PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, Number 9, September 2020, Pages 3835–3846
https://doi.org/10.1090/proc/15060
Article electronically published on May 22, 2020

DISTINCT DEGREES IN INDUCED SUBGRAPHS

MATTHEW JENSSEN, PETER KEEVASH, EOIN LONG, AND LIANA YEPREMYAN

(Communicated by Patricia L. Hersh)

Abstract. An important theme of recent research in Ramsey theory has been establishing pseudorandomness properties of Ramsey graphs. An N-vertex graph is called *C-Ramsey* if it has no homogeneous set of size $C \log N$. A theorem of Bukh and Sudakov, solving a conjecture of Erdős, Faudree, and Sós, shows that any C-Ramsey N-vertex graph contains an induced subgraph with $\Omega_C(N^{1/2})$ distinct degrees. We improve this to $\Omega_C(N^{2/3})$, which is tight up to the constant factor.

We also show that any N-vertex graph with $N > (k-1)(n-1)$ and $n \geq n_0(k) = \Omega(k^9)$ either contains a homogeneous set of order n or an induced subgraph with k distinct degrees. The lower bound on N here is sharp, as shown by an appropriate Turán graph, and confirms a conjecture of Narayanan and Tomon.

1. Introduction

A major open problem in Ramsey theory is the construction of explicit graphs that are approximately tight for Ramsey's theorem; all known constructions involve some randomness, which motivates a substantial literature establishing that Ramsey graphs have certain pseudorandomness properties. Given a graph G, we call $U \subset V(G)$ *homogeneous* if the induced subgraph $G[U]$ is complete or empty. Ramsey's theorem states that $\hom(G) \to \infty$ as $N := |V(G)| \to \infty$. In a more quantitative form, we have $\frac{1}{2}\log_2 N \leq \hom(G) \leq 2\log_2 N$, where the lower bound is due to Erdős and Szekeres [8] and the upper bound to Erdős [6] (the birth of the probabilistic method in combinatorics). It is remarkable that in the 70+ years since these results there have only been improvements to the lower order terms (see the survey [3]). Furthermore, there is no known explicit construction of an N-vertex graph G with $\hom(G) = O(\log N)$, despite intense interest in this question and the related notions of randomness extraction/dispersion in computer science; the best known explicit construction due to Li [13] gives $\hom(G) = (\log N)^{O(\log\log\log N)}$.

Motivated by both the difficulty in providing explicit constructions and the challenge in improving the bounds for the Ramsey problem, an important theme of recent research in Ramsey theory has been establishing properties of Ramsey graphs supporting the intuition that they should be "random-like". This indirect study has been very fruitful, and it is now known that N-vertex Ramsey graphs display

Received by the editors October 8, 2019, and, in revised form, February 3, 2020.
2010 *Mathematics Subject Classification.* Primary 05D10; Secondary 05D40, 05C69.
Key words and phrases. Ramsey theory, probabilistic methods, induced subgraphs.
This research was supported in part by ERC Consolidator Grant 647678.
This research was supported by Marie Sklodowska Curie Global Fellowship, H2020-MSCA-IF-2018:846304.

similar behaviour to the Erdős-Renyi random graph $G_{N,1/2}$ in many respects: the edge density by Erdős and Szemerédi [9]; universality of small induced subgraphs by Prömel and Rödl [16]; the number of non-isomorphic induced subgraphs by Shelah [17]; the sizes and orders of induced subgraphs by Kwan and Sudakov [10, 11], and Narayanan, Sahasrabudhe, and Tomon [15].

Here we consider a problem of Erdős, Faudree, and Sós [5] concerning induced subgraphs with many distinct degrees. Given a graph G, we let

$$f(G) := \max\big\{k \in \mathbb{N} : G \text{ has an induced subgraph with } k \text{ distinct degrees}\big\}.$$

Bukh and Sudakov [2] showed that any N-vertex graph G with $\hom(G) \leq C \log N$ has $f(G) = \Omega_C(N^{1/2})$, thus confirming a conjecture in [5] motivated by the observation that $f\big(G_{N,1/2}\big) = \Omega(N^{1/2})$ with high probability (whp); they noted however the lack of a corresponding upper bound, and showed that whp $f\big(G_{N,1/2}\big) = O(N^{2/3})$. An unpublished result of Conlon, Morris, Samotij, and Saxton [4] shows that whp $f\big(G_{N,1/2}\big) = \Omega(N^{2/3})$, so this in fact gives the correct order. Our first theorem establishes the same lower bound for Ramsey graphs, which is therefore tight up to the constant factor.

Theorem 1.1. *Let G be an N-vertex C-Ramsey graph. Then $f(G) = \Omega_C\big(N^{2/3}\big)$.*

Moreover, we establish this lower bound on $f(G)$ using only the combinatorially simpler "diversity" property (see [2, 17]) that many vertices have dissimilar neighbourhoods: we say $U \subset V(G)$ is *δ-diverse* if $|N_G(u) \triangle N_G(u')| \geq \delta|V(G)|$ for any distinct u, u' in U.

Theorem 1.2. *Given $\delta > 0$ there is $c > 0$ such that any N-vertex graph G with a δ-diverse set of size $N^{2/3}$ has an induced subgraph with at least $cN^{2/3}$ distinct degrees.*

Theorem 1.2 implies Theorem 1.1 as the hypotheses of the former follow from those of the latter by results of Kwan and Sudakov [11] (see subsection 2.3).

It is also natural to investigate the relationship between $\hom(G)$ and $f(G)$ in more generality. Narayanan and Tomon [14] showed for any $k \in \mathbb{N}$, $\varepsilon > 0$ and $N \geq N_0(k, \varepsilon)$ that any N-vertex graph G has $f(G) \geq k$ or $\hom(G) \geq N/(k-1+\varepsilon)$. They conjectured that the optimal relationship between $\hom(G)$ and $f(G)$ when $|V(G)| \gg f(G)$ should be given by the $(k-1)$-partite Turán graph on $N = (k-1)(n-1)$ vertices, which has $f(G) = k-1$ and $\hom(G) = n-1 = N/(k-1)$. We confirm this conjecture, thus obtaining an exact result, and, moreover, we only require a lower bound on n that is polynomial in k (in [14] an exponential lower bound is assumed).

