2}$, *i.e. there is an r-uniform hypergraph containing no non-trivial level as a trace, with at least* $\binom{r+k-2}{k-2}$ *edges.*

(iii) $Tr^{(r)}([k]^{(1)}, [k]^{(2)}, \ldots, [k]^{(k-2)}) \geqslant \Omega(r^{k-1})$, *i.e. there is an r-uniform hypergraph with at least* $\Omega(r^{k-1})$ *edges containing no level* $[k]^{(i)}$ *with* $1 \leqslant i \leqslant k-2$ *as a trace.*

Define the hypergraph of complements $C(\mathcal{H})$ to have edges $\{V(\mathcal{H})\backslash A : A \in \mathcal{H}\}$. Note that $\mathcal{H}$ has a k-co-singleton trace if and only if $C(\mathcal{H})$ has a k-singleton trace. It follows that $\mathcal{H}$ has the k-singleton and k-co-singleton forbidden as traces exactly when it is impossible to disjointly represent any set of k edges or their complements; hence the title of this paper.

Next, we consider the problem of excluding singletons and co-singletons in more detail. The smallest non-trivial case is $k = 3$. Here we are able to obtain exact results and classify the extremal examples. We will use the notation $[x, y]$ for the set of integers i such that $x \leqslant i \leqslant y$. Define

$$\mathcal{A}_r = \{[1, r], [2, r+1], \ldots, [r+1, 2r]\},$$

$$C_4 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\},$$

$$D_1 = \{\{1, 2, 5\}, \{2, 3, 5\}, \{3, 4, 5\}, \{1, 4, 5\}\},$$

$$D_2 = \{\{1, 3, 4\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 5, 6\}\}.$$

Theorem 1.2. *Define* $f(r) = \begin{cases} r+1 & r \neq 2 \\ 4 & r = 2 \end{cases}$. *Then the size of the largest r-uniform hypergraph without a* 3*-singleton or* 3*-co-singleton trace is* $Tr^{(r)}([3]^{(1)}, [3]^{(2)}) = f(r)$. *Up to isomorphism, the only extremal examples are as follows.* $\mathcal{A}_r$ *is extremal for any r except*

$r = 2$, when C_4 is extremal, and $\mathcal{A}_2$ is the only example with 3 edges. D_1 and D_2 are also extremal for $r = 3$. Furthermore, if we include the restriction that the ground set has n vertices, then we have $Tr^{(r)}(n, [3]^{(1)}, [3]^{(2)}) = \min\{f(r), f(n-r)\}$. The extremal examples are obtained from those above, possibly adding some vertices to all sets.

We use this theorem to deduce its non-uniform version, for which we obtain an exact result and find the extremal example. Let $\mathcal{B}_n$ be the hypergraph consisting of all intervals $I \subset [n]$ which contain at least one of $\lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor + 1$, and also the empty set.

Theorem 1.3. *The size of the largest hypergraph on n vertices without a 3-singleton or 3-co-singleton trace is $Tr(n, [3]^{(1)}, [3]^{(2)}) = \lfloor n^2/4 \rfloor + n + 1$. Equality is achieved only by a hypergraph isomorphic to $\mathcal{B}_n$.*

We remark that the first part of this theorem also follows from a result that was proved independently by Alon [2] and Frankl [4]. They showed that if $\mathcal{H}$ is a set system on *n* vertices with $|\mathcal{H}| > \lfloor n^2/4 \rfloor + n + 1$, then there is a set system $\mathcal{F}$ on 3 vertices with at least 7 edges for which $\mathcal{H}$ has $\mathcal{F}$ as a trace. Such an $\mathcal{F}$ clearly contains a 3-singleton or 3-co-singleton. The significance of our theorem is that we are able to characterise the extremal structures (which does not follow from the work of Alon and Frankl). This is rather unusual for a trace problem. Exact results and characterization of the extremal constructions have always been of interest in extremal combinatorics, and there have been many recent results in which characterization of extremal or approximately extremal structures has played an important role.

We also consider some variations on the above problems. First, we consider the asymmetric generalisation $Tr^{(r)}([k]^{(1)}, [\ell]^{(\ell-1)})$. We focus on the cases $k = 3$ or $\ell = 3$, for which we can obtain the following bounds.

Theorem 1.4. (i) *For $\ell \geqslant 4$, the size of the largest r-uniform hypergraph without a 3-singleton or ℓ-co-singleton trace satisfies*

$$\frac{r\ell}{4} - \frac{\ell^2}{2} + 2\ell \leqslant Tr^{(r)}([3]^{(1)}, [\ell]^{(\ell-1)}) \leqslant \frac{r\ell}{4} + \left(\frac{3}{4} + \frac{1}{\ell+3}\right) r + 1.$$

(ii) *For $k \geqslant 4$, the size of the largest r-uniform hypergraph without a 3-co-singleton or k-singleton trace satisfies*

$$\frac{(k^2-2k)(r-1)}{4} \leqslant Tr^{(r)}([k]^{(1)}, [3]^{(2)}) \leqslant \frac{k(k+1)r}{4}.$$

Note that both parts of the above theorem are asymptotically tight as $\ell \to \infty$ or $k \to \infty$, with $r \gg \ell$ or $r \gg k$. Next, we give a very short proof of the following recent result of Balogh and Bollobás [3]. They define the *k-chain* as $\mathcal{C}_k = \{\emptyset, \{1\}, [1, 2], \ldots, [1, k-1]\}$ and show that there is a finite bound on the number of sets in any hypergraph without a *k*-singleton, *k*-co-singleton or *k*-chain trace, independently of the number of vertices or the size of the edges. They give a recursion which provides a doubly exponential bound. We obtain a similar bound with the following theorem.

Theorem 1.5. *A set system of size at least $2^{2^{2k}}$ has at least one of a k-singleton, k-co-singleton or k-chain as a trace, i.e.* $Tr([k]^{(1)}, [k]^{(k-1)}, \mathcal{C}_k) \leqslant 2^{2^{2k}}$.

One can ask a number of other natural forbidden trace questions involving chains. The most interesting seems to be that of determining the maximum size of a r-uniform hypergraph with no k-singleton or k-chain trace. Concerning this, we have the following results.

Theorem 1.6. (i) *The size of the largest r-uniform hypergraph without a k-singleton or* 3*-chain trace is* $Tr^{(r)}([k]^{(1)}, \mathcal{C}_3) = \max\{k-1, r+1\}$, *for* $k \geqslant 3$.

(ii) *The size of the largest r-uniform hypergraph without a k-singleton or k-chain trace* (*where* $r \geqslant k-2$) *satisfies*

$$\binom{r+k-2}{k-2} \leqslant Tr^{(r)}([k]^{(1)}, \mathcal{C}_k) \leqslant \binom{(k-1)r}{k-2}.$$

The rest of this paper is organised as follows. Section 2 contains the proofs of our first three theorems on singleton and co-singleton traces. In Section 3 we study the forementioned variations, starting with the asymmetric singleton and co-singleton problem, where we prove Theorem 1.4. Then we introduce chains and prove Theorems 1.5 and 1.6. The final section is devoted to some concluding remarks and open problems.

Notation. For the convenience of the reader we collect here some notation that we use in this paper. We write $[x, y]$ for the set of integers i such that $x \leqslant i \leqslant y$, where x, y can be any reals, but will usually be integers. Note that if $y < x$ then $[x, y] = \emptyset$. We also write $[n] = [1, n]$. For any set X the i^{th} level of X is the set of all subsets of X of size i, which we denote $X^{(i)}$. We also write 2^X for the set of all subsets of X and $X^{(\leqslant i)} = \bigcup_{j=0}^{i} X^{(j)}$. Given two sets A and B we write $A \backslash B$ for the set of points in A that are not in B, and $A \Delta B$ for the symmetric difference $(A\backslash B) \cup (B \backslash A)$. For a hypergraph $\mathcal{H}$ the hypergraph of complements $C(\mathcal{H})$ has edges $\{V(\mathcal{H})\backslash A : A \in \mathcal{H}\}$. For a graph G we let $N_G(x)$ denote the neighbourhood of a vertex x, i.e. the set of vertices adjacent to x. We write $d_G(x) = |N_G(x)|$ for the degree of x.

