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The generalised Oberwolfach problem ☆

Peter Keevash, Katherine Staden

Mathematical Institute, University of Oxford, Oxford, UK

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ABSTRACT

We prove that any quasirandom dense large graph in which all degrees are equal and even can be decomposed into any given collection of two-factors (2-regular spanning subgraphs). A special case of this result gives a new solution to the Oberwolfach problem.

1. Introduction

At meals in the Oberwolfach Mathematical Institute, the participants are seated at circular tables. At an Oberwolfach meeting in 1967, Ringel (see [17]) asked whether there must exist a sequence of seating plans so that every pair of participants sit next to each other exactly once. We assume, of course, that there are an odd number of participants, as each participant sits next to two others in each meal. The tables may have various sizes, which we assume are the same at each meal.

Oberwolfach problem (Ringel). Let F be any two-factor (i.e. 2-regular graph) on n vertices, where n is odd. Can the complete graph K_n be decomposed into copies of F?

We obtain a new solution of this problem for large n, with a theorem that is more general in three respects: (a) we can decompose any dense quasirandom graph that is

☆ Research supported in part by ERC Consolidator Grant 647678.
E-mail addresses: keevash@maths.ox.ac.uk (P. Keevash), staden@maths.ox.ac.uk (K. Staden).

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regular of even degree (not just K_n for n odd), (b) we can decompose into any prescribed collection of two-factors (not just copies of some fixed two-factor F), (c) our theorem applies to directed graphs (digraphs).

We start by stating our result for undirected graphs. We require the following quasirandomness definition. We say that a graph G on n vertices is (ε, t)-typical if every set S of at most t vertices has $((1 \pm \varepsilon)d(G))^{|S|}n$ common neighbours, where $d(G) = e(G)\binom{n}{2}^{-1}$ is the density of G.

Theorem 1.1. *For all $\alpha > 0$ there exist t, ε, n_0 such that any (ε, t)-typical graph on $n \geq n_0$ vertices that is $2r$-regular for some integer $r > \alpha n$ can be decomposed into any family of r two-factors.*

Theorem 1.1 implies some variant forms of the Oberwolfach problem that have appeared in the literature, such as the Hamilton–Waterloo Problem (two types of two-factors), or that if n is even then K_n can be decomposed into a perfect matching and any specified collection of $n/2 - 1$ two-factors. More generally, with parameters as in Theorem 1.1, it is easy to deduce that any (ε, t)-typical graph on $n \geq n_0$ vertices that is $(2r + 1)$-regular for some integer $r > \alpha n$ can be decomposed into a perfect matching and any family of r two-factors.

We will deduce Theorem 1.1 from the directed version below. First we extend our definitions to digraphs. We say that a digraph G on n vertices is (ε, t)-typical if for every set $S = S^- \cup S^+$ of at most t vertices there are $((1 \pm \varepsilon)d(G))^{|S|}n$ vertices which are both common inneighbours of S^- and outneighbours of S^+, where $d(G) = e(G)\binom{n}{2}^{-1}$ is the density of G. We say that G is r-regular if $d^+_G(v) = d^-_G(v) = r$ for all $v \in V(G)$. A *one-factor* is a 1-regular digraph; equivalently, it is a union of vertex-disjoint oriented cycles.

Theorem 1.2. *For all $\alpha > 0$ there exist t, ε, n_0 such that any (ε, t)-typical digraph on $n \geq n_0$ vertices that is r-regular for some integer $r > \alpha n$ can be decomposed into any family of r one-factors.*

Theorem 1.1 follows from Theorem 1.2 and the observation that for any typical graph that is regular of even degree there exists an orientation which is a regular typical digraph. To see this, one can orient edges independently at random and make a few modifications to obtain the required orientation. (See Lemma 9.1 below for a similar argument.)

While we were preparing this paper, the Oberwolfach problem (for large n) was solved by Glock, Joos, Kim, Kühn and Osthus [9]. They also obtained a more general result that covers the other undirected applications just mentioned, but our result is more general than theirs in the three respects mentioned above: (a) we can decompose any dense typical regular graph (whereas their result only applies to almost complete graphs), (b) we can decompose into any collection of two-factors (whereas they can allow for a collection of two-factors provided that some fixed F occurs $\Omega(n)$ times), (c) our result also applies to digraphs (whereas theirs is for undirected graphs).

There is a large literature on the Oberwolfach Problem, of which we mention just a few highlights (a more detailed history is given in [9]). The problem was solved for infinitely many n by Bryant and Scharaschkin [6], in the case when F consists of two cycles by Traetta [20], and for cycles of equal length by Alspach, Schellenberg, Stinson and Wagner [3]. A related conjecture of Alspach that K_n can be decomposed into any collection of cycles each of length $\leq n$ and total size $\binom{n}{2}$ was solved by Bryant, Horsley and Pettersson [5].

There are several recent general results on approximate decompositions that imply an approximate solution to the generalised Oberwolfach Problem, i.e. that any given collection of two-factors can be embedded in a quasirandom graph provided that a small fraction of the edges can be left uncovered: we refer to the papers of Allen, Böttcher, Hladký and Piguet [1], Ferber, Lee and Mousset [8] and Kim, Kühn, Osthus and Tyomkyn [15].

Notation.

Given a graph $G = (V, E)$, when the underlying vertex set V is clear, we will also write G for the set of edges. So $|G|$ is the number of edges of G. Usually $|V| = n$. The *edge density* $d(G)$ of G is $|G|/\binom{n}{2}$. We write $N_G(x)$ for the neighbourhood of a vertex x in G. The degree of x in G is $d_G(x) = |N_G(x)|$. For $A \subseteq V(G)$, we write $N_G(A) := \bigcap_{x \in A} N_G(x)$; note that this is the common neighbourhood of all vertices in A, not the neighbourhood of A.

In a directed graph J with $x \in V(J)$, we write $N^+_J(x)$ for the set of out-neighbours of x in G and $N^-_G(x)$ for the set of in-neighbours. We let $d^\pm_G(A) := |N^\pm_G(A)|$. We define common out/in-neighbourhoods $N^\pm_J(A) = \bigcap_{x \in A} N^\pm_J(A)$.

We say G is (ε, t)-*typical* if $d_G(S) = ((1 \pm \varepsilon)d(G))^{|S|}n$ for all $S \subseteq V(G)$ with $|S| \leq t$.