Theorem 1.3. *Suppose G is an N-vertex graph with $N > (n-1)(k-1)$, where $n = \Omega(k^9)$. Then $f(G) \geq k$ or $\hom(G) \geq n$.*

We prove Theorems 1.1 and 1.2 in the next section and Theorem 1.3 in the following section. The final section contains some concluding remarks.

2. Distinct degrees in Ramsey graphs

Our proof that any sufficiently diverse graph contains an induced subgraph with many distinct degrees naturally splits into two pieces.

In the first subsection we give a new perspective: we reduce the problem to a continuous relaxation (in a similar spirit to [12, Section 3]). We show that it is sufficient to define a probability distribution on the vertex set, with respect to which a random induced subgraph has a large set of vertices whose expected degrees are well-separated.

While this change of perspective creates a larger and more flexible solution space, the existence of the required distribution is still quite subtle. In the second subsection we show its existence via an additional randomisation, in which the probabilities themselves are randomly generated according a distribution that takes into account the neighbourhood structure of our graph.

In the final subsection of this section we combine the two above ingredients to prove our result on diverse graphs (Theorem 1.2) and deduce (via results of Kwan and Sudakov) our result on Ramsey graphs (Theorem 1.1).

2.1. **A continuous relaxation.** Let G be a graph with vertex partition $V(G) = U \cup V$. Given $\mathbf{p} = (p_v)_{v \in V} \in [0,1]^V$, let $G(\mathbf{p}) = G[U \cup W]$ denote the random induced subgraph where W contains each $v \in V$ independently with probability p_v. The main result of this subsection is the following lemma, showing that separation of expected degrees in $G(\mathbf{p})$ guarantees an induced subgraph with distinct degrees.

Lemma 2.1. *Given $\delta > 0$ there is $c > 0$ so that the following holds. Let G be a graph with vertex partition $V(G) = U \cup V$ where $|V| = N$. Suppose also that $U' \subset U$ and $\mathbf{p} \in [0.1, 0.9]^V$ such that any distinct u, u' in U' satisfy*

$$\left|\mathbb{E}\big(d_{G(\mathbf{p})}(u)\big) - \mathbb{E}\big(d_{G(\mathbf{p})}(u')\big)\right| \geq \delta \quad \textit{and} \quad \left|\big(N_G(u) \triangle N_G(u')\big) \cap V\right| \geq \delta N.$$

Then there is $W \subset V$ so that $G[U \cup W]$ has at least $c|U'|$ distinct degrees.

The idea of the proof is that in $G(\mathbf{p})$ a vertex typically has degree within $O(\sqrt{N})$ of its expectation, and if we restrict to the set B of such "balanced" vertices, then a pair of vertices $u, u' \in U'$ can only have equal degrees when their expected degrees differ by $O(\sqrt{N})$. The separation of expected degrees implies that B has only $O_\delta(|U'|\sqrt{N})$ such pairs. Each has equal degrees with probability $O_\delta(1/\sqrt{N})$, by diversity and an anti-concentration estimate, so we can ensure that B has only $O_\delta(|U'|)$ pairs with equal degree in U'; then Turán's theorem will provide the required conclusion. The required anti-concentration estimate is the following generalisation of the well-known Erdős-Littlewood-Offord inequality [7]; this is not a new result, but for completeness and the convenience of the reader we will give a simple deduction from [7], namely the case that all $p_i = 1/2$.

Lemma 2.2. *Fix non-zero $a_1, \ldots, a_n \in \mathbb{R}$ and $p_1, \ldots, p_n \in [0.1, 0.9]$. Suppose also that $X_1, \ldots, X_n$ are independent Bernoulli random variables with $X_i \sim Be(p_i)$, i.e., $\mathbb{P}(X_i = 1) = p_i$ and $\mathbb{P}(X_i = 0) = 1 - p_i$. Then*

$$\max_{x \in \mathbb{R}} \mathbb{P}\Big(\sum_{i \in [n]} a_i X_i = x\Big) = O(n^{-1/2}).$$

Proof. For each i we fix $w_i, z_i \in [0,1]$ with $p_i = w_i/2 + (1 - w_i)z_i$ and write $X_i = W_i Y_i + (1 - W_i)Z_i$, where $Y_i \sim Be(1/2)$, $W_i \sim Be(w_i)$, and $Z_i \sim Be(z_i)$ are independent. We make this choice so that each $w_i \geq 0.2$; e.g., if $p_i \leq 1/2$ let $z_i = 0$ and $w_i = 2p_i$, or if $p_i > 1/2$ let $z_i = 1$ and $w_i = 2(1 - p_i)$. We condition on any choice C of the W_i's and Z_i's, which determines $I := \{i : W_i = 1\}$. By Chebyshev's inequality, $\mathbb{P}(|I| < n/10) < O(n^{-1})$, so it suffices to bound $\mathbb{P}(\sum_{i \in [n]} a_i X_i = x \mid C)$ for any C such that $|I| \geq n/10$; the required bound $O(n^{-1/2})$ holds by [7] applied to $(Y_i : i \in I)$. $\square$

We also use the following version of Turán's theorem (see, e.g., Chapter 6 in [1]).

Theorem 2.3. *Any n-vertex graph G with average degree d contains an independent set of size at least $n/(d+1)$.*

Proof of Lemma 2.1. Note that we can assume N is large, by taking $c > 0$ small enough. Let H be a random induced subgraph according to $G(\mathbf{p})$ and

$$\begin{aligned} B &= \{u \in U' : \big|d_H(u) - \mathbb{E}\big(d_{G(\mathbf{p})}(u)\big)\big| \le \sqrt{N}\},\\ P &= \big\{\{u,u'\} \subset U' : \big|\mathbb{E}\big(d_{G(\mathbf{p})}(u)\big) - \mathbb{E}\big(d_{G(\mathbf{p})}(u')\big)\big| \le 2\sqrt{N}\big\}, \text{ and}\\ J &= \{\{u,u'\} \in P : d_H(u) = d_H(u')\}. \end{aligned}$$

We claim that with positive probability we have both $|B| \ge |U'|/2$ and $|J| = O_\delta(|U'|)$. This claim implies the lemma, as by Turán's theorem $J[B]$ contains an independent set of size $\Omega_\delta(|U'|)$, which must consist of vertices with distinct degrees, as if u, u' are in B and $d_H(u) = d_H(u')$, then $\{u,u'\} \in P$, so $\{u,u'\} \in J$.