2. Singleton and co-singleton traces

We start with an observation from [5]. Suppose that $\mathcal{H} = \{A_1, \ldots, A_m\}$ is r-uniform and has no k-singleton trace. Since $\mathcal{H}$ is r-uniform $A_i \backslash A_j \neq \emptyset$ for every $i \neq j$. For each i, let B_i be a minimal subset of $\left(\bigcup_{j=1}^{m} A_j\right) \backslash A_i$ for which $B_i \cap A_j \neq \emptyset$ for all $j \neq i$. Note that $B_i \neq B_j$ for $i \neq j$. For each $x \in B_i$ there is some A_j for which $A_j \cap B_i = \{x\}$, by minimality. Thus the trace of $\mathcal{H}$ on B_i contains all of its singletons, and we must have $|B_i| \leqslant k-1$.

Proof of Theorem 1.1. (i) Suppose $\mathcal{H} = \{A_1, \ldots, A_m\}$ is an r-uniform hypergraph with no k-singleton trace and $m \geqslant kr^{k-2}$. We will show that there is a k-co-singleton trace. For each i, let B_i be a minimal subset of $\left(\bigcup_{j=1}^{m} A_j\right)\setminus A_i$ for which $B_i \cap A_j \neq \emptyset$ for all $j \neq i$. Then $|B_i| \leqslant k-1$ for all i, as noted above. Now $B_i \cap A_1 \neq \emptyset$ for every $i > 1$, so there is some $x_1 \in A_1$ so that at least $\lceil \frac{kr^{k-2}-1}{r} \rceil = kr^{k-3}$ of the B_i contain x_1. We can iterate this process as follows. At the t^{th} stage we have points $x_1, \ldots, x_t$ and a set of indices I_t of size at least kr^{k-2-t}, so that $\{x_1, \ldots, x_t\} \subset B_i$ for all $i \in I_t$. Now we pick some $i \in I_t$ and note that A_i intersects all B_j with $i \neq j \in I_t$ yet is disjoint from B_i. Let x_{t+1} be a point in A_i belonging to as many B_j with $j \in I_t$ as possible, and let $I_{t+1} = \{j \in I_t : x_{t+1} \in B_j\}$. Then $|I_{t+1}| \geqslant kr^{k-3-t}$. Also $x_{t+1} \notin \{x_1, \ldots, x_t\}$, as $\{x_1, \ldots, x_t\} \subset B_i$, and x_{t+1} belongs to A_i which is disjoint from B_i.

After stage $k-3$ we have points $\{x_1, \ldots, x_{k-2}\}$ and an index set I_{k-2} of size at least k such that the sets B_j for $j \in I_{k-2}$ have the form $B_j = \{x_1, \ldots, x_{k-2}, y_j\}$. (No B_j can be equal to $\{x_1, \ldots, x_{k-2}\}$ by minimality.) Let $Y = \{y_j : j \in I_{k-2}\}$. Now A_j is disjoint from B_j and intersects B_k for each $k \in I_{k-2}\setminus j$, so $A_j \cap Y = Y \setminus y_j$. Thus we have a k-co-singleton trace, as required.

(ii) Let $\mathcal{H} = \left\{ \bigcup_{i=0}^{k-2}[ir, ir + a_i - 1] : a_i \geqslant 0, \sum_{i=0}^{k-2} a_i = r \right\}$, i.e. each edge of $\mathcal{H}$ is a union of $(k-1)$ intervals whose leftmost points are multiples of r, and whose total length is r. Then $|\mathcal{H}| = \binom{r+k-2}{k-2}$. Consider any set $K \subset [0, (k-1)r-1]$ of size k. Then there is some $0 \leqslant i \leqslant k-2$ for which K has at least two points in $[ir, (i+1)r-1]$. Suppose they are a and b, with $a < b$. Then any set of $\mathcal{H}$ that contains b must also contain a. Any non-trivial level of K separates all pairs of points, so cannot appear as a trace of $\mathcal{H}$.

(iii) Let X be a set of size $r+k-1$ and let $X = X_1 \cup \cdots \cup X_{k-1}$ be a partition into parts that are as equal in size as possible, i.e. $|X_i| = \lfloor \frac{r+k-2+i}{k-1} \rfloor$. Define $\mathcal{H}$ to be the r-uniform hypergraph whose edges are the complements of transversals of the partition, i.e. $\mathcal{H} = \left\{ X\setminus\{x_1, \ldots, x_{k-1}\} : x_i \in X_i \ \forall i \right\}$. Then $|\mathcal{H}| = \prod_{i=1}^{k-1} \lfloor \frac{r+k-2+i}{k-1} \rfloor = \Omega(r^{k-1})$. Consider any set $K \subset X$ of size k. There is some i for which K contains at least two points of X_i, say they are a and b. Then any set of $\mathcal{H}$ contains at least one of a and b. However, any level $K^{(i)}$ with $1 \leqslant i \leqslant k-2$ contains a set not meeting $\{a, b\}$, so cannot appear as a trace of $\mathcal{H}$. □

We remark that a very similar construction to that in part (ii) of the above proof gives an example of a non-uniform hypergraph on $[n]$ with $\Omega(n^{k-1})$ edges and no non-trivial layer as a trace. We take $\mathcal{H} = \{\bigcup_{i=0}^{k-2}[in/(k-1), in/(k-1) + a_i] : 0 \leqslant a_i < n/(k-1)\}$. Then $\mathcal{H}$ has $(n/(k-1))^{k-1}$ edges and no non-trivial layer as a trace (as explained above).

Next we need the following lemma.

Lemma 2.1. *Let $\mathcal{H}$ be an r-uniform hypergraph with no k-singleton trace. Choose edges $A_1, \ldots, A_{k-1}$ to maximise the size of $\bigcup_{i=1}^{k-1} A_i$. Then $A \subset \bigcup_{i=1}^{k-1} A_i$ for every edge A.*

Proof. Suppose there is a point $x \in A \setminus \bigcup_{i=1}^{k-1} A_i$. For every $1 \leqslant i \leqslant k-1$ we can find a point $x_i \in A_i \setminus \bigcup_{j \neq i} A_j$ that does not belong to A. Otherwise we would have $\bigcup_{j=1}^{k-1} A_j \cup \{x\} \subset \bigcup_{j \neq i} A_j \cup A$, which contradicts the maximum property of $A_1, \ldots, A_{k-1}$. However, this gives a k-singleton trace on $\{x_1, \ldots, x_{k-1}, x\}$, which is a contradiction. It follows that $\bigcup_{i=1}^{k-1} A_i$ contains all edges of $\mathcal{H}$. □

Now we give the proof of Theorem 1.2, which classifies the extremal r-uniform hypergraphs with no 3-singleton or 3-co-singleton as a trace. The general example is the interval system $\mathcal{A}_r = \{[1, r], [2, r + 1], \ldots, [r + 1, 2r]\}$. To see that this contains no 3-singleton or 3-co-singleton as a trace consider any three points $\{a, b, c\} \subset [1, 2r]$. One of the intervals $[1, r]$ and $[r + 1, 2r]$ contains at least two of these points. Without loss of generality suppose $a, b \in [1, r]$ and $a < b$. Then any set of $\mathcal{A}_r$ that contains a must also contain b, and we are done. Additional examples are C_4 for $r = 2$, and $D_1 = \{\{1, 2, 5\}, \{2, 3, 5\}, \{3, 4, 5\}, \{1, 4, 5\}\}$ and $D_2 = \{\{1, 3, 4\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 5, 6\}\}$ for $r = 3$. The reader can easily check that these do not have a 3-singleton or 3-co-singleton as a trace.