We say that an event E holds with high probability (whp) if $\mathbb{P}(E) > 1 - \exp(-n^c)$ for some $c > 0$ and $n > n_0(c)$. We note that by a union bound for any fixed collection $\mathcal{E}$ of such events with $|\mathcal{E}|$ of polynomial growth whp all $E \in \mathcal{E}$ hold simultaneously.

We omit floor and ceiling signs for clarity of exposition.

We write $a \ll b$ to mean $\forall\ b > 0\ \exists\ a_0 > 0\ \forall\ 0 < a < a_0$.

We write $a \pm b$ for an unspecified number in $[a - b, a + b]$.

Throughout the vertex set V will come with a cyclic order, which we usually identify with the natural cyclic order on $[n] = \{1, \ldots, n\}$. For any $x \in V$ we write x^+ for the successor of x, so if $x \in [n]$ then x^+ is $x + 1$ if $x \neq n$ or 1 if $x = n$. We define the predecessor x^- similarly. Given x, y in $[n]$ we write $d(x, y)$ for their cyclic distance, i.e. $d(x, y) = \min\{|x - y|, n - |x - y|\}$.

2. Overview of the proof

We will illustrate the ideas of our proof by starting with a special case and becoming gradually more general. Suppose first that we wish to decompose a typical dense (undirected) $2r$-regular graph G on n vertices into r triangle-factors (i.e. two-factors in which each cycle is a triangle – we require $3 \mid n$ for this question to make sense). The existence of such a decomposition (also known as a resolvable triangle-decomposition of G) follows from a recent result of the first author [12] generalising the existence of designs (see [11]) to many other 'design-like' problems. The proof in [12] goes via the following auxiliary decomposition problem, which also plays an important role in this paper.

Let J be an auxiliary graph with $V(J)$ partitioned as $V \cup W$, where $V = V(G)$ and $|W| = r$. Let $J[V] = G$, $J[V,W] = V \times W$ and $J[W] = \emptyset$. Note that a decomposition of G into triangle-factors is equivalent to a decomposition of J into copies of K_4 each having 3 vertices in V and 1 vertex in W. Indeed, given such a decomposition of J, for each $w \in W$ we define a triangle-factor of G by removing w from all copies of K_4 containing w in the decomposition; clearly every edge of G appears in exactly one of these triangle-factors. Conversely, any decomposition of G into triangle-factors can be converted into a suitable K_4-decomposition of J by adding each $w \in W$ to one of the triangle-factors (according to an arbitrary matching).

The auxiliary construction described above is quite flexible, so a similar argument covers many other cases of our problem. For example, decomposing G into C_ℓ-factors (two-factors in which each cycle has length ℓ) is equivalent to decomposing J into 'wheels' W_ℓ with 'rim' in V and 'hub' in W. (We obtain W_ℓ from C_ℓ, which is called the rim, by adding a new vertex, called the hub, joined to every other vertex, by edges that we call spokes.) Such a decomposition exists by [12].

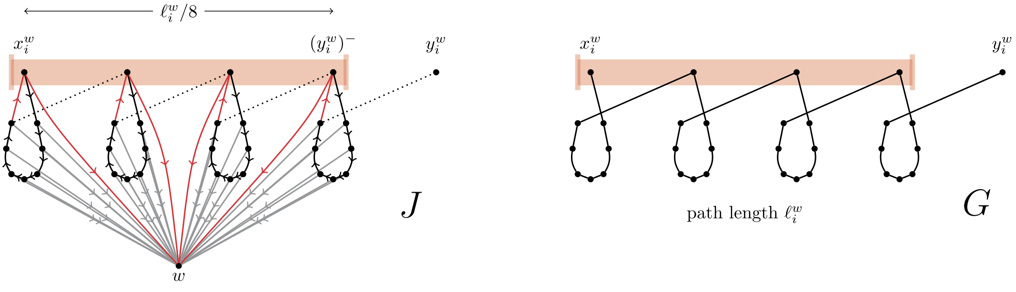
We can encode our generalised Oberwolfach Problem in full generality by introducing colours on the edges. For each possible cycle length ℓ we introduce a colour, which we also call ℓ. For each $w \in W$, we denote its corresponding factor by F_w, and suppose that it has n^w_ℓ cycles of length ℓ (where $\sum_\ell \ell n^w_\ell = n$). We colour J so that each $w \in W$ is incident to exactly n^w_ℓ edges of colour ℓ, and all other edges are uncoloured. We colour each W_ℓ so that exactly one spoke has colour ℓ and all other edges are uncoloured. Then a decomposition of G into $\{F_w : w \in W\}$ is equivalent to a decomposition of J into wheels with this colouring with rim in V and hub in W. Note that this equivalence does not depend on which edges of J we colour, but to apply [12] we will require the colouring to be suitably quasirandom. Another important constraint in applying [12] is that the number of colours and the size of the wheels should be bounded by an absolute constant. Thus our generalised Oberwolfach Problem can only be solved by direct reduction to [12] in the case that all factors have all cycle lengths bounded by some absolute constant.

This now brings us to the crucial issue for this paper: how can we encode two-factors with cycles of arbitrary length by an auxiliary construction to which [12] applies? Before describing this, we pass to an auxiliary problem of decomposing a subgraph G' of G into graphs $(G_w : w \in W)$, where each G_w is a vertex-disjoint union of paths with prescribed endpoints, lengths and vertex set. More precisely, for each $w \in W$ we are given specified lengths $(\ell^w_i : i \in I_w)$, vertex-pairs $((x^w_i, y^w_i) : i \in I_w)$, a forbidden set Z_w, and we want each G_w to be a union of vertex-disjoint $x^w_i y^w_i$-paths of length ℓ^w_i for each $i \in I_w$ with $V(G_w) = V(G) \setminus Z_w$. We will arrive at this problem having embedded some subgraphs $F'_w \subseteq F_w$ of each $w \in W$, so the prescribed endpoints will be endpoints of paths in F'_w that need to be connected up to form cycles, and Z_w will consist of all vertices of degree

2 in F'_w. We assume that all lengths ℓ_i^w are divisible by 8 (which is easy to ensure for long cycles).