To prove the claim, we first estimate $|B|$. Notice that for any $u \in U'$ we have $\mathbb{V}ar\big(d_{G(\mathbf{p})}(u)\big) \le \sum_{v\in V} p_v(1-p_v) \le N/4$ and so Chebyshev's inequality gives $\mathbb{P}(u \notin B) \le \frac{N/4}{\sqrt{N}^2} = 1/4$. Thus $\mathbb{E}(|U' \setminus B|) \le |U'|/4$, so by Markov's inequality $\mathbb{P}\big(|U' \setminus B| \ge |U'|/2\big) \le 1/2$, i.e., $\mathbb{P}\big(|B| \ge |U'|/2\big) \ge 1/2$.

To estimate $|J|$, we first note that by the degree separation property we have $|P| \le 2\delta^{-1}|U'|N^{1/2}$. Each $\{u,u'\} \in P$ belongs to J with probability

$$\mathbb{P}\big(d_H(u) - d_H(u') = 0\big) = O((\delta N)^{-1/2})$$

by Lemma 2.2, which can be applied by the diversity property and the assumption that all $p_v \in [0.1, 0.9]$. Thus $\mathbb{E}|J| = O_\delta(|U'|)$, so $\mathbb{P}\big(|J| = O_\delta(|U'|)\big) > 1/2$ by Markov's inequality. This proves the claim and so the lemma. □

2.2. **Solving the relaxation in diverse graphs.** The following lemma shows how to find the distribution $\mathbf{p}$ required to apply Lemma 2.1.

Lemma 2.4. *Given $\delta > 0$ there is $c > 0$ such that the following holds. Let G be a graph with vertex partition $V(G) = U \cup V$ where U is δ-diverse, $|U| \le N^{2/3}$, and $|V| = N$. Then there are $\mathbf{p} \in [0.1, 0.9]^V$ and $U' \subset U$ with $|U'| \ge c|U|$ so that $|\mathbb{E}\big(d_{G(\mathbf{p})}(u)\big) - \mathbb{E}\big(d_{G(\mathbf{p})}(u')\big)| \ge 1$ for all distinct $u, u' \in U'$.*

The key idea is that our construction of the probability vector $\mathbf{p}$ is itself random, with a distribution depending on the neighbourhood structure of G. We start by sketching a simplified proof of the lemma under the stronger assumption $|U| = O(N^{2/3}/\log^{1/3} N)$. For each $u \in U$ we define a "signed neighbourhood vector" $\mathbf{u} \in \{-1,1\}^V$ by $\mathbf{u}_v = 1$ if $uv \in E(G)$ or $\mathbf{u}_v = -1$ otherwise. Let $\mathbf{1} \in [0,1]^V$ denote the "all-1" vector. We randomly select integers $m_u \in \big[-|U|, |U|\big]$ uniformly and independently for all $u \in U$. Then we let

$$\mathbf{p} := \frac{1}{2}\mathbf{1} + \sum_{u\in U} \Big(\frac{m_u}{N}\Big)\mathbf{u}. \tag{2.1}$$

The variance of each coordinate p_v of $\mathbf{p}$ is at most $|U|^3/N^2 = O(\log N)^{-1}$ and so, by a standard concentration argument, with high probability $\mathbf{p} \in [0.1, 0.9]^V$ (for an appropriate choice of the implicit constant in the stronger assumption on $|U|$). On the other hand, our definition of $\mathbf{p}$ in terms of the neighbourhood structure relates expected degree differences to our diversity assumption, as follows. For any distinct u, u' in U, as $\mathbb{E}\big(d_{G(\mathbf{p})}(u)\big) = d_{G[U]}(u) + (\mathbf{1}+\mathbf{u})\cdot\mathbf{p}/2$, we have

$$\mathbb{E}\big(d_{G(\mathbf{p})}(u)\big) - \mathbb{E}\big(d_{G(\mathbf{p})}(u')\big) = d_{G[U]}(u) - d_{G[U]}(u') + (\mathbf{u}-\mathbf{u}')\cdot\mathbf{p}/2. \tag{2.2}$$

Let $\mathcal{E}_{u,u'}$ denote the event that $|\mathbb{E}\big(d_{G(\mathbf{p})}(u)\big) - \mathbb{E}\big(d_{G(\mathbf{p})}(u')\big)| \le 1$. Conditional on any choice of $\mathbf{m} = (m_w)_{w\neq u}$, we see from (2.1) and (2.2) that there is some interval I of length 4 (depending on $\mathbf{m}$) such that $\mathcal{E}_{u,u'}$ holds if and only if $(\mathbf{u}-\mathbf{u}')\cdot\frac{m_u}{N}\mathbf{u} \in I$. As $(\mathbf{u}-\mathbf{u}')\cdot\mathbf{u} = 2\big|\big(N_G(u)\triangle N_G(u')\big)\cap V\big| \ge 2\delta N$, this corresponds to a choice of m_u in an interval of length $O(\delta^{-1})$, which occurs with probability $O(\delta^{-1}|U|^{-1})$. By Markov's inequality, we can therefore choose $\mathbf{p}\in[0.1,0.9]^V$ so that only $O(\delta^{-1}|U|)$ such $\mathcal{E}_{u,u'}$ hold. Then by Turán's theorem there is $U'\subset U$ of size $\Omega(\delta|U|)$ within which no such event $\mathcal{E}_{u,u'}$ holds, as required.

The actual proof is similar to the above sketch, except that we cannot rely on concentration of measure to ensure $\mathbf{p}\in[0.1,0.9]^V$; instead, we "truncate the outliers".

Proof of Lemma 2.4. By taking c small enough we may assume that $N \ge N_0(\delta)$. Secondly, replacing U with a subset if necessary, we can also assume that $|U| \le \delta N^{2/3}/5$. Let $(m_u)_{u\in U}$ and $\mathbf{p}$ be as in (2.1). For $u\in U$ we write $\mathbf{q}^u = \mathbf{p} - \frac{m_u}{N}\mathbf{u}$, and note that $\mathbf{q}^u$ is independent of m_u. We call u *good* if there are at most $\delta N/2$ coordinates $v\in V$ with $q^u_v \notin [0.2, 0.8]$, and bad otherwise. We also write U^g for the set of good vertices in U.