Proof of Theorem 1.2. Let $\mathcal{H}$ be an r-uniform hypergraph with no 3-singleton or 3-co-singleton as a trace. We argue by induction on r. The case $r = 1$ is trivial. In the case $r = 2$, $\mathcal{H}$ is a triangle-free graph of maximum degree 2. Furthermore, if $\mathcal{H}$ contains two disjoint edges ab, cd then any other edge must meet both of them. It follows that the extremal example is achieved when $\mathcal{H} = C_4$, and has 4 edges. Note also that the only example with 3 edges is a path of length 3, which is isomorphic to $\mathcal{A}_2$.

Now we consider the general case. Suppose first that there is some x which belongs to every set in $\mathcal{H}$. Then $\mathcal{H}' = \{X\backslash x : X \in \mathcal{H}\}$ is an $(r - 1)$-uniform hypergraph with no 3-singleton or 3-co-singleton as a trace. Now we have $|\mathcal{H}| = |\mathcal{H}'| \leqslant Tr^{(r-1)}([3]^{(1)}, [3]^{(2)})$ by induction. This is strictly less than $r + 1$ except when $r = 3$. Therefore, if $|\mathcal{H}| \geqslant r + 1$ then $r = 3$, $|\mathcal{H}| = 4$, $\mathcal{H}' \cong C_4$ and $\mathcal{H} \cong D_1$.

Now we can suppose that the sets of $\mathcal{H}$ do not have a common point. Choose $A, B \in \mathcal{H}$ to maximise $|A \cup B|$. Then any $C \in \mathcal{H}$ is contained in $A \cup B$ by Lemma 2.1. We claim that A and B are disjoint. For suppose $x \in A \cap B$. Then there is an edge C of $\mathcal{H}$ not containing x. Since $C \subset A \cup B$ and $|A| = |B| = |C| = r$, there are $a \in C \cap A\backslash B$ and $b \in C \cap B\backslash A$. Then $\{A, B, C\}$ has a 3-co-singleton trace on $\{a, b, x\}$, which is a contradiction.

Let $\mathcal{H}_0 = \mathcal{H}\backslash\{A\}$. Suppose that the sets of $\mathcal{H}_0$ do not have a common point. Then we can repeat the above analysis: if we pick $C, D \in \mathcal{H}_0$ to maximise $|C \cup D|$ then C and D are disjoint. Now we claim that $\mathcal{H} = \{A, B, C, D\}$. For suppose that $\mathcal{H}$ contains another set E. Note that $A \cup B$ and $C \cup D$ are both partitions of the ground set. Now we see that E must intersect both $A \cap C$ and $B \cap D$ or intersect both $A \cap D$ and $B \cap C$; otherwise it would be contained in one of A, B, C, D, which is impossible as $\mathcal{H}$ is r-uniform. Without loss of generality E intersects the sets $A \cap C$ and $B \cap D$ and does not contain $B \cap C$. (If it contains both $B \cap C$ and $A \cap D$ then it cannot contain $A \cap C$ or $B \cap D$, so we can rename the sets to arrive at the same situation.) Take $x \in E \cap A \cap C$, $y \in E \cap B \cap D$ and $z \in (B \cap C)\backslash E$. Then $\{C, B, E\}$ has a 3-co-singleton trace on $\{x, y, z\}$, which is a contradiction. We deduce that $\mathcal{H} = \{A, B, C, D\}$. If $\mathcal{H}$ is extremal we must have $r = 2$ or $r = 3$. When $r = 2$ we see that $\mathcal{H} \cong C_4$ and when $r = 3$ it is easy to check that $\mathcal{H} \cong D_2$.

Now we are reduced to the situation when there is some x that belongs to every set in $\mathcal{H}_0$. Then $\mathcal{H}_0' = \{X\backslash x : X \in \mathcal{H}_0\}$ is an $(r - 1)$-uniform hypergraph with no 3-singleton or 3-co-singleton as a trace. Therefore $|\mathcal{H}| = |\mathcal{H}_0'| + 1 \leqslant Tr^{(r-1)}([3]^{(1)}, [3]^{(2)}) + 1$ by induction.

Consider the case $r \geqslant 5$. If $|\mathcal{H}| \geqslant r + 1$ then $|\mathcal{H}_0'| \geqslant r$, and then by induction we must have $\mathcal{H}_0' \cong \mathcal{A}_{r-1}$. Then $\mathcal{H}_0$ is isomorphic to a system obtained by adding x to all sets of

$\mathcal{A}_{r-1}$. We can choose notation so that $\mathcal{H}_0 = \{[1, r], [2, r+1], \ldots, [r, 2r-1]\}$. Since r belongs to every set in $\mathcal{H}_0$ we have $r \notin A$. There cannot be $i, j \in A$ with $1 \leqslant i \leqslant r-1$ and $r+1 \leqslant j \leqslant 2r-1$, otherwise $\{A, [1, r], [r, 2r-1]\}$ would have a 3-co-singleton trace on $\{i, r, j\}$, which is a contradiction. By symmetry we can suppose that $A \cap [r-1] = \emptyset$. It follows that there is some $y \in A \backslash [1, 2r-1]$. Note that there cannot be $r+1 \leqslant i \leqslant 2r-1$ with $i \notin A$, otherwise $\{[1, r], [r, 2r-1], A\}$ would have a 3-singleton trace on $\{1, i, y\}$, which is a contradiction. Therefore $A = \{r+1, \ldots, 2r-1, y\}$. Renaming y as $2r$ we see that $\mathcal{H} \cong \mathcal{A}_r$, as required.

Next consider the case $r = 4$. If $|\mathcal{H}| \geqslant 5$ then $|\mathcal{H}_0'| \geqslant 4$, and then by induction it is isomorphic to one of $\mathcal{A}_3$, D_1 or D_2. If $\mathcal{H}_0' \cong \mathcal{A}_3$ then the same argument as in the previous paragraph shows that $\mathcal{H} \cong \mathcal{A}_4$, and we are done. If $\mathcal{H}_0' \cong D_1$ then we can choose notation so that $\mathcal{H}_0 = \{\{1, 2, 5, 6\}, \{2, 3, 5, 6\}, \{3, 4, 5, 6\}, \{1, 4, 5, 6\}\}$. Without loss of generality $B = \{1, 2, 5, 6\}$. Since A and B are disjoint and their union is equal to the ground set we can write $A = \{3, 4, 7, 8\}$. Now $\{\{1, 4, 5, 6\}, \{2, 3, 5, 6\}, \{3, 4, 7, 8\}\}$ has a 3-singleton trace on $\{1, 2, 7\}$, which is a contradiction. If $\mathcal{H}_0' \cong D_2$ then we can choose notation so that $\mathcal{H}_0 = \{\{1, 3, 4, 7\}, \{1, 5, 6, 7\}, \{2, 3, 4, 7\}, \{2, 5, 6, 7\}\}$. Without loss of generality $B = \{1, 3, 4, 7\}$. Since A and B are disjoint and their union is equal to the ground set we can write $A = \{2, 5, 6, 8\}$. Now $\{\{1, 5, 6, 7\}, \{2, 3, 4, 7\}, \{2, 5, 6, 8\}\}$ has a 3-singleton trace on $\{1, 3, 8\}$, which is a contradiction.

Finally we consider the case $r = 3$. If $|\mathcal{H}| \geqslant 4$ then $|\mathcal{H}_0'| \geqslant 3$ so $\mathcal{H}_0'$ is either $\mathcal{A}_2$ or C_4, as noted at the beginning of the proof. If $\mathcal{H}_0' \cong \mathcal{A}_2$ then previous analysis shows that $\mathcal{H} \cong \mathcal{A}_3$. If $\mathcal{H}_0' \cong C_4$ then we can take $\mathcal{H}_0 = \{\{1, 2, 5\}, \{2, 3, 5\}, \{3, 4, 5\}, \{1, 4, 5\}\}$. Without loss of generality $B = \{1, 2, 5\}$. Since A and B are disjoint and their union is equal to the ground set we can write $A = \{3, 4, 6\}$. Now $\{\{1, 4, 5\}, \{2, 3, 5\}, \{3, 4, 6\}\}$ has a 3-singleton trace on $\{1, 2, 6\}$, which is a contradiction. This completes the proof of the first part of the theorem.