We will translate the above path factor problem into an equivalent problem of decomposing a certain auxiliary two-coloured directed graph J, with $V(J) = V \cup W$ as in the previous construction. We call the two colours '0' (which means 'uncoloured') and 'K' (which means 'special'). Again, $J[W] = \emptyset$. For now we defer discussion of $J[V, W]$ and describe the arcs of $J[V]$, which are in bijection with the edges of G. For colour 0 this bijection simply corresponds to a choice of orientation for edges, but for colour K we employ the following 'twisting' construction. We fix throughout a cyclic order of V, and require that each arc $\overrightarrow{xy}$ of colour K in J comes from an edge xy^+ of G, where y^+ denotes the successor of y in the cyclic order.

Consider any directed 8-cycle C in J with vertex sequence $x_1 \dots x_8$, such that all arcs have colour 0 except that $\overrightarrow{x_7x_8}$ has colour K. The edges in G corresponding to C form a path with vertex sequence $x_8x_1 \dots x_7x_8^+$. Now suppose we have a family of such cycles $\mathcal{C} = (C^i : i \in I)$ where each C^i has vertex sequence $x_1^i \dots x_8^i$. Call $\mathcal{C}$ compatible if (i) its cycles are mutually vertex-disjoint, and (ii) if any $(x_8^i)^+$ is used by a cycle in $\mathcal{C}$ then it is some x_8^j. Suppose $\mathcal{C}$ is compatible and let $([x_j, y_j] : j \in J)$ denote the family of maximal cyclic intervals contained in $\{x_8^i : i \in I\}$. Then the edges of G corresponding to the cycles of $\mathcal{C}$ form a family of vertex-disjoint paths $(P_j : j \in J)$, where each P_j is an $x_jy_j^+$-path whose vertex sequence is the concatenation of vertex sequences of the 8-paths as described above for each cycle of $\mathcal{C}$ using a vertex of $[x_j, y_j]$.



The above construction allows us to pass from the path factor problem to finding certain edge-disjoint compatible cycle families in J. In order for our path factor problem to obey the constraints of this encoding we require the prescribed vertex-pairs for each w to define disjoint cyclic intervals $([x_i^w, (y_i^w)^-] : i \in I_w)$ of lengths $\ell_i^w/8$ (and also that no successor y_i^w is contained in any of the other intervals for w, where a successor of an interval is the successor of its largest member). We are thus introducing extra constraints into the path factor problem that may affect up to $n/8$ vertices for each w, but the flexibility on the remaining vertices will be sufficient.

Now we can complete the description of the auxiliary graph J and the decomposition problem that encodes the compatible cycle family problem. We define $J[V]$ as above, and $J[V, W]$ so that all arcs are directed towards W, each in-neighbourhood $N_J^-(w)$ is obtained from $V(G) \setminus Z_w$ by deleting the interval successors $\{y_i^w : i \in I_w\}$, all arcs

$\overrightarrow{xw}$ with x in an interval $[x_i^w, (y_i^w)^-]$ are coloured K, and all other arcs of $J[V, W]$ are coloured 0. Finally, the compatible cycle family problem is equivalent to decomposing J into coloured directed wheels $\overrightarrow{W}_8^K$, obtained from W_8 by directing the rim cyclically, directing all spokes towards the hub w, giving colour K to one rim edge $\overrightarrow{xy}$ and one spoke $\overrightarrow{yw}$, and colouring the other edges by 0. The deduction from [12] of the existence of wheel decompositions is given in section 3.

We now describe the strategy for the proof of Theorem 1.2. The goal is to embed some parts of our two-factors so that the remaining problem is of one of two special types that has an encoding suitable for applying [12], either a path factor problem encoded as $\overrightarrow{W}_8^K$-decomposition or a C_ℓ-factor problem encoded as $\overrightarrow{W}_\ell$-decomposition (obtained from the coloured wheel W_ℓ discussed above for C_ℓ-factors by introducing directions as in $\overrightarrow{W}_8^K$, which are not necessary but convenient for giving a unified analysis). We call a factor 'long' if it has at least $n/2$ vertices in cycles of length at least K (as well as denoting the special colour, K is also used as a large constant length threshold, above which we treat cycles using the special twisting encoding as above). We call the other factors 'short'.

We start by reducing to the case that all factors are long or all factors are short. To do so, suppose first that there are $\Omega(n)$ long factors and $\Omega(n)$ short factors. Then we can randomly partition G into typical graphs G^L and G^S, each of which is regular of the correct degree (twice the number of long factors for G^L and twice the number of short factors for G^S). If there are $o(n)$ factors of either type then these can be embedded one-by-one (by the blow-up lemma [16]), and then the remaining problem still satisfies the conditions of Theorem 1.2 (with slightly weaker typicality). The short factor problem can be further reduced to the case that there is some length ℓ^* such that each factor has $\Omega(n)$ cycles of length ℓ^*. Indeed, we can divide the factors into a constant number of groups according to some choice of cycle length that appears $\Omega(n)$ times in each factor of the group. Any group of $o(n)$ factors can be embedded greedily, so after taking a suitable random partition, it suffices to show that the remaining groups can each be embedded in a graph that is typical and regular of the correct degree.

Thus we can assume that we are in one of the following cases. Case K: all factors are long, our goal is to reduce to $\overrightarrow{W}_8^K$-decomposition. Case ℓ^*: all factors have $\Omega(n)$ cycles of length ℓ^*, our goal is to reduce to $\overrightarrow{W}_{\ell^*}$-decomposition. In any case, the reduction is achieved by applying an approximate decomposition result in a suitable random subgraph, in which we embed a subgraph of each of our factors. At this step, in Case ℓ^* we embed all cycles of length $\neq \ell^*$, and in Case K we embed all short cycles and some parts of the long cycles as needed to reduce to a suitable path factor problem.