We claim that $\mathbb{P}(|U^g| \ge |U|/2) > 1/2$. To see this, we note for any u and v that $q^u_v - 1/2 = N^{-1}\sum_{u'\neq u} \pm m_{u'}$ $\big($where the $\pm$ sign is determined by $(\mathbf{u}')_v$ in (2.1)$\big)$ is a random variable with mean 0 and variance at most $N^{-2}|U|^3 < 0.01\delta$, so by Chebyshev's inequality $\mathbb{P}(|q^u_v - 1/2| > 0.3) < 0.01\delta/0.3^2 = \delta/9$. Thus the expected number of v with $q^u_v \notin [0.2,0.8]$ is at most $\delta N/9$, so by Markov's inequality u is bad with probability less than $1/4$. Now the expected number of bad u is less than $|U|/4$, so by Markov's inequality more than half of U is bad with probability less than $1/2$. The claim follows.

Now we define $\mathbf{p}'\in[0.1,0.9]^V$ by truncating $\mathbf{p}$: for each $v\in V$, if $p_v < 0.1$ let $p'_v = 0.1$, if $p_v > 0.9$ let $p'_v = 0.9$, or let $p'_v = p_v$ otherwise. We let $\mathcal{E}_{u,u'}$ denote the event that $|\mathbb{E}\big(d_{G(\mathbf{p}')}(u)\big) - \mathbb{E}\big(d_{G(\mathbf{p}')}(u')\big)| \le 1$.

We claim for any distinct u,u' in U that $\mathbb{P}(\mathcal{E}_{u,u'} \mid u\in U^g) < 4\delta^{-1}|U|^{-1}$. To see this, we condition on any choice of $\mathbf{m} = (m_w)_{w\neq u}$ such that u is good. We let V_0 be the set of $v\in V$ such that $q^u_v\notin[0.2,0.8]$, so that $|V_0|\le \delta N/2$. For each $v\in V\setminus V_0$ we have $p_v = q^u_v + N^{-1}m_u u_v = q^u_v \pm N^{-1}|U| \in [0.1,0.9]$, so $p'_v = p_v$ for any choice of m_u. Given $\mathbf{m}$, we can consider $f(m_u) := \mathbb{E}\big(d_{G(\mathbf{p}')}(u)\big) - \mathbb{E}\big(d_{G(\mathbf{p}')}(u')\big) = d_{G[U]}(u) - d_{G[U]}(u') + (\mathbf{u}-\mathbf{u}')\cdot\mathbf{p}'/2$ as a function of the random variable m_u. As in the sketch above $\mathcal{E}_{u,u'}$ can only occur if, conditioned on $\mathbf{m}$, $f(m_u)$ lies in an interval I of length 2 (again, depending on $\mathbf{m}$). To control this probability, note that for any $i\in[-|U|, |U|-1]$ we can write

$$f(i+1) - f(i) = \sum_{v\in V}(u_v - u'_v)N^{-1}u_v g_{i,v}/2,$$

where $g_{i,v} \in [0,1]$ and $g_{i,v} = 1$ for all $v \in V \setminus V_0$; the interpretation of $g_{i,v}$ is the proportion of the total change in p_v that is contained in $[0.1, 0.9]$. In particular,

$$\begin{aligned} f(i+1) - f(i) &\geq \sum_{v \in V \setminus V_0} (u_v - u'_v) N^{-1} u_v / 2 \\ &\geq N^{-1} |(N_G(u) \triangle N_G(u')) \cap (V \setminus V_0)| \\ &\geq N^{-1} \big(|N_G(u) \triangle N_G(u')| - |V_0| - |U| \big) > \delta/4. \end{aligned}$$

As $\mathcal{E}_{u,u'}$ only occurs if $f(m_u)$ lies in the interval I of length 2, we see that $\mathcal{E}_{u,u'}$ only occurs if m_u lies in an interval of length at most $8\delta^{-1}$; the claim follows.

The conclusion is similar to that in the above sketch. Indeed, letting J be the graph on U^g where uu' is an edge if $\mathcal{E}_{u,u'}$ holds, we have $\mathbb{E}[e(J)] < 8\delta^{-1}|U|/2$, so $\mathbb{P}(e(J) > 8\delta^{-1}|U|) < 1/2$. Thus with positive probability both $|U^g| \geq |U|/2$ and $e(J) \leq 8\delta^{-1}|U|$. By Turán's theorem, J has an independent set U' with $|U'| \geq \delta|U|/32$, as required. □

2.3. **Proof of Theorems 1.1 and 1.2.** We start with Theorem 1.2, which follows from Lemmas 2.1 and 2.4. To see this, again note that by taking c sufficiently small we may assume $N \geq N_0(\delta)$. Fix a δ-diverse set U of size $\frac{1}{2}N^{2/3}$ and set $V = V(G) \setminus U$. Applying Lemma 2.4 we obtain $\mathbf{p} \in [0.1, 0.9]^V$ and $U' \subset U$ with $|U'| \geq c|U|$ such that $|\mathbb{E}\big(d_{G(\mathbf{p})}(u)\big) - \mathbb{E}\big(d_{G(\mathbf{p})}(u')\big)| \geq 1$ for all distinct $u, u' \in U'$. Then Lemma 2.1 gives $W \subset V$ so that $G[U \cup W]$ has at least $c|U'|$ distinct degrees, as required.

To deduce Theorem 1.1 it suffices to show that if G is an N-vertex C-Ramsey graph, then G satisfies the hypotheses of Theorem 1.2, i.e., has a δ-diverse set U of size $N^{2/3}$ with $\delta = \Omega_C(1)$. We can deduce this from results of Kwan and Sudakov [11] as follows. Combining their Lemma 3 part 1 and Lemma 4, setting their δ equal to 1/4, we obtain $W \subset V(G)$ with $|W| = \Omega_C(N)$ such that for any $u \in W$ there are at most $|W|^{1/4}$ vertices $u' \in W$ with $|N_{G[W]}(u) \triangle N_{G[W]}(u')| < O_C(|W|)$. By Turán's theorem, W contains an $\Omega_C(1)$-diverse set U with $|U| = \Omega_C(N^{3/4}) > N^{2/3}$, as required.