Now suppose that we fix the number of vertices n. When $n \geqslant 2r$ we see from the first part of the theorem that there is no change, i.e. $Tr^{(r)}(n, [3]^{(1)}, [3]^{(2)}) = Tr^{(r)}([3]^{(1)}, [3]^{(2)})$. Now suppose that $\mathcal{H}$ is an r-uniform hypergraph on n vertices with no 3-singleton or 3-co-singleton trace, and that $n < 2r$. Now it is no longer possible to have two disjoint sets, so it follows from the first part of the proof that the sets of $\mathcal{H}$ have a common point x. Then $\mathcal{H}' = \{A \backslash x : A \in \mathcal{H}\}$ is an $(r-1)$-uniform hypergraph on $(n-1)$ vertices with no 3-singleton or 3-co-singleton trace. We can repeat this process until the number of vertices is at least twice the size of the edges. This occurs when we have removed $2r - n$ vertices, reaching an $(n-r)$-uniform hypergraph $\mathcal{H}^*$ on $2(n-r)$ vertices. Now we have $|\mathcal{H}^*| \leqslant n - r + 1$, unless $n - r = 2$ when we can have $|\mathcal{H}^*| = 4$. We deduce that $Tr^{(r)}(n, [3]^{(1)}, [3]^{(2)}) = \min\{f(r), f(n-r)\}$. The extremal examples are as before, possibly adding some vertices to all sets. $\square$

Now we can give the proof of Theorem 1.3, which states that $Tr(n, [3]^{(1)}, [3]^{(2)}) = \lfloor n^2/4 \rfloor + n + 1$, and the only extremal example is $\mathcal{B}_n = \{[a, b] : a \leqslant \lfloor n/2 \rfloor + 1, b \geqslant \lfloor n/2 \rfloor\}$. The argument that this contains no 3-singleton or 3-co-singleton trace is the same as for $\mathcal{A}_r$. Indeed, if $\{a, b, c\} \subset [n]$ one of $[1, \lfloor n/2 \rfloor]$ and $[\lfloor n/2 \rfloor + 1, n]$ contains at least two points. We can suppose $a, b \in [1, \lfloor n/2 \rfloor]$ with $a < b$. Then any set of $\mathcal{B}_n$ that contains a also contains b, and we are done.

Proof of Theorem 1.3. Note that the cases $1 \leqslant n \leqslant 3$ are trivial. Suppose $\mathcal{H}$ is a hypergraph on $n \geqslant 4$ vertices with no 3-singleton or 3-co-singleton trace. Let $\mathcal{H}^r$ be the edges of $\mathcal{H}$ of size r. Then by Theorem 1.2 we have $|\mathcal{H}^r| \leqslant \min\{f(r), f(n-r)\}$, where $f(r) = \begin{cases} r+1 & r \neq 2 \\ 4 & r = 2 \end{cases}$. It is convenient to consider the hypergraph whose edges are the complements of the edges of $\mathcal{H}$, which we denote $C(\mathcal{H}) = \{[n]\backslash X : X \in \mathcal{H}\}$. This also has no 3-singleton or 3-co-singleton trace. Observe that $C(\mathcal{H})^r = C(\mathcal{H}^{n-r})$.

Next we see that $|\mathcal{H}^1| + |\mathcal{H}^2| \leqslant 5$, with equality only when $\mathcal{H}^2 \cong \mathcal{A}_2$ and $\mathcal{H}^1 \cong \mathcal{A}_1$. Otherwise we would have $|\mathcal{H}^1| \geqslant 1$ and $|\mathcal{H}^2| = 4$. Then we must have $\mathcal{H}^2 \cong C_4$, say $\mathcal{H}^2 = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$. Consider $\{i\} \in \mathcal{H}^1$. If $i \in [1,4]$ we can suppose $i = 1$, and then $\{\{2,3\},\{3,4\},\{1\}\}$ has a 3-singleton trace on $\{2,4,1\}$. If $i \notin [1,4]$ we can suppose $i = 5$, and then $\{\{1,2\},\{3,4\},\{5\}\}$ has a 3-singleton trace on $\{1,3,5\}$. Either way we get a contradiction, so $|\mathcal{H}^1| + |\mathcal{H}^2| \leqslant 5$. Applying the same argument to $C(\mathcal{H})$ gives $|C(\mathcal{H}^1)| + |C(\mathcal{H}^2)| \leqslant 5$, i.e. $|\mathcal{H}^{n-1}| + |\mathcal{H}^{n-2}| \leqslant 5$. We conclude that $|\mathcal{H}| \leqslant \sum_{r=0}^{n} \min\{r+1, n-r+1\} = \lfloor n^2/4 \rfloor + n + 1$. (This last equality is easy to see by considering the cases of n even and n odd separately.)

Suppose now that $|\mathcal{H}| = \lfloor n^2/4 \rfloor + n + 1$. Then $|\mathcal{H}^r| = \min\{r+1, n-r+1\}$ for all r. We claim that $\mathcal{H}^r \cong \mathcal{A}_r$ for $r \leqslant \lfloor n/2 \rfloor$ and $\mathcal{H}^r \cong C(\mathcal{A}_{n-r})$ for $r \geqslant \lfloor n/2 \rfloor + 1$. This follows from Theorem 1.2 except when $r = 3$ or $n - r = 3$. For $r = 3$ we need to show that we cannot have $\mathcal{H}^3 \cong D_1$ or $\mathcal{H}^3 \cong D_2$. First suppose that $\mathcal{H}^3 = D_1 = \{\{1,2,5\},\{2,3,5\},\{3,4,5\},\{1,4,5\}\}$. Since $|\mathcal{H}^1| = 2$ we have $\{i\} \in \mathcal{H}^1$ with $i \neq 5$. The same argument as given for C_4 in the previous paragraph now gives a contradiction here. Similarly if $\mathcal{H}^3 = D_2 = \{\{1,3,4\},\{1,5,6\},\{2,3,4\},\{2,5,6\}\}$, then any singleton gives a 3-singleton trace, which is a contradiction. This deals with the case $r = 3$, and the case $n - r = 3$ follows by taking complements.

To complete the proof we need to show that these interval hypergraphs only fit together by forming a copy of $\mathcal{B}_n$. We need the following claim.

Claim. (i) *Suppose* $\mathcal{A} = \{[1,r],[2,r+1],\ldots,[r+1,2r]\}$, $\mathcal{B}$ *is an* $(r-1)$*-uniform hypergraph and* $\mathcal{A} \cup \mathcal{B}$ *has no* 3*-singleton or* 3*-co-singleton trace. Then* $\mathcal{B} \subset \{[2,r],[3,r+1],\ldots,[r+1,2r-1]\}$.

(ii) *Suppose* $\mathcal{A} = \{[1,r],[2,r+1],\ldots,[r,2r-1]\}$, $\mathcal{B}$ *is an* $(r-1)$*-uniform hypergraph and* $\mathcal{A} \cup \mathcal{B}$ *has no* 3*-singleton or* 3*-co-singleton trace. Then* $\mathcal{B} \subset \{[1,r-1],[2,r],\ldots,[r+1,2r-1]\}$.