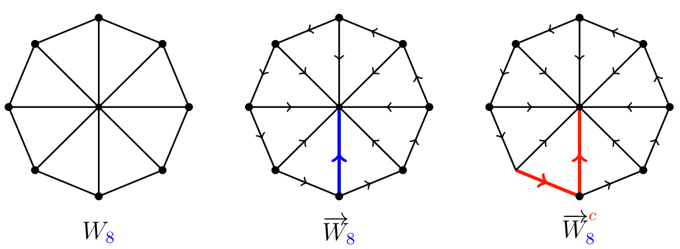
This approximate decomposition result is superficially similar to the maximum degree 2 case of the blow-up lemma for approximate decompositions due to Kim, Kühn, Osthus and Tyomkyn [15]. However, it does not suffice to use their result, as we require a decomposition that is compatible with the conditions of our final decomposition problem (into $\overrightarrow{W}_8^K$ or $\overrightarrow{W}_{\ell^*}$), so the sets of vertices of the partial factors embedded in this step must be suitably quasirandom and avoid the intervals needed for Case K. Furthermore, we obtain the required approximate decomposition by similar arguments to those for the exact decomposition, which does not add much extra work.

The technical heart of the paper is a randomised algorithm (presented in section 4), which gives a unified treatment of the cases described above. It simultaneously (a) partitions almost all of G into two graphs G_1 and G_2, and (b) sets up auxiliary digraphs J_1 and J_2 such that (i) an approximate wheel decomposition of J_2 gives an approximate decomposition of G_2 into the partial factors described above, and (ii) the graph G_1' of edges that are unused by the approximate decomposition has an auxiliary digraph that is a sufficiently small perturbation of J_1 that it can still be used for the exact decomposition step. The analysis of the algorithm falls naturally into two parts: the choice of intervals (section 5), then regularity properties of an auxiliary hypergraph defined by wheels (section 6). The results of this analysis are applied to show the existence of the various partial factor decompositions discussed above: the approximate step is in section 7 and the exact step in section 8. Section 9 combines all the ingredients prepared in the previous sections to produce the proof of our main theorem (illustrated in Fig. 1). The final section contains some concluding remarks.

3. Wheel decompositions

In this section we describe the results we need on wheel decompositions and how they follow from [12]. We start by recalling the coloured wheels described in section 2.

For any $c \geq 3$, the uncoloured c-wheel consists of a directed c-cycle (called the rim), another vertex (called the hub), and an arc from each rim vertex to the hub. We obtain the coloured c-wheel $\overrightarrow{W}_c$ by giving all arcs colour 0 except that one of the spokes has colour c. We obtain the special c-wheel $\overrightarrow{W}_c^K$ by giving all arcs colour 0 except that one rim edge $\overrightarrow{xy}$ and one spoke $\overrightarrow{yw}$ have colour K. As discussed in section 2, we will only use $\overrightarrow{W}_c^K$ with $c = 8$, but here we will consider the general configuration so that the decomposition problems are quite similar. We start by stating the result for $\overrightarrow{W}_c$.



Theorem 3.1. *Let $n^{-1} \ll \delta \ll \omega \ll c^{-1}$ and $h = 2^{50c^3}$. Let $J = J^0 \cup J^c$ be a digraph with arcs coloured 0 or c, with $V(J)$ partitioned as (V, W) where $\omega n \leq |V|, |W| \leq n$. Then J has a $\overrightarrow{W}_c$-decomposition such that every hub lies in W if the following hold:*

Divisibility: *all arcs in $J[V]$ have colour 0, all arcs in $J[V, W]$ point towards W, $d_J^-(v, V) = d_J^+(v, V) = d_J^+(v, W)$ for all $v \in V$, and $d_J^-(w) = cd_{J^c}^-(w)$ for all $w \in W$.*

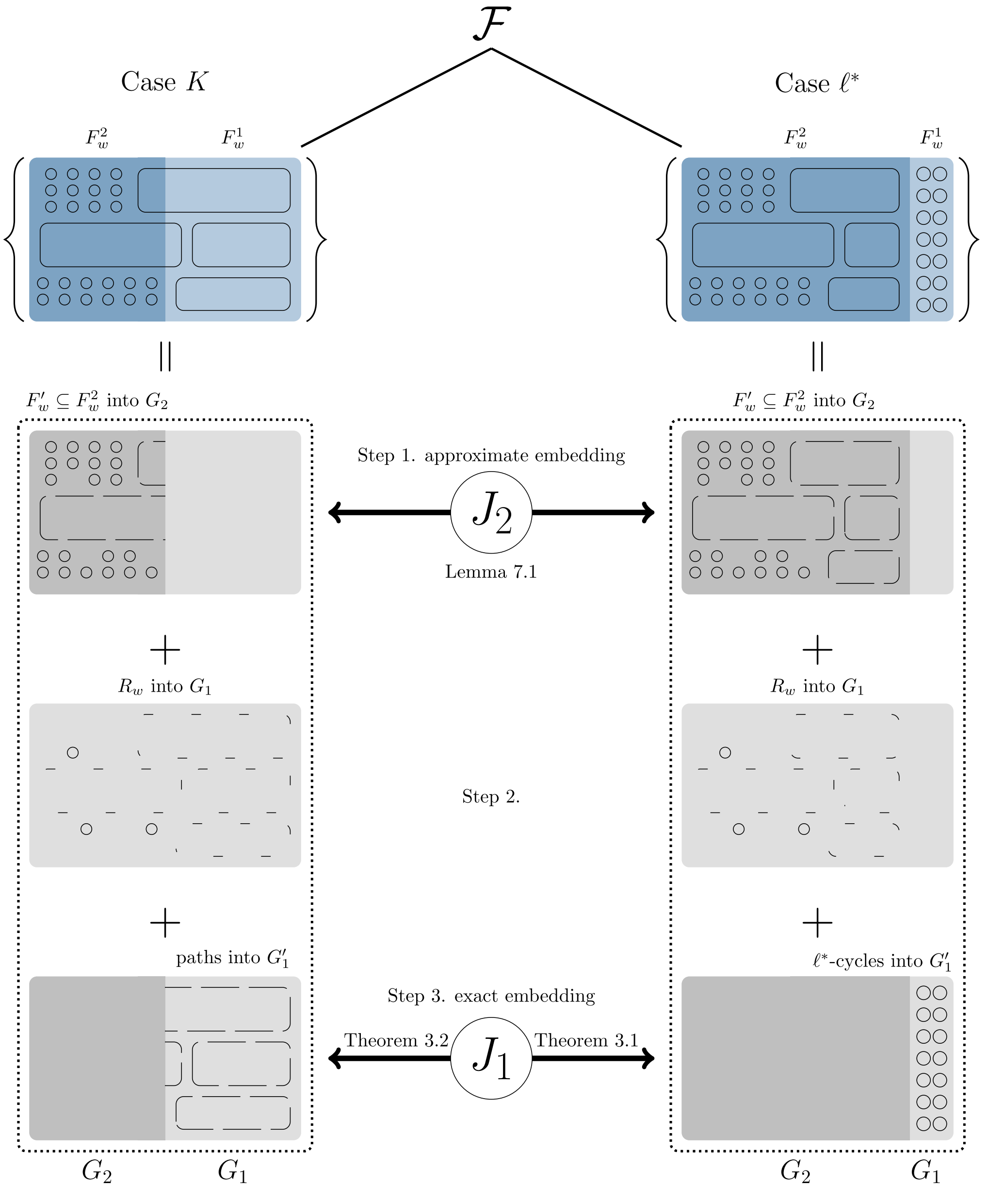


Fig. 1. An overview of the proof.