3. Optimal homogeneous sets

In this section we will prove Theorem 1.3, which gives an optimal bound on $\hom(G)$ when $|V(G)| \gg f(G)$. In the first subsection we analyse the approximate structure of graphs G with $f(G)$ bounded. The second subsection introduces control graphs which are graphs with a special structure that facilitates finding induced subgraphs with many distinct degrees. The theorem itself is proved in the final subsection.

3.1. **Approximate structure.** Our first lemma, which is similar to [2, Lemma 2.3], shows that if a graph does not have an induced subgraph with many distinct degrees, then we can partition its vertices into a few parts so that vertices within any part have similar neighbourhoods. Below we will sometimes write (X_1, X_2) to represent a partition of a set X, i.e., $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. Similarly write $(X_1, \ldots, X_K)$ for a partition of $X = \bigcup_{i \in [K]} X_i$ into K pieces.

Lemma 3.1. *Suppose that G is an N-vertex graph with $f(G) < k$. Then there is a partition $(V_1, \ldots, V_L)$ of $V(G)$ with $L < 4k$ so that for all $i \in [L]$ and $u, u' \in V_i$ we have $\big|N_G(u) \triangle N_G(u')\big| \leq 2^{11}k^2$.*

Proof. Take a maximal set $S = \{v_1, \dots, v_L\} \subset V(G)$ such that $|N_G(v_i) \triangle N_G(v_j)| \geq 2^{10}k^2$ for all distinct i, j. We claim that $L < 4k$. This will suffice to prove the lemma; indeed, for any $u \in V(G)$ we can assign u to some part V_i such that $|N_G(u) \triangle N_G(v_i)| \leq 2^{10}k^2$, which exists by maximality of S.

To prove the claim, suppose for contradiction we have $S' \subset S$ with $|S'| = 4k$. We select $W \subset V(G)$ uniformly at random and consider the random graph J on S' consisting of all pairs $\{v_i, v_j\} \subset S' \cap W$ with the same degree in $G[W]$. Fix any $\{v_i, v_j\} \subset S'$, write $D = |N_G(v_i) \setminus N_G(v_j)|$ and $D' = |N_G(v_j) \setminus N_G(v_i)|$, say with $D \geq D'$. Conditional on any intersection of W with $N_G(v_j) \setminus N_G(v_i)$, we can bound $\mathbb{P}(\{v_i, v_j\} \in J)$ by

$$\max_j \mathbb{P}(Bin(D, 1/2) = j) \leq D^{-1/2} \leq 2|N_G(v_i) \triangle N_G(v_j)|^{-1/2} \leq (16k)^{-1}.$$

Thus $\mathbb{E}e(J) \leq \binom{4k}{2}(16k)^{-1} < k/2$, so $\mathbb{P}(e(J) \leq k) > 1/2$. As $\mathbb{P}\big(|W \cap S'| \geq 2k\big) \geq 1/2$, we can fix W with $|W \cap S'| \geq 2k$ and $e(J) \leq k$. Turán's theorem then gives $I \subset W \cap S'$ of size k that is independent in J, i.e., its vertices have distinct degrees in $G[W]$. This contradiction proves the claim, and so the lemma. $\square$

Our next lemma shows that neighbourhood similarity as in Lemma 3.1 implies an essentially homogeneous graph structure between parts and within parts (for the latter we will apply it with $V_1 = V_2$).

Lemma 3.2. *Let G be a graph with subsets V_1 and V_2 of $V(G)$ such that $|V_1| \geq 2D$ and $|N_G(v) \triangle N_G(v')| \leq D$ if $\{v, v'\} \subset V_1$ or $\{v, v'\} \subset V_2$. Then one of the following hold:*

(1) *each vertex in V_1 has at most $5D$ neighbours in V_2, or*
(2) *each vertex in V_1 has at least $|V_2| - 5D$ neighbours in V_2.*

Proof. Pick $v \in V_1$ and set $A = N(v) \cap V_2$ and $B = V_2 \setminus N(v)$. It suffices to show that $|A| \leq 4D$ or $|B| \leq 4D$. Indeed as $|N_G(v) \triangle N_G(v')| \leq D$ for every $v' \in V_1$, this gives either (1) or (2).

For each $v' \in V_1$, neighbourhood similarity again gives $|N_G(v') \cap A| \geq |A| - D$ and $|N_G(v') \cap B| \leq D$. Suppose for contradiction that $|A|, |B| > 4D$. Then $e(V_1, A) \geq |V_1|(|A| - D) > |V_1| \cdot 3|A|/4$, so there is $a \in A$ with $|N_G(a) \cap V_1| > 3|V_1|/4$. Similarly, $e(V_1, B) \leq |V_1|D < |V_1| \cdot |B|/4$, so there is $b \in B$ with $|N_G(b) \cap V_1| < |V_1|/4$. However, this gives the contradiction $|N_G(a) \triangle N_G(b)| > |V_1|/2 \geq D$. $\square$

In combination, Lemmas 3.1 and 3.2 show that if $f(G)$ is bounded, then G has the approximate structure of a blowup, in the sense of the next definition. The accompanying lemma applies a merging process to also guarantee that this blowup is non-degenerate, in that it is not also a blowup with fewer parts.

Definition 3.3. Let H be a graph and let $\mathcal{P}$ be a partition of $V(H)$. Given parts X, Y of $\mathcal{P}$, we let $H[X, Y]$ be the graph on $X \cup Y$ with edges $\{xy \in E(H) : x \in X, y \in Y\}$.

We call H a *$\mathcal{P}$-blowup* if each such $H[X, Y]$ (allowing $X = Y$) is empty or complete.

We call a $\mathcal{P}$-blowup H *non-degenerate* if it is not also a $\mathcal{P}'$-blowup for some partition $\mathcal{P}'$ of $V(H)$ with fewer parts than $\mathcal{P}$.

We call a graph G on $V(H)$ a *Δ-perturbation* of H if for any parts X and Y of $\mathcal{P}$ and v, v' in X we have $|N_G(v, Y) \triangle N_H(v', Y)| \leq \Delta$.