Proof. (i) Consider $B \in \mathcal{B}$. Suppose there is some point $x \in B\backslash[1,2r]$. Then, since $|B| = r-1$, we can find $y \in [1,r]\backslash B$ and $z \in [r+1,2r]\backslash B$. Now $\{B,[1,r],[r+1,2r]\}$ has a 3-singleton trace on $\{x,y,z\}$, which is a contradiction. It follows that $B \subset [1,2r]$. We cannot have $1 \in B$. Otherwise we can take $i \in [2,r] - B$ and $j \in [r+2,2r] - B$, and then $(B,[2,r+1],[r+1,2r])$ has a 3-singleton trace on $(1,i,j)$. Similarly $2r \notin B$. Now suppose that B is not an interval. By symmetry we may assume that there are $i < j$ such that $i \in B \cap [1,r-1]$ and $j \in [1,r]\backslash B$. Then $\{B,[j,j+r-1],[r+1,2r]\}$ has a 3-singleton trace on $\{i,j,2r\}$. This contradiction shows that B is an interval.

(ii) Consider $B \in \mathcal{B}$. There cannot be a point $x \in B\backslash[1,2r-1]$, or we can take $y \in [1,r-1]\backslash B$ and $z \in [r+1,2r-1]\backslash B$, and then $\{B,[1,r],[r,2r-1]\}$ has a 3-singleton trace on $\{x, y, z\}$, which is a contradiction. Therefore $B \subset [1, 2r-1]$. Let $\mathcal{A}'$ consist of the intervals of length $r-1$ in the ordered set $\{1, \ldots, r-1, r+1, \ldots, 2r-1\}$. Then $\mathcal{A} = \{A \cup \{r\} : A \in \mathcal{A}'\}$. If $r \in B$ then by part (i) of the claim we see that $B \setminus \{r\}$ is an interval of length $r-2$ in $\{1, \ldots, r-1, r+1, \ldots, 2r-1\}$ not containing 1 or $2r-1$, so B is an interval of length $r-1$ in $[2r-1]$ not containing 1 or $2r-1$. On the other hand, if $r \notin B$ then B cannot contain points i, j with $i < r$ and $j > r$. Otherwise $\{B, [1, r], [r, 2r-1]\}$ would have a 3-co-singleton trace on $\{i, j, r\}$. Thus in the case $r \notin B$, B must be either $[1, r-1]$ or $[r+1, 2r-1]$, so we are done. $\square$

Now we can complete the proof of the theorem. For simplicity we just consider the case when $n = 2m$ is even, the odd case being similar. We can renumber so that $\mathcal{H}^m = \{[1, m], [2, m+1], \ldots, [m+1, 2m]\}$. Repeatedly applying part (i) of the claim shows that $\mathcal{H}^r = \{[m-r+1, m], [m-r+2, m+1], \ldots, [m+1, m+r]\}$ for every $r < m$. Note $C(\mathcal{H})^m = C(\mathcal{H}^m)$ is isomorphic to $\mathcal{H}^m$: it consists of all intervals of length m in the ordered set $\{m+1, m+2, \ldots, 2m, 1, 2, \ldots, m\}$. Applying the same argument in $C(\mathcal{H})$ we see that $C(\mathcal{H})^r = \{\{2m-r+1, \ldots, 2m\}, \{2m-r+2, \ldots, 2m, 1\}, \ldots, \{2m, 1, \ldots, r-1\}, \{1, \ldots, r\}\}$ for $r < m$. Therefore $\mathcal{H}^{2m-r} = C(C(\mathcal{H})^r) = \{[1, 2m-r], [2, 2m-r+1], \ldots, [r+1, 2m]\}$, as required. $\square$

3. Related problems

In this section we describe some variations on our main problem. A natural extension is the asymmetric version, defined by forbidding k-singleton and ℓ-co-singleton traces, for any k and ℓ. We will focus on the cases when $k = 3$ or $\ell = 3$, for which we can obtain asymptotically tight bounds. Next we describe the effect of introducing a chain $\mathcal{C}_k = \{\emptyset, \{1\}, [1, 2], \ldots, [1, k-1]\}$ as a forbidden trace. This system is in some sense the opposite of a level $[k]^{(i)}$, as instead of having all sets the same size it has one set of each possible size.

3.1. The asymmetric version

In this subsection we prove Theorem 1.4, which gives bounds for the functions $Tr^{(r)}([3]^{(1)}, [\ell]^{(\ell-1)})$ and $Tr^{(r)}([k]^{(1)}, [3]^{(2)})$.

Proof of Theorem 1.4. (i) For $r < \ell - 1$ we note that $Tr^{(r)}([3]^{(1)}, [\ell]^{(\ell-1)}) = Tr^{(r)}([3]^{(1)})$, which is greater than the desired lower bound. For larger r the lower bound is given by the following construction. Choose a positive integer t so that $r' = t(\ell - 2)$ satisfies $r - (\ell - 3) \leqslant r' \leqslant r$. We define a hypergraph $\mathcal{H}$ on $[1, t(\ell-1) + r - r'] = [1, r+t]$ as follows. For every $1 \leqslant i \leqslant \ell - 1$ we let the complement of $[(i-1)t+1, it]$ be an edge. Also, for every $1 \leqslant i \leqslant (\ell-1)/2 < j \leqslant \ell - 1$ and for every $1 \leqslant s \leqslant t-1$ we let the complement of $[(i-1)t+1, (i-1)t+s] \cup [(j-1)t+1, (j-1)t+t-s]$ be an edge. Then $\mathcal{H}$ is an r-uniform hypergraph and since $\ell \geqslant 4$ we have

$$
\begin{aligned}
|\mathcal{H}| &= (t-1)\lfloor(\ell-1)^2/4\rfloor + \ell - 1 \geqslant \left(\frac{r-\ell+3}{\ell-2} - 1\right)\left(\frac{\ell^2 - 2\ell}{4}\right) + \ell - 1 \\
&= r\ell/4 - \ell^2/2 + 9\ell/4 - 1 \geqslant r\ell/4 - \ell^2/2 + 2\ell.
\end{aligned}
$$

We claim that $\mathcal{H}$ does not have a 3-singleton or ℓ-co-singleton trace. To see this, note first that we can ignore the points $[t(\ell-1)+1, t(\ell-1)+r-r']$, as they belong to all edges. Also, if $(i-1)t+1 \leqslant x < y \leqslant it$ for some i then any edge that contains x must also contain y. It follows that any set on which we have a singleton trace or co-singleton trace can contain at most one point from each interval $[(i-1)t+1, it]$. This immediately shows that there is no ℓ-co-singleton trace. Also, if there is a 3-singleton trace on some set then its points must belong to 3 different intervals. Say the set is $\{x_1, x_2, x_3\}$ with $x_i \in [(a_i-1)t+1, a_i t]$ and a_1, a_2, a_3 pairwise distinct. By symmetry we can suppose that a_1 and a_2 are both at most $(\ell-1)/2$. Now by definition there is no edge that misses both x_1 and x_2. This shows that there is no 3-singleton trace.

For the upper bound we argue by induction on r, the case $r = 1$ being trivial. Suppose $\mathcal{H} = \{A_1, \ldots, A_m\}$ is an r-uniform hypergraph with no 3-singleton or ℓ-co-singleton trace. For each i, let B_i be a minimal subset of $\left(\bigcup_{j=1}^m A_j\right) \setminus A_i$ for which $B_i \cap A_j \neq \emptyset$ for all $j \neq i$. As noted at the beginning of Section 2, the B_i are distinct and $|B_i| \leqslant 2$ for all i. In fact, we can assume that $|B_i| = 2$ for all i. For if $B_i = \{x\}$ for some i then every edge except A_i contains $\{x\}$, and applying the induction hypothesis hypergraph to $\mathcal{H}' = \{A_j \setminus \{x\} : j \neq i\}$ gives

$$\begin{aligned} |\mathcal{H}| \leqslant |\mathcal{H}'| + 1 &\leqslant \frac{(r-1)\ell}{4} + \left(\frac{3}{4} + \frac{1}{\ell+3}\right)(r-1) + 1 + 1 \\ &\leqslant \frac{r\ell}{4} + \left(\frac{3}{4} + \frac{1}{\ell+3}\right) r + 1. \end{aligned}$$

Let G be the graph with edge set $\{B_1, \ldots, B_m\}$. We claim that G is triangle-free. For suppose we have $B_i = \{x, y\}$, $B_j = \{y, z\}$ and $B_k = \{z, x\}$. Since A_i is disjoint from B_i and meets B_j and B_k we have $A_i \cap \{x, y, z\} = \{z\}$. Similarly $A_j \cap \{x, y, z\} = \{x\}$ and $A_k \cap \{x, y, z\} = \{y\}$, so we have a 3-singleton trace on $\{x, y, z\}$, which is a contradiction.