Regularity: *each copy of* $\overrightarrow{W}_c$ *in* J *has a weight in* $[\omega n^{1-c}, \omega^{-1} n^{1-c}]$ *such that for any arc* $\overrightarrow{e}$ *there is total weight* $1 \pm \delta$ *on wheels containing* $\overrightarrow{e}$.

Extendability: *for all disjoint* $A, B \subseteq V$ *and* $C \subseteq W$ *each of size* $\leq h$ *we have* $|N^+_{J^0}(A) \cap N^+_{J^c}(B) \cap W| \geq \omega n$ *and* $|N^+_{J^0}(A) \cap N^-_{J^0}(B) \cap N^-_{J^{c'}}(C)| \geq \omega n$ *for both* $c' \in \{0, c\}$.

Before stating our result on $\overrightarrow{W}_8^K$-decompositions, we recall that V has a cyclic order, which we can identify with the natural cyclic order on $[n]$, and define the following separation properties.

Definition 3.2. For $1 \leq x < y \leq n$ the cyclic distance is $d(x,y) = \min\{y-x, n+x-y\}$. We say that $S \subseteq [n]$ is d-separated if $d(a,a') \geq d$ for all distinct a, a' in S. For disjoint $S, S' \subseteq [n]$ we say (S, S') is d-separated if $d(a,a') \geq d$ for all $a \in S$, $a' \in S'$. For a (di)graph H whose vertex set is a subset of $[n]$ we say H is d-separated if $V(H)$ is d-separated.

Now we state our result on $\overrightarrow{W}_8^K$-decompositions. We note that it only concerns digraphs J such that $d(x,y) \geq d$ for all $\overrightarrow{xy} \in J[V]$, as this is implied by the regularity assumption. Our proof of Theorem 1.2 will require us to only consider such J, so that we can satisfy the extendability assumption.

Theorem 3.3. *Let $n^{-1} \ll \delta \ll \omega \ll c^{-1}$. Let $h = 2^{50c^3}$ and $d \ll n$. Let $J = J^0 \cup J^K$ be a digraph with arcs coloured 0 or K, with $V(J)$ partitioned as (V, W) where $\omega n \leq |V|, |W| \leq n$, such that all arcs in $J[V,W]$ point towards W and $J[W] = \emptyset$. Then J has a $\overrightarrow{W}_c^K$-decomposition such that every hub lies in W if the following hold:*

Divisibility: *$d_J^-(w) = c d_{J^K}^-(w)$ for all $w \in W$, and for all $v \in V$ we have $d_J^-(v,V) = d_J^+(v,V) = d_J^+(v,W)$ and $d_{J^K}^-(v,V) = d_{J^K}^+(v,W)$.*

Regularity: *each $3d$-separated copy of $\overrightarrow{W}_c^K$ in J has a weight in $[\omega n^{1-c}, \omega^{-1} n^{1-c}]$ such that for any arc $\overrightarrow{e}$ there is total weight $1 \pm \delta$ on wheels containing $\overrightarrow{e}$.*

Extendability: *for all disjoint $A, B \subseteq V$ and $L \subseteq W$ each of size $\leq h$, for any $a, b, \ell \in \{0, K\}$ we have $|N_{J^a}^+(A) \cap N_{J^b}^-(B) \cap N_{J^\ell}^-(L)| \geq \omega n$, and furthermore, if (A,B) is $3d$-separated then $|N_{J^0}^+(A) \cap N_{J^K}^+(B) \cap W| \geq \omega n$.*

For the remainder of this section we will explain how Theorem 3.3 follows from [12] (we omit the similar and simpler details for Theorem 3.1). We follow the exposition in [13], which deduces from [12] a general result on coloured directed designs that we will apply here.

3.1. The functional encoding

We encode any digraph J (or H) by a set of functions $\mathfrak{J}$ (or $\mathfrak{H}$), where for each arc $\overrightarrow{ab} \in J$ we include in $\mathfrak{J}$ the function $(1 \mapsto a, 2 \mapsto b)$, i.e. the function $f : [2] \to V(J)$ with $f(1) = a$ and $f(2) = b$ (and similarly for $H, \mathfrak{H}$). We will identify $\mathfrak{J}$ with its characteristic vector, i.e. $\mathfrak{J}_f = 1_{f \in \mathfrak{J}}$; if we want to emphasise the vector interpretation we write $\underline{\mathfrak{J}}$. If J has coloured arcs, and ℓ is a colour, we write J^ℓ for the digraph in colour ℓ, which is encoded by $\mathfrak{J}^\ell$.

We will consider decompositions by a coloured digraph H defined as follows. We start with $\overrightarrow{W}_c^K$ on the vertex set $[c+1]$, where we label the rim cycle by $[c]$ cyclically (so

$c+1$ is the hub) so that, writing $c_- = c-1$ and $c_+ = c+1$, $\overrightarrow{c_-c}$ and $\overrightarrow{cc_+}$ have colour K and all other arcs have colour 0. We let $\mathcal{P}$ be the partition $([c], \{c_+\})$ of $[c+1]$. We introduce new colours $0'$ and K', and change the colours of $\overrightarrow{cc_+}$ to K' and of the other spokes to $0'$. We do this so that H is '$(\mathcal{P}, \mathrm{id})$-canonical' in the sense of [13, Definition 7.1]; specialised to our setting, the relevant properties are that H is an oriented graph (with no multiple edges or 2-cycles) and that for each colour all of its arcs have one fixed pattern with respect to $\mathcal{P}$ (specifically, for colours 0 and K all arcs are contained in $[c]$, and for colours $0'$ and K' all arcs are directed from $[c]$ to $\{c_+\}$).