Lemma 3.4. *Suppose that G is an N-vertex graph with a partition $(V_1, \ldots, V_L)$ of $V(G)$ such that $|N_G(u) \triangle N_G(u')| \leq D_1$ for all $i \in [L]$ and u, u' in V_i. Let $L, T, \Delta \in \mathbb{N}$ with $T \geq 5\Delta \geq 300L^2 D_1$. Then there are partitions (W, R) of $V(G)$ and $\mathcal{P}$ of W such that $|R| \leq LT$, each part of $\mathcal{P}$ has size at least T, and $G[W]$ is a Δ-perturbation of a non-degenerate $\mathcal{P}$-blowup.*

Proof. We let R be the union of all V_i with $|V_i| \leq T$ (so clearly $|R| \leq LT$) and let $W = V(G) \setminus R$. Next we define a partition $\mathcal{P}$ of W by starting with that defined by restricting $(V_1, \ldots, V_L)$ and repeatedly merging any two parts X and Y if there are some $x \in X$ and $y \in Y$ with $|N_{G[W]}(x) \triangle N_{G[W]}(y)| \leq D_2 := 10LD_1$ (note that we measure the neighbourhood differences here according to $G[W]$ rather than G). This process terminates with some partition $\mathcal{P}$ whose parts have size at least T (by definition of R), so that for any distinct parts X, Y and $x \in X$, $y \in Y$ we have $|N_{G[W]}(x) \triangle N_{G[W]}(y)| > D_2$.

We claim that for any part X of $\mathcal{P}$ we have

$$|N_{G[W]}(x) \triangle N_{G[W]}(x')| \leq L(D_1 + D_2) \leq \Delta/5$$

for any x, x' in X. To see this, we show by induction on $t \geq 1$ that if X is a merger of t of the V_i's, then $|N_{G[W]}(x) \triangle N_{G[W]}(x')| \leq tD_1 + (t-1)D_2$ for any x, x' in X. When $t = 1$ this holds by our assumptions. Now suppose $t > 1$ and X was obtained by merging X_1 and X_2 with $|N_{G[W]}(w_1) \triangle N_{G[W]}(w_2)| \leq D_2$ for some $w_i \in X_i$. If each X_i is a merger of t_i of the V_i's, where $t = t_1 + t_2$, then by induction hypothesis $|N_{G[W]}(x_i) \triangle N_{G[W]}(x_i')| \leq t_i D_1 + (t_i - 1)D_2$ for any x_i, x_i' in X_i. Then for any x, x' in X we can bound $|N_{G[W]}(x) \triangle N_{G[W]}(x')|$ by $\big(t_1 D_1 + (t_1 - 1)D_2\big) + \big(t_2 D_1 + (t_2 - 1)D_2\big) + D_2 = tD_1 + (t-1)D_2$. This proves the claim.

It follows from Lemma 3.2 that $G[W]$ is a Δ-perturbation of some $\mathcal{P}$-blowup H. To show that H is non-degenerate, we need to show that for any distinct parts X and Y of $\mathcal{P}$ there is some part Z (possibly equal to X or Y) such that one of $H[X, Z]$ and $H[Y, Z]$ is complete and the other is empty.

To see this, we fix any $x \in X$ and $y \in Y$, and note by the merging rule that $|N_{G[W]}(x) \triangle N_{G[W]}(y)| > D_2 = 10LD_1$, so there is some part V_i of the original partition with $|(N_{G[W]}(x) \triangle N_{G[W]}(y)) \cap V_i| > 10D_1$. We must have $V_i \subset W$, so $|V_i| \geq T$ by definition of R. By Lemma 3.2, for any $u \in W$ we have $|N_{G[W]}(u) \cap V_i| \leq 5D_1$ or $|N_{G[W]}(u) \cap V_i| \geq |V_i| - 5D_1 \geq T - 5D_1$. We deduce that one of $|N_{G[W]}(x) \cap V_i|$ and $|N_{G[W]}(y) \cap V_i|$ is $\leq 5D_1$ and the other is $\geq |V_i| - 5D_1$, so they differ by at least $T - 10D_1 > 2\Delta$. Let Z be the part of $\mathcal{P}$ containing V_i. As $G[W]$ is a Δ-perturbation of H, we cannot have $H[X, Z]$ and $H[Y, Z]$ both complete or both empty. Thus H is non-degenerate, as required. □

3.2. **Control graphs.** Our strategy for proving Theorem 1.3 in the next subsection will be to find an induced subgraph as in the next definition; the following lemma shows that this will indeed have an induced subgraph with many distinct degrees.

Definition 3.5. We call a graph F a *k-control graph* if there are partitions (A, B, C) of $V(F)$ and $(C_1, \ldots, C_t)$ of C, where $A = \{a_1, \ldots, a_k\}$ and each $|C_i| \geq k^2 - 1$, such that

(i) given $(i, j) \in [k] \times [t]$ the bipartite graph $F[a_i, C_j]$ is either empty or complete, and
(ii) if $N_F(a_i) \cap C = N_F(a_j) \cap C$ and $i \neq j$, then $d_{F[A \cup B]}(a_i) \neq d_{F[A \cup B]}(a_j)$.

Lemma 3.6. *If F is a k-control graph, then $f(F) \geq k$.*

Proof. With notation as in Definition 3.5, we randomly select integers $m_i \in [0, |C_i|]$ uniformly and independently for each $i \in [t]$, fix $C'_i \subset C_i$ with each $|C'_i| = m_i$, and consider the induced subgraph $F' = F[A \cup B \cup C']$ with $C' = \bigcup_{i \in [t]} C'_i$. We will show that with positive probability, the vertices in A have distinct degrees in F', and so $f(F) \geq f(F') \geq k$.

Consider any distinct a, a' in A. If $N_F(a) \cap C = N_F(a') \cap C$, then by property (ii) we have $d_{F'}(a) \neq d_{F'}(a')$ regardless of the choice of $C'_1, \ldots, C'_t$. On the other hand, if $N_F(a) \cap C \neq N_F(a') \cap C$, then there is some C_i such that (say) $C_i \subset N_F(a)$ and $C_i \cap N_F(a') = \emptyset$. Conditional on any choices of $\{C'_j\}_{j \neq i}$, there is at most one choice of m_i that gives $d_{F'}(a) = d_{F'}(a')$, which occurs with probability $(|C_i|+1)^{-1} \leq k^{-2}$. We deduce $\mathbb{P}(f(F') < k) \leq \binom{k}{2} k^{-2} < 1/2$, so the lemma follows. □

In the proof of Theorem 1.3 we will obtain control graphs in each set of the partition from Lemma 3.4 using the following lemma, and combine these to form a k-control graph.