Next, we note that we cannot have two edges of G 'sticking out' of the same point of some edge of $\mathcal{H}$, i.e. for any A_i and $x \in A_i$ there is at most one edge of G incident to x with the other endpoint not in A_i. For suppose $B_j = \{x, y\}$ and $B_k = \{x, z\}$ with $y, z \notin A_i$. Then, since $A_j \cap B_j = A_k \cap B_k = \emptyset$, we see that A_i, A_j, A_k have a 3-singleton trace on $\{x, y, z\}$, which is impossible.

This implies the following observation concerning any pair of intersecting edges. If we have $B_i = \{x, y\}$ and $B_j = \{x, z\}$ then every edge of $\mathcal{H}$ meets $\{y, z\}$. Indeed, suppose A_k is disjoint from $\{y, z\}$. We may assume $k \neq i$, and then since A_k meets B_i we have must have $x \in A_k$. However this situation contradicts the previous paragraph. Now if $B_i = \{x, y\}$ then $A_i \cap N_G(x) = N_G(x) \setminus \{y\}$, so there is a co-singleton trace on the neighbourhood of x. It follows that $d_G(x) \leqslant \ell - 1$ for every x. Moreover, for any edge $B_i = \{x, y\}$ we have $d_G(x) + d_G(y) \leqslant \ell + 1$. For suppose that $d_G(x) + d_G(y) \geqslant \ell + 2$. Let $X = N_G(x)$ and $Y = N_G(y)$. Since G is triangle-free we have $X \cap Y = \emptyset$. Therefore $Z = X \cup Y \setminus \{x, y\}$ contains at least ℓ points. By the above observation, any set A_j can miss at most one point from each of X and Y. Consider any $B_j = \{x, z\}$ with $z \neq y$. Then A_j does not contain z, so contains $X \setminus \{z\}$. Therefore $\{x, y\}$ is an edge of G sticking out of A_j at y. As noted before it must be the only such edge, so $Y \setminus \{x\} \subset A_j$. This shows that $A_j \cap Z = Z \setminus \{z\}$. Arguing similarly for edges of G incident with y, we see that for every $z \in Z$ there is some edge A_j with $A_j \cap Z = Z \backslash \{z\}$. Since $|Z| \geqslant \ell$ we have an ℓ-co-singleton trace, which is a contradiction. Therefore $d_G(x) + d_G(y) \leqslant \ell + 1$.

Now we bound the number of edges of G as follows. There is one edge B_1 disjoint from A_1, and at most r edges that meet A_1 in exactly one point (at most one sticking out of each of the r points of A_1). The remaining edges form a graph G' on A_1. Write e for the number of edges in G' and for each $x \in A_1$ let I_x be an indicator function that is 1 if there is an edge sticking out of A_1 at x, and 0 otherwise. Then $m = e(G) = e + 1 + \sum_{x \in A_1} I_x$.

Let $d'(x) = d_G(x) - I_x$ denote the degree of x in G'. Recalling that $d_G(x) + d_G(y) \leqslant \ell + 1$ when xy is an edge, and applying the Cauchy–Schwartz inequality we have

$$\begin{aligned}(\ell+1)e &\geqslant \sum_{\{x,y\}\in G'} \big(d_G(x)+d_G(y)\big) = \sum_{\{x,y\}\in G'} \big(d'(x)+d'(y)+I_x+I_y\big)\\ &= \sum_{x\in A_1} d'(x)^2 + \sum_{x\in A_1} d'(x) I_x\\ &\geqslant r\left(\frac{1}{r}\sum_{x\in A_1} d'(x)\right)^2 + \sum_{x\in A_1} d'(x)I_x = 4e^2/r + \sum_{x\in A_1} d'(x)I_x.\end{aligned}$$

Writing $S = \sum_{x \in A_1} d'(x) I_x$ we have $4e^2 - r(\ell+1)e + rS \leqslant 0$, so $e \leqslant \frac{1}{8}\Big(r(\ell+1) + \sqrt{r^2(\ell+1)^2 - 16rS}\Big)$. Using the inequality $\sqrt{a-b} \leqslant \sqrt{a} - \frac{b}{2\sqrt{a}}$ gives $e \leqslant \frac{r(\ell+1)}{4} - \frac{S}{\ell+1}$. Therefore

$$m = e + 1 + \sum_{x\in A_1} I_x \leqslant \frac{r(\ell+1)}{4} + 1 + \sum_{x\in A_1} I_x\left(1 - \frac{d'(x)}{\ell+1}\right).$$

Since $d'(x) < \ell + 1$ for all x we have $I_x\left(1 - \frac{d'(x)}{\ell+1}\right) \leqslant 1 - \frac{d'(x)}{\ell+1}$, so

$$\begin{aligned}m &\leqslant \frac{r(\ell+1)}{4} + 1 + \sum_{x\in A_1}\left(1 - \frac{d'(x)}{\ell+1}\right) = \frac{r(\ell+1)}{4} + 1 + r - \frac{2e}{\ell+1}\\ &\leqslant \frac{r(\ell+1)}{4} + 1 + r - \frac{2(m-r-1)}{\ell+1}.\end{aligned}$$

This gives $\frac{(\ell+3)m}{(\ell+1)} \leqslant \frac{r(\ell+5)}{4} + \frac{2r}{\ell+1} + \frac{\ell+3}{\ell+1}$, so $m \leqslant \frac{\ell^2+6\ell+13}{4(\ell+3)} r + \frac{\ell+3}{\ell+3} = \frac{r(\ell+3)}{4} + \frac{r}{\ell+3} + 1$.

(ii) The construction for the lower bound is essentially the complement of that in part (i). We define a hypergraph $\mathcal{H}$ on $[(k-1)r]$ as follows. For every $1 \leqslant i \leqslant k-1$ we let the interval $[(i-1)r+1, ir]$ be an edge. Also, for every $i \leqslant (k-1)/2 < j$ and for every $1 \leqslant s \leqslant r-1$ we let $[(i-1)r+1, (i-1)r+s] \cup [(j-1)r+1, (j-1)r+r-s]$ be an edge. Then $\mathcal{H}$ is an r-uniform hypergraph and $|\mathcal{H}| = (r-1)\lfloor (k-1)^2/4 \rfloor + k - 1 > \frac{(k^2-2k)(r-1)}{4}$. Note that it is the complement of the construction in part (i) with edges of size $(k-2)r$. Since that construction had no k-co-singleton or 3-singleton trace, this construction has no k-singleton or 3-co-singleton trace.