Now we translate the H-decomposition problem for a digraph J into its functional encoding. We will have a partition $\mathcal{Q} = (V, W)$ of $V(J)$, and wish to decompose J by copies $\phi(H)$ of H such that $\phi([c]) \subseteq V$ and $\phi(c_+) \in W$ (i.e. wheels with hub in W), and $\phi([c])$ is $3d$-separated (in which case we will say that the graph $\phi(H)$ is $3d$-separated). We think of the functional encoding $\mathfrak{J}$ as living inside a 'labelled complex' Φ of all possible partial embeddings of H: we define $\Phi = (\Phi_B : B \subseteq [c+1])$, where each Φ_B consists of all injections $\phi : B \to V(J)$ such that $\phi(B \cap [c]) \subseteq V$, $\phi(B \cap \{c_+\}) \subseteq W$ and $Im(\phi)$ is $3d$-separated. The set of functional encodings of possible embeddings of H (if present in $\mathfrak{J}$) is then

$$H(\Phi) := \{\phi\mathfrak{H} : \phi \in \Phi_{[c+1]}\}, \quad \text{where } \phi\mathfrak{H} := \{\phi \circ \theta : \theta \in \mathfrak{H}\}.$$

The H-decomposition problem for J is equivalent to finding $\mathcal{H} \subseteq H(\Phi)$ with $\sum\{\underline{\mathfrak{H}'} : \mathfrak{H}' \in \mathcal{H}\} = \underline{\mathfrak{J}}$, or equivalently $\bigcup \mathcal{H} = \mathfrak{J}$ (where if $\mathfrak{J}$ has multiple edges we consider a multiset union). We call such $\mathcal{H}$ an H-decomposition in Φ.

3.2. Regularity

Now we will describe the hypotheses of the theorem that will give us an H-decomposition in Φ. We start with regularity, which is simply the functional encoding of the regularity assumption in Theorem 3.3. Specifically, we say J is (H, δ, ω)-regular in Φ if there are weights $y_\phi \in [\omega n^{1-c}, \omega^{-1} n^{1-c}]$ for each $\phi \in \Phi_{[c+1]}$ with $\phi\mathfrak{H} \subseteq \mathfrak{J}$ such that $\sum_\phi y_\phi \underline{\phi\mathfrak{H}} = (1 \pm \delta)\underline{\mathfrak{J}}$.

3.3. Extendability

Next we consider extendability, which we discuss in a simplified setting that suffices for our purposes, following [13, Definition 7.3]. The idea is that for any vertex x of H there should be many ways to extend certain sets of partial embeddings of $H - x$ to embeddings of H. Specifically, we say (Φ, J) is (ω, h, H)-vertex-extendable if for any $x \in [c+1]$ and disjoint $A_i \subseteq V \cup W$ for $i \in [c+1] \setminus \{x\}$ each of size $\leq h$ such that $(i \mapsto v_i : i \in [c+1] \setminus \{x\}) \in \Phi$ whenever each $v_i \in A_i$, there are at least ωn vertices v such that

i. $(i \mapsto v_i : i \in [c+1]) \in \Phi$ whenever $v_x = v$ and $v_i \in A_i$ for each $i \neq x$, and

ii. each $\mathfrak{J}^\ell$ with $\ell \in \{0, K, 0', K'\}$ contains all $(1 \mapsto v_1, 2 \mapsto v_2)$ where for some $\theta \in \mathfrak{H}^\ell$ we have $(v_1 = v \ \& \ v_2 \in A_{\theta(2)})$ or $(v_2 = v \ \& \ v_1 \in A_{\theta(1)})$.

Note that by definition of Φ this only concerns maps ϕ such that $Im(\phi)$ is $3d$-separated. To interpret (ii) we consider 4 cases according to the position of x in the wheel.

$x = c+1$. For any pairwise $3d$-separated $A_i \subseteq V$, $i \in [c]$ of sizes $\leq h$ there are at least ωn vertices v such that $\overrightarrow{v_c v} \in J^{K'}$ for all $v_c \in A_c$ and $\overrightarrow{v_i v} \in J^{0'}$ for all $v_i \in A_i$, $i \neq c$. Equivalently, for any disjoint $A, B \subseteq V$ with $|A| \leq h$ and $|B| \leq (c-1)h$ such that (A, B) is $3d$-separated we have $|N^+_{J^{K'}}(A) \cap N^+_{J^{0'}}(B)| \geq \omega n$.

$x = c$. For any pairwise $3d$-separated $A_i \subseteq V$, $i \in [c-1]$ and $A_{c+1} \subseteq W$ of sizes $\leq h$ there are at least ωn vertices v such that $\overrightarrow{v v_{c+1}} \in J^{K'}$ for all $v_{c+1} \in A_{c+1}$, $\overrightarrow{v_{c-1} v} \in J^K$ for all $v_{c-1} \in A_{c-1}$, and $\overrightarrow{v v_1} \in J^0$ for all $v_1 \in A_1$. Equivalently, for any disjoint $A, B \subseteq V$ and $C \subseteq W$ of sizes $\leq h$ such that (A, B) is $3d$-separated we have $|N^+_{J^K}(A) \cap N^-_{J^0}(B) \cap N^-_{J^{K'}}(C)| \geq \omega n$.

$x = c-1$. For any pairwise $3d$-separated $A_i \subseteq V$, $i \in [c] \setminus \{c-1\}$ and $A_{c+1} \subseteq W$ of sizes $\leq h$ there are at least ωn vertices v such that $\overrightarrow{v v_{c+1}} \in J^{0'}$ for all $v_{c+1} \in A_{c+1}$, $\overrightarrow{v v_c} \in J^K$ for all $v_c \in A_c$, and $\overrightarrow{v_{c-2} v} \in J^0$ for all $v_{c-2} \in A_{c-2}$. Equivalently, for any disjoint $A, B \subseteq V$ and $C \subseteq W$ of sizes $\leq h$ such that (A, B) is $3d$-separated we have $|N^-_{J^K}(A) \cap N^+_{J^0}(B) \cap N^-_{J^{0'}}(C)| \geq \omega n$.