Lemma 3.7. *Let $\Delta, k, n, N \in \mathbb{N}$ with $n \geq 4k\Delta$ and $N > (k-1)(n-1)$.*

Suppose G is an N-vertex graph with independence number $\alpha(G) < n$ and a partition (U, W) of $V(G)$ with $|U| \leq n/2$ and $|N_G(v) \cap W| \leq \Delta$ for all $v \in V(G)$.

Then G contains a k-control graph with vertex partition $(A = \{a_1, \ldots, a_k\}, B, C)$, where $G[A]$, $G[A, C]$ are empty and $|C| \geq |W| - k^2\Delta$, and B has a partition $(B_1, \ldots, B_k)$ with each $|B_i| = i-1$ so that each $G[\{a_i\}, B_j]$ is complete if $i = j$ or empty if $i \neq j$.

Proof. If $k = 1$, then the result is clear, taking a_1 to be any vertex from $V(G)$, $B_1 = \emptyset$, and $C = W \setminus N_G(a_1)$. For $k > 1$ we argue by induction. By Turán's theorem, G contains a vertex $a \in V(G)$ with degree at least $k-1$. Let $a_k = a$ and $B_k \subset N_G(a)$ with $|B_k| = k-1$. Let G' be obtained from G by deleting U, a and $N_G(B_k \cup \{a\})$. We delete at most $|U| + 1 + \big(\Delta + (k-1)(\Delta - 1)\big) \leq k\Delta + n/2 \leq n-1$ vertices, so $|V(G')| \geq N - (n-1) > (k-2)(n-1)$. By induction G' contains $(A' = \{a_1, \ldots, a_{k-1}\}, B', C)$, where $G[A']$, $G[A', C]$ are empty and $|C| \geq (|W| - k\Delta) - (k-1)^2\Delta \geq |W| - k^2\Delta$, and B' has a partition $(B_1, \ldots, B_{k-1})$ with each $|B_i| = i-1$ so that each $G[\{a_i\}, B_j]$ is complete if $i = j$ or empty if $i \neq j$. We obtain A, B from A', B' by adding a_k, B_k; then (A, B, C) is as required, as there are no edges between $B_k \cup \{a_k\}$ and $V(G')$. □

Remark 3.8. The following simplified consequence of Lemma 3.7 will often be convenient to apply. Let G be an N-vertex graph G with $\hom(G) < n$ that is a Δ-perturbation of a one-part blowup (i.e., a complete or empty graph). Suppose $k = \phi(N) := \lceil \frac{N}{n-1} \rceil \leq n/4\Delta$ and $N > k^2\Delta + K$ with $K \geq k^2$. Then G has a k-control graph with partition (A, B, C) where $|C| = K$.

3.3. **Proof of Theorem 1.3.** To begin, we fix parameters, for reference during the proof. Set

$$D_1 = 2^{11}k^2; \quad \Delta = 2^{25}k^4; \quad \Delta_1 = 2^5\Delta k; \quad T = 2^4\Delta_1 k^2; \quad n_0 = 2^9\Delta_1 k^4 = 2^{45}k^9.$$

Let G be an N-vertex graph where $N = (k-1)(n-1)+1$ and $n \geq n_0$. We suppose for a contradiction that $\hom(G) < n$ and $f(G) < k$. Lemma 3.1 gives a partition $V(G) = V_1 \cup \cdots \cup V_L$ with $L \leq 4k$ such that $|N_G(u) \triangle N_G(u')| \leq D_1$ for all $u, u' \in V_i$.

Lemma 3.4 then gives partitions (W,R) of $V(G)$ and $\mathcal{P}=(W_1,\dots,W_M)$ of W such that $|R|\le LT$, each part of $\mathcal{P}$ has size at least T, and $G[W]$ is a Δ-perturbation of a non-degenerate $\mathcal{P}$-blowup H.

Our aim is to find a k-control graph, which by Lemma 3.6 will give the required contradiction that $f(G)\ge k$. This control graph will have partition (A,B,C) obtained by combining k_i-control graphs on vertex set $E_i\subset W_i$ with partitions (A_i,B_i,C_i) for each $i\in[M]$, where $\sum_i k_i=k$ and each $G[E_i,E_{i'}]$ with $i\ne i'$ is complete or empty according to H. We may also need an additional k_0-control graph with partition $(a_0,\emptyset,C_0)$ where $k_0=1$ and $a_0\in R$. We will ensure that all parts C_i^j of each C_i have size at least k^2-1, and the non-degeneracy of H will guarantee that vertices in distinct A_i's have distinct neighbourhoods in C, so this construction will indeed give a control graph on (A,B,C).

Next we will describe an algorithm that finds a k-control graph in some cases; we will later show how it can be modified to cover the remaining cases.

Algorithm. *We proceed in M rounds numbered by $i\in[M]$. At the start of round i we have sets $W_j^i\subset W_j$ for each $j\in[M]$, where each $W_j^1=W_j$ and we will obtain each W_j^{i+1} from W_j^i by deleting at most $2k^2\Delta$ vertices. As $G[W]$ is a Δ-perturbation of H, and $|W_i^i|\ge|W_i|-2Mk^2\Delta>2k^2\Delta$, we can apply Remark 3.8 to $G[W_i^i]$ with $K=k^2$, thus obtaining a k_i-control graph on a set E_i with partition (A_i,B_i,C_i) where $|C_i|=k^2$ and $k_i=\phi(|W_i^i|)=\lceil\frac{|W_i^i|}{n-1}\rceil\le n/4\Delta$. As $G[W]$ is a Δ-perturbation of H, for each $j>i$ we can remove $|E_i|\Delta\le 2k^2\Delta$ vertices from W_j^i to obtain W_j^{i+1} such that $G[E_i,W_j^{i+1}]$ is complete or empty according to $H[W_i,W_j]$. After all rounds are complete we obtain a k'-control graph with parts (A,B,C) where $A=\bigcup A_i$, $B=\bigcup B_i$, $C=\bigcup C_i$ and $k'=\sum k_i$.*

Now we consider what conditions guarantee $k'=k$ in the algorithm. To analyse this, we associate vertices of R with parts W_i according to any neighbourhood similarity. Specifically, we fix vertices $w_i\in W_i$ for each $i\in[M]$ and let

$$U_i:=\big\{v\in R:|N_G(v,W)\triangle N_G(w_i,W)|\le\Delta_1\big\}.$$

We start by considering the case that $\bigcup_{i\in[M]}U_i=R$.