For the upper bound, suppose $\mathcal{H}$ is an r-uniform hypergraph with no 3-co-singleton or k-singleton trace. Choose $k-1$ edges of H so that their union has maximum possible size. By Lemma 2.1 all edges of $\mathcal{H}$ are contained in this union, which has size at most $(k-1)r$. Consider the hypergraph of complements $C(\mathcal{H})$, which has edges $\{V(\mathcal{H}) \backslash A : A \in \mathcal{H}\}$. Then

$C(\mathcal{H})$ is an s-uniform hypergraph with $s \leqslant (k-2)r$ and has no 3-singleton or k-co-singleton trace. By the first part of this theorem we have $|\mathcal{H}| = |C(\mathcal{H})| \leqslant \frac{k(k-2)r}{4} + \left(\frac{3}{4} + \frac{1}{k+3}\right)(k-2)r + 1 < \frac{k(k+1)r}{4}$. This completes the proof of the theorem. $\square$

Similar arguments can be applied for the general asymmetric function $Tr^{(r)}([k]^{(1)}, [\ell]^{(\ell-1)})$; we will just summarise the results and leave the details to the reader. For $k > \ell$ we have

$$(1 - o_\ell(1)) \binom{k-1}{\ell-1} \binom{r+\ell-2}{\ell-2} \leqslant Tr^{(r)}([k]^{(1)}, [\ell]^{(\ell-1)}) < k(k-2)^{\ell-2} r^{\ell-2},$$

so $Tr^{(r)}([k]^{(1)}, [\ell]^{(\ell-1)})$ is of order $k^{\ell-1} r^{\ell-2}$, and the uncertainty in the constant is approximately $(\ell-1)!(\ell-2)!$ for large k. For $k < \ell$ we have

$$(1 - o_k(1)) \binom{\ell-1}{k-1} \binom{r+k-2}{k-2} \leqslant Tr^{(r)}([k]^{(1)}, [\ell]^{(\ell-1)}) \leqslant \ell r^{k-2},$$

so $Tr^{(r)}([k]^{(1)}, [\ell]^{(\ell-1)})$ is of order ℓr^{k-2}, and the uncertainty in the constant is approximately $(k-1)!(k-2)!$ for large ℓ.

3.2. Chains

Define the *k-chain* as $\mathcal{C}_k = \{\emptyset, \{1\}, [1,2], \ldots, [1, k-1]\}$. We start this subsection with a very short proof of Theorem 1.5, which states that $Tr([k]^{(1)}, [k]^{(k-1)}, \mathcal{C}_k) \leqslant 2^{2^{2k}}$.

First, we recall that the Ramsey number $R(k, \ell)$ is the smallest integer t for which any graph on t vertices must contain a clique of size k or an independent set of size ℓ. We use the well-known bound $R(k,k) \leqslant \binom{2k-2}{k-1}$ (see, e.g., [8]).

Proof of Theorem 1.5. Let $t = R(k,k)$. Suppose $\mathcal{H}$ is a hypergraph with no k-singleton trace and at least $(k-1)^t$ edges. We will show that there is a k-co-singleton or a k-chain trace. This suffices, as the above bound easily gives $(k-1)^t \leqslant 2^{2^{2k}}$. We will find sequences of sets $A_1, \ldots, A_t$ in $\mathcal{H}$ and points $x_1, \ldots, x_t$ so that, setting $\mathcal{H}_i = \{A \in \mathcal{H} : \{x_1, \ldots, x_i\} \subset A\}$, we have $|\mathcal{H}_i| \geqslant (k-1)^{t-i}$, $A_i \in \mathcal{H}_{i-1}$ and $x_i \notin A_i$ for all $1 \leqslant i \leqslant t$. Note that $\mathcal{H}_0 = \mathcal{H}$.

To do this, suppose we have already found $A_1, \ldots, A_i$ and $x_1, \ldots, x_i$, for some $0 \leqslant i \leqslant t-1$. Let I be the intersection of all of the sets in $\mathcal{H}_i$ and let B be a minimal set disjoint from I that meets every set of $\mathcal{H}_i$ (except I if $I \in \mathcal{H}_i$). Since there is no k-singleton trace, the observation at the beginning of Section 2 gives $|B| \leqslant k-1$. Choose a point $x_{i+1} \in B$ that belongs to as many sets of $\mathcal{H}_i$ as possible. Then $|\mathcal{H}_{i+1}| \geqslant \lceil (|\mathcal{H}_i| - 1)/(k-1) \rceil \geqslant (k-1)^{t-(i+1)}$. Now B is disjoint from I, so x_{i+1} does not belong to every set of $\mathcal{H}_i$, and we can choose $A_{i+1} \in \mathcal{H}_i$ so that $x_{i+1} \notin A_{i+1}$.

Thus we have $A_1, \ldots, A_t$ and $x_1, \ldots, x_t$ so that $x_i \notin A_i$ and $x_i \in A_j$ for all $1 \leqslant i < j \leqslant t$. Define a graph on $\{1, \ldots, t\}$ by joining i to j if $i > j$ and $x_i \in A_j$. Since $t = R(k,k)$ this graph contains either a clique or an independent set of size k. It is easy to verify that if S is a clique of size k then the trace of $\{A_s : s \in S\}$ on $\{x_s : s \in S\}$ is a k-co-singleton, and if

S is an independent set of size k then the trace of $\{A_s : s \in S\}$ on $\{x_s : s \in S\}$ is a k-chain. This proves the theorem. $\square$

One can ask a number of other forbidden trace questions involving chains and levels. These questions are easy for non-uniform hypergraphs. Since $\mathcal{C}_k \subset 2^{[k-1]}$ it follows immediately from the Sauer–Shelah theorem that $Tr(n, \mathcal{C}_k) = \sum_{i=0}^{k-2} \binom{n}{i}$. Note that $[n]^{(\leqslant k-2)}$ contains no k-chain or k-co-singleton, and its hypergraph of complements contains no k-chain or k-singleton. Therefore we see also that $Tr(n, \mathcal{C}_k, [k]^{(1)}) = Tr(n, \mathcal{C}_k, [k]^{(k-1)}) = \sum_{i=0}^{k-2} \binom{n}{i}$.

For uniform hypergraphs the situation is much less clear. Here the interesting question is to determine the maximum size of an r-uniform hypergraph with no k-singleton or k-chain trace. (The problem of excluding just k-co-singleton and k-chain traces seems less natural, as in this case we need to bound the ground set, or we can take as many disjoint edges as we please.) For this problem, we will prove Theorem 1.6, which shows that $Tr^{(r)}([k]^{(1)}, \mathcal{C}_3) = \max\{k-1, r+1\}$ and that $Tr^{(r)}([k]^{(1)}, \mathcal{C}_k)$ is of order r^{k-1}. First we need the following lemma on hypergraphs without a 3-chain trace.

Lemma 3.1. *Let $\mathcal{H}$ be an r-uniform hypergraph without a 3-chain trace. Choose edges $A, B \in \mathcal{H}$ so that their union is as large as possible. Say that another edge C is of type 1 if $C \cap (A \cup B) = A \cap B$ and of type 2 if $A \Delta B \subset C \subset A \cup B$. Then any other edge is either of type 1 or of type 2, and furthermore all other edges have the same type.*

Proof. Choose edges $A, B \in \mathcal{H}$ so that their union is as large as possible. First consider any edge C that is disjoint from $A\Delta B$. Since $|A \cup C| = 2r - |A \cap C|$, by maximality of $|A \cup B|$ we must have $A \cap B \subset C$, i.e. $C \cap (A \cup B) = A \cap B$, so C is of type 1. Now any other edge C intersects $A\Delta B$. By symmetry we can assume it intersects $A\backslash B$. Take $x \in C \cap A\backslash B$. There cannot be $y \in A\backslash(B \cup C)$, otherwise $\{B, C, A\}$ would have a 3-chain trace on $\{x, y\}$, which is impossible. Therefore $A\backslash B \subset C$. By maximality of $|A \cup B|$ we now have $C \subset A \cup B$. Since $C \neq A$ we see that C intersects $B\backslash A$. Then repeating the above argument gives $B\backslash A \subset C$. Therefore $A\Delta B \subset C \subset A \cup B$, i.e. C is of type 2. Furthermore, there cannot be an edge C of type 1 and an edge D of type 2. Then we could pick $x \in A\backslash B$, $y \in B\backslash A$, and $\{C, A, D\}$ has a 3-chain trace on $\{x, y\}$, which is a contradiction. $\square$

Define a *sunflower* of size s to be a system of sets $A_1, \ldots, A_s$ for which there is some set B so that $A_i \cap A_j = B$ for all $i \neq j$. We call B the *centre* of the sunflower.