$x \in [c-2]$. For any pairwise $3d$-separated $A_i \subseteq V$, $i \in [c] \setminus \{x\}$ and $A_{c+1} \subseteq W$ of sizes $\leq h$ there are at least ωn vertices v such that $\overrightarrow{v v_{c+1}} \in J^{0'}$ for all $v_{c+1} \in A_{c+1}$, $\overrightarrow{v v_{x+1}} \in J^0$ for all $v_{x+1} \in A_{x+1}$, and $\overrightarrow{v_{x-1} v} \in J^0$ for all $v_{x-1} \in A_{x-1}$, where $A_0 := A_c$. Equivalently, for any disjoint $A, B \subseteq V$ and $C \subseteq W$ of sizes $\leq h$ such that (A, B) is $3d$-separated we have $|N^-_{J^0}(A) \cap N^+_{J^0}(B) \cap N^-_{J^{0'}}(C)| \geq \omega n$.

All of these conditions follow from the extendability assumption in Theorem 3.3 (after renaming colours 0 and K in $J[V, W]$ as $0'$ and K', and replacing h with $(c-1)h$).

3.4. Divisibility

It remains to consider divisibility; we follow [13, Definition 7.2]. For integers $s \leq t$ we write I^s_t for the set of injections from $[s]$ to $[t]$. We identify $V \cup W$ with $[n']$ for some n'. For $0 \leq i \leq 2$, $\psi \in I^i_{n'}$, $\theta \in I^i_{c+1}$, we define index vectors in $\mathbb{N}^2$ describing types with respect to the partitions $\mathcal{P}$ or $\mathcal{Q}$: we write $i_{\mathcal{P}}(\theta) = (|Im(\theta) \cap [c]|, |Im(\theta) \cap \{c_+\}|)$ and $i_{\mathcal{Q}}(\psi) = (|Im(\psi) \cap V|, |Im(\psi) \cap W|)$. For example, for $\theta = (1 \mapsto c_-, 2 \mapsto c) \in \mathfrak{H}$ we have $i_{\mathcal{P}}(\theta) = (2, 0)$. We define degree vectors $\mathfrak{H}(\theta)^*$ and $\mathfrak{J}(\psi)^*$ in $\mathbb{N}^{C \times I^i_2}$ by

$$\mathfrak{H}(\theta)^*_{\ell\pi} = |\mathfrak{H}^\ell(\theta\pi^{-1})| \quad \text{and} \quad \mathfrak{J}(\psi)^*_{\ell\pi} = |\mathfrak{J}^\ell(\psi\pi^{-1})|,$$

where e.g. $\mathfrak{H}^\ell(\theta\pi^{-1})$ denotes the set of $\theta' \in \mathfrak{H}^\ell$ having $\theta\pi^{-1}$ as a restriction. Letting $\langle \cdot \rangle$ denote the integer span of a set of vectors, we say J is H-divisible in Φ if

$$\mathfrak{J}(\psi)^* \in \langle \mathfrak{H}(\theta)^* : i_{\mathcal{P}}(\theta) = i_{\mathcal{Q}}(\psi) \rangle \quad \text{for all } \psi \in \Phi.$$

We refer to the divisibility conditions for index vectors (i_1, i_2) with $i_1 + i_2 = j$ as j-divisibility conditions, where we assume $0 \le j \le 2$, as otherwise they are vacuous. We describe these conditions concretely as follows.

2-divisibility. These conditions simply say that the arcs of J have the same types with respect to $\mathcal{Q}$ as those of H do with respect to $\mathcal{P}$, i.e. all arcs of $J[V]$ have colour 0 or K, all arcs of $J[V,W]$ have colour $0'$ or K', and $J[W] = \emptyset$. To see this, consider any degree vector $\mathfrak{H}(\theta)^*$ with $\theta \in I^2_{c+1}$. We write id $= (1 \mapsto 1, 2 \mapsto 2)$ and $(12) = (1 \mapsto 2, 2 \mapsto 1)$. For any $\ell \in C$, $\pi \in \{\mathrm{id}, (12)\}$ we have $\mathfrak{H}(\theta)^*_{\ell\pi}$ equal to 1 if (ℓ, π) is the pair such that $\theta \circ \pi^{-1} \in \mathfrak{H}^\ell$ (there is at most one such pair) or equal to 0 otherwise. For example, if $\theta = (1 \mapsto c, 2 \mapsto c_-)$ then $\mathfrak{H}(\theta)^*_{\ell\pi}$ is 1 if $(\ell, \pi) = (K, (12))$, otherwise 0. Thus $\mathfrak{H}\langle(i_1, i_2)\rangle := \langle \mathfrak{H}(\theta)^* : i_{\mathcal{P}}(\theta) = (i_1, i_2)\rangle$ consists of all integer vectors supported in coordinates with colours in $\{0, K\}$ if $(i_1, i_2) = (2, 0)$ or $\{0', K'\}$ if $(i_1, i_2) = (1, 1)$, whereas $\mathfrak{H}\langle(0, 2)\rangle$ only contains the all-0 vector. Therefore, the 2-divisibility conditions say that $\mathfrak{J}(\psi)^*$ can be non-zero only at coordinates with colours in $\{0, K\}$ if $i_{\mathcal{Q}}(\psi) = (2, 0)$ or $\{0', K'\}$ if $i_{\mathcal{Q}}(\psi) = (1, 1)$, and $\mathfrak{J}(\psi)^* = 0$ if $i_{\mathcal{Q}}(\psi) = (0, 2)$, i.e. J has the same arc types with respect to $\mathcal{Q}$ as H with respect to $\mathcal{P}$.

0-divisibility. Writing $\emptyset$ for the function with empty domain, all $\mathfrak{H}(\emptyset)^*_{\ell\emptyset} = |\mathfrak{H}^\ell| = |H^\ell|$, and similarly for J, so the 0-divisibility condition is that for some integer m all $|J^\ell| = m|H^\ell|$. For our specific H, this is equivalent to $|J[V]| = |J[V,W]| = c|J^c[V]| = c|J^c[V,W]|$.