As ϕ is superadditive, we have $\sum_{i\in[M]}\phi(|W_i\cup U_i|)\ge\phi(N)=k$. If we have

$$\phi(|W_i\cup U_i|)=\phi\big(|W_i|-4M\Delta_1k^2\big)$$

for all i, then we deduce

$$|A|=\sum_{i\in[M]}\phi(|W_i^i|)\ge\sum_{i\in[M]}\phi\big(|W_i|-4M\Delta_1k^2\big)=\sum_{i\in[M]}\phi\big(|W_i\cup U_i|\big)\ge k.$$

Thus we can assume (possibly by relabelling) that

$$\phi(|W_1\cup U_1|)>\phi\big(|W_1|-4M\Delta_1k^2\big).$$

If $|W_1|<n/2$ we estimate

$$|A|\ge 1+\sum_{i\in[2,M]}\phi\big(|W_i|-4M\Delta_1k^2\big)\ge 1+\phi(N-|R|-|W_1|)\ge 1+\phi(N-(n-1))=k.$$

Thus we can assume $|W_1|\ge n/2$.

To complete the analysis of this case, we modify round 1 of the algorithm by setting W_1^1 equal to $W_1\cup U_1$ rather than W_1. By definition of U_1 we can apply

Lemma 3.7 to either $G[W_1^1]$ or $\overline{G}[W_1^1]$, now with $W = W_1$, $U = U_1$, and $2\Delta_1$ in place of Δ, which is valid as $|U_1| \leq |R| \leq 4kT \leq n/2$, and $|W_1^1| - (2\Delta_1)k^2 \geq n/2 - (2\Delta_1)k^2 \geq k^2$ as $n \geq 5\Delta_1 k^2$. Thus in round 1 we find a k_1-control graph with $k_1 = \phi(|W_1 \cup U_1|) > \phi\big(|W_1| - 4M\Delta_1 k^2\big)$. The remainder of the algorithm is the same. Now we estimate

$$|A| = \sum_{i\in[M]} \phi(|W_i^i|) \geq 1 + \sum_{i\in[M]} \phi\big(|W_i| - 4M\Delta_1 k^2\big) \geq 1 + \phi(N - |R| - 4M^2\Delta_1 k^2) \geq k.$$

It remains to consider the case $\bigcup_{i\in[M]} U_i \neq R$. Here before applying the algorithm we first fix $a_0 \in R \setminus \big(\bigcup_{i\in[M]} U_i\big)$ and choose an extra 1-control graph $(a_0, \emptyset, C_0)$ as follows. For each $i \in [M]$, by definition of U_i we have

$$|N_G(a_0, W) \triangle N_G(w_i, W)| > \Delta_1 = 32\Delta k \geq 4M(k^2 + \Delta).$$

Thus we can greedily choose disjoint sets $C_{1,0}, \ldots, C_{M,0}$ so that each $C_{i,0}$ has size k^2, is contained in some $W_{j(i)}$, and is contained in $N_G(a_0, W) \setminus N_H(w_i, W)$ or $N_H(w_i, W) \setminus N_G(a_0, W)$ (recall that $G[W]$ is a Δ-perturbation of H). We let $C_0 = \bigcup_{i\in[M]} C_{i,0}$. Then we apply the algorithm as before, except that we now let $W_{i,1}$ be the set of $w \in W_i \setminus C_0$ with $N_G(w, C_0) = N_H(w, C_0)$, noting that $|W_{i,1}| \geq |W_i| - (\Delta + 1)|C_0| \geq |W_i| - 2\Delta M k^2$, as $G[W]$ is a Δ-perturbation of H. We still obtain a control graph, as the neighbourhood of a_0 differs from the neighbourhoods of all vertices in E_i on $C_{i,0}$. Furthermore, $|A| = |\{a_0\}| + \sum_{i\in[M]} |A_i| \geq k$. This completes the proof. $\square$

4. Concluding remarks

This paper was concerned with the minimum possible value of $f(G)$ in two regimes for $\hom(G)$. For Ramsey graphs, i.e., $\hom(G) = O(\log N)$, in Theorem 1.1 we showed $f(G) = \Omega(N^{2/3})$, which gives the correct order of magnitude (as shown by a random graph); it would be interesting (but no doubt very difficult) to obtain an asymptotic result.

At the other extreme, when $\hom(G)$ is large we have obtained an exact result, thus proving a conjecture of Narayanan and Tomon [14]. This also makes progress on another of their conjectures that $\hom(G) \geq N^{1/2}$ guarantees $f(G) = \Omega\big(\frac{N}{\hom(G)}\big)$; indeed, Theorem 1.3 proves this in a strong form provided $\hom(G) \geq \Omega(N^{9/10})$. The exponent here can be reduced by taking more care with the exceptional set R in the proof, but it seems that new ideas are needed to reduce the exponent to 1/2.

Finally, it would be particularly interesting to determine the minimum order of magnitude of $f(G)$ in the intermediate range of $\hom(G)$.

Acknowledgment

Thanks to the referee for a number of helpful comments.

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School of Mathematicss, University of Birmingham, Birmingham, United Kingdom
Email address: m.jenssen@bham.ac.uk

Mathematical Institute, University of Oxford, Oxford, United Kingdom
Email address: keevash@maths.ox.ac.uk

School of Mathematics, University of Birmingham, Birmingham, United Kingdom
Email address: e.long@bham.ac.uk

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, Illinois—and—London School of Economics, Department of Mathematics, London, UK
Email address: lyepre2@uic.edu, l.yepremyan@lse.ac.uk