Lemma 3.2. *Let $\mathcal{H}$ be an r-uniform hypergraph without a 3-chain trace and not containing any 3 edges that form a sunflower. Then $|\mathcal{H}| \leqslant r+1$, with equality only when $\mathcal{H} \cong [r+1]^{(r)}$.*

Proof. We argue by induction on r, the case $r = 1$ being trivial. Choose edges $A, B \in \mathcal{H}$ so that their union is as large as possible. There cannot be an edge C with $C \cap (A \cup B) = A \cap B$, as then $\{A, B, C\}$ forms a sunflower of size 3 with centre $A \cap B$, which is contrary to assumption.

From Lemma 3.1 it follows that for any other edge C we have $A\Delta B \subset C \subset A \cup B$. If $|A \cup B| = r + 1$ then we immediately have $|\mathcal{H}| \leqslant r + 1$, with equality only when $\mathcal{H} =$

$(A \cup B)^{(r)}$. Otherwise $|A\Delta B| \geqslant 4$. Applying the induction hypothesis to the hypergraph $\mathcal{H}' = \{C\backslash(A\Delta B) : C \in \mathcal{H}\backslash\{A, B\}\}$, which is s-uniform for some $s \leqslant r - 4$, we get $|\mathcal{H}| \leqslant |\mathcal{H}'| + 2 \leqslant \big((r-4)+1\big) + 2 < r + 1$. This completes the proof. $\square$

Finally, we need the following result of Frankl and Pach [5] which is a uniform version of the Sauer–Shelah theorem.

Lemma 3.3. *For any $k \leqslant r \leqslant n$ we have $Tr^{(r)}(n, 2^{[k]}) \leqslant \binom{n}{k-1}$.*

Proof of Theorem 1.6. (i) For the lower bound we either take $k-1$ disjoint r-tuples or $[r+1]^{(r)}$, whichever is larger. It is clear that neither construction has a k-singleton or 3-chain trace. For the upper bound we argue by induction on r, the case $r = 1$ being trivial. Consider an r-uniform hypergraph $\mathcal{H}$ with no k-singleton or 3-chain trace. Choose edges $A, B \in \mathcal{H}$ so that their union is as large as possible. By Lemma 3.1 the other edges are either all of type 1 or all of type 2.

In the type 1 case $\mathcal{H}$ forms a sunflower with centre $A \cap B$. Since $\mathcal{H}$ does not have a k-singleton trace we immediately have $|\mathcal{H}| \leqslant k - 1$. In the type 2 case we claim that there is no sunflower of size 3. For suppose that C, D, E form a sunflower. We cannot have A or B in the sunflower, as the other sets differ only inside $A \cap B$. The centre is some set F with $A\Delta B \subset F$. Pick $x \in C\backslash F$ and $y \in D\backslash F$. By definition $x, y \notin E$, and also $x, y \in A \cap B$. Then $\{E, C, A\}$ has a 3-chain trace on $\{x, y\}$, which is a contradiction. Therefore there is no sunflower of size 3. Now then Lemma 3.2 shows that $|\mathcal{H}| \leqslant r + 1$, which completes the proof of the first part of the theorem.

(ii) The lower bound is given by $[r+k-2]^{(r)}$. Every k-set is met by any edge in at least 2 points so there is no k-singleton trace, and for every $(k-1)$-set there is no edge that is disjoint from it, so there is no k-chain trace. For the upper bound, consider an r-uniform hypergraph $\mathcal{H}$ with no k-singleton or k-chain trace. Lemma 2.1 shows that the ground set of $\mathcal{H}$ contains at most $(k-1)r$ points. Since $\mathcal{C}_k \subset 2^{[k-1]}$ there is no $2^{[k-1]}$ trace and by Lemma 3.3 we have $|\mathcal{H}| \leqslant \binom{(k-1)r}{k-2}$. This completes the proof of the theorem. $\square$

Our proof shows that in the first part of the above theorem equality can only occur for a sunflower of size $k-1$ or for $[r+1]^{(r)}$.

4. Concluding remarks

- From Theorem 1.1 we know that $Tr^{(r)}([k]^{(1)}, [k]^{(k-1)})$ is of order r^{k-2}, but the uncertainty in the constant is of order $(k-1)!$. It would be interesting to determine the asymptotics of this constant for large k. The construction that we use for the lower bound is also a lower bound for $Tr^{(r)}([k]^{(1)}, [k]^{(2)}, \ldots, [k]^{(k-1)})$, so it may be that there is a better construction that works just for $Tr^{(r)}([k]^{(1)}, [k]^{(k-1)})$.
- We remarked after the proof of Theorem 1.1 that $Tr(n, [k]^{(1)}, [k]^{(2)}, \ldots, [k]^{(k-1)})$ is of order n^{k-1}. Our construction for the lower bound gives a constant $1/(k-1)^{k-1}$, whereas the upper bound from the Sauer–Shelah theorem gives a constant of $1/(k-1)!$. Füredi and Quinn showed that for excluding just one layer as a trace the Sauer–Shelah bound is

tight, i.e. $Tr(n, [k]^{(\ell)}) = \sum_{i=0}^{k-1} \binom{n}{i}$ for any fixed ℓ. It would be interesting to determine whether the constant changes when we forbid more layers.

- Theorem 1.4 is asymptotically tight for $r \gg \ell$ and $\ell \to \infty$ but it would be interesting to obtain an asymptotically tight result for fixed ℓ and $r \to \infty$. Note that in the case $\ell = r + 2$ the condition that there is no ℓ-co-singleton trace places no restriction on an r-uniform hypergraph, so we have $Tr^{(r)}([3]^{(1)}, [r+2]^{(r+1)}) = Tr^{(r)}([3]^{(1)})$. Frankl and Pach showed that this is equal to $\lfloor (r+2)^2/4 \rfloor$, which we can write as $\ell r/4 + \lfloor r/2 \rfloor + 1$. On the basis of this one might think that $Tr^{(r)}([3]^{(1)}, [\ell]^{(\ell-1)}) = \ell r/4 + r/2 + o(r)$ for fixed ℓ and $r \to \infty$.
- The same proof as in Theorem 1.5 gives the bound $Tr([k]^{(1)}, [\ell]^{(\ell-1)}, \mathcal{C}_m) \leqslant (k-1)^{R(\ell,m)}$ when we forbid singleton, co-singleton and chain traces of various sizes. We obtained a doubly exponential upper bound for $Tr([k]^{(1)}, [k]^{(k-1)}, \mathcal{C}_k)$, but can only find an exponential lower bound. (This is achieved by a naïve random construction, and one can also give explicit examples, such as $[2k-4]^{(k-2)}$.) It would be interesting to determine the true behaviour of this function.
- The best known lower bound for $Tr^{(r)}([k]^{(1)})$, due to Frankl and Pach, is obtained by the complement hypergraph of a $(k-1)$-uniform hypergraph on $r+k-1$ vertices with as many edges as possible subject to not containing a copy of the complete $(k-1)$-uniform hypergraph on k vertices. This does not have a $\mathcal{C}_\ell$ trace for $\ell > k$, so it may be that $Tr^{(r)}([k]^{(1)}, \mathcal{C}_\ell) = Tr^{(r)}([k]^{(1)})$ for $\ell > k$. For $\ell \leqslant k$ we can use the proof of Theorem 1.6 to see that $Tr^{(r)}([k]^{(1)}, \mathcal{C}_\ell)$ is bounded above by $\binom{(k-1)r}{\ell-2}$ and below by $\binom{r+l-2}{\ell-2}$. This shows that $Tr^{(r)}([k]^{(1)}, \mathcal{C}_\ell)$ is of order $r^{\ell-2}$, although the uncertainty in the constant is about $(k-1)^{(\ell-2)}$. It would be interesting to determine the asymptotics of the constant. In the case $\ell = k$ it seems that the lower bound $\binom{r+k-2}{k-2}$ may be asymptotically tight.

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