1-divisibility. Given $\theta = (1 \mapsto a) \in I^1_{c+1}$ and $\ell \in C = \{0, K, 0', K'\}$, the two coordinates of $\mathfrak{H}(\theta)^*$ corresponding to colour ℓ are the in/outdegrees of a in H^ℓ: we have $\mathfrak{H}(\theta)^*_{\ell \mathrm{id}} = |\mathfrak{H}(1 \mapsto a)| = d^+_{H^\ell}(a)$ and $\mathfrak{H}(\theta)^*_{\ell(12)} = |\mathfrak{H}(2 \mapsto a)| = d^-_{H^\ell}(a)$. Similarly, for $\psi = (1 \mapsto v) \in I^1_{n'}$ the coordinates of $\mathfrak{J}(\psi)^*$ corresponding to colour ℓ are $d^\pm_{J^\ell}(v)$. We compute:

$\mathfrak{H}(1 \mapsto a)^*$	$d^+_{H^0}(a)$	$d^-_{H^0}(a)$	$d^+_{H^K}(a)$	$d^-_{H^K}(a)$	$d^+_{H^{0'}}(a)$	$d^-_{H^{0'}}(a)$	$d^+_{H^{K'}}(a)$	$d^-_{H^{K'}}(a)$
$a = c_+$	0	0	0	0	0	$c-1$	0	1
$a = c$	1	0	0	1	0	0	1	0
$a = c_-$	0	1	1	0	1	0	0	0
$a \in [c-2]$	1	1	0	0	1	0	0	0

so $\langle \mathfrak{H}(1 \mapsto c_+)^* \rangle = \{\boldsymbol{v} \in \mathbb{Z}^8 : v_1 = v_2 = v_3 = v_4 = v_5 = v_7 = 0, v_6 = (c-1)v_8\}$, and

$$\langle \mathfrak{H}(1 \mapsto a)^* : a \in [c]\rangle = \{\boldsymbol{v} \in \mathbb{Z}^8 : v_2 = v_5, v_4 = v_7, v_1 + v_3 = v_2 + v_4, v_6 = v_8 = 0\}.$$

For $w \in W$ the 1-divisibility condition is $\mathfrak{J}(1 \mapsto w)^* \in \langle \mathfrak{H}(1 \mapsto c_+)^*\rangle$, i.e. $d^-_{J^{0'}}(w) = (c-1)d^-_{J^{K'}}(w)$, or equivalently $d^-_J(w) = cd^-_{J^{K'}}(w)$. For $v \in V$ the 1-divisibility condition is $\mathfrak{J}(1 \mapsto v)^* \in \langle \mathfrak{H}(1 \mapsto a)^* : a \in [c]\rangle$, which is equivalent to $d^-_{J^K}(v) = d^+_{J^{K'}}(v)$ and $d^+_J(v, V) = d^-_J(v, V) = d^+_J(v, W)$.

All of these divisibility conditions follow from the divisibility assumption in Theorem 3.3 (after renaming colours 0 and K in $J[V,W]$ as $0'$ and K'). By the above discussion, Theorem 3.3 follows from the following special case of [13, Theorem 7.4].

Theorem 3.4. *Let $n^{-1} \ll \delta \ll \omega \ll c^{-1}$. Let $h = 2^{50c^3}$ and $d \ll n$. Let J be a digraph with $V(J)$ partitioned as (V, W) where $\omega n \leq |V|, |W| \leq n$, such that $J[W] = \emptyset$, all arcs in $J[V, W]$ point towards W, all arcs in $J[V]$ are coloured 0 or K and all arcs in $J[V, W]$ are coloured $0'$ or K'. Let $\Phi = (\Phi_B : B \subseteq [c + 1])$, where Φ_B consists of all injections $\phi : B \to V(J)$ such that $\phi(B \cap [c]) \subseteq V$, $\phi(B \cap \{c_+\}) \subseteq W$ and $Im(\phi)$ is $3d$-separated. Suppose J is H-divisible in Φ and (H, δ, ω)-regular in Φ and (Φ, J) is (ω, h, H)-vertex-extendable. Then J has an H-decomposition in Φ.*

4. The algorithm

Suppose we are in the setting of Theorem 1.2: we are given a (ε, t)-typical αn-regular digraph G on n vertices, where $n^{-1} \ll \varepsilon \ll t^{-1} \ll \alpha$, and we need to decompose G into some given family $\mathcal{F}$ of αn oriented one-factors on n vertices. In this section we present an algorithm that partitions almost all of G into two digraphs G_1 and G_2, and each factor F_w into subfactors F^1_w and F^2_w, and also sets up auxiliary digraphs J_1 and J_2, such that (i) an approximate wheel decomposition of J_2 gives an approximate decomposition of G_2 into partial factors that are roughly $\{F^2_w\}$, (ii) given the approximate decomposition of G_2, we can set up (via a small additional greedy embedding) the remaining problem to be finding an exact decomposition of a small perturbation G'_1 of G_1 into partial factors that are roughly $\{F^1_w\}$, corresponding to a wheel decomposition of a small perturbation J'_1 of J_1. For most of the section we will describe and motivate the algorithm; we then conclude with the formal statement.

We fix additional parameters with hierarchy

$$n^{-1} \ll \varepsilon \ll t^{-1} \ll K^{-1} \ll d^{-1} \ll \eta \ll s^{-1} \ll L^{-1} \ll \alpha. \tag{1}$$

For convenient reference later, we also make some comments here regarding the roles of these additional parameters: η will be used to bound the number of vertices embedded greedily, we consider a cycle 'long' if it has length at least K, and the cyclic intervals used to define the special colour K will have sizes $d_i = d/(2s)^{i-1}$ with $i \in [2s+1]$. By the reductions in section 9.1, we will be able to assume that we are in one of the following cases:

Case K: each $F \in \mathcal{F}$ has at least $n/2$ vertices in cycles of length at least K,

Case ℓ^* with $\ell^* \in [3, L]$: each $F \in \mathcal{F}$ has $\geq L^{-3}n$ cycles of length ℓ^*.

We write $\mathcal{F} = (F_w : w \in W)$, so $|W| = \alpha n$. We partition each F_w as $F^1_w \cup F^2_w$ as follows. In Case ℓ^* we let F^1_w consist of exactly $L^{-3}n$ cycles of length ℓ^* (and then $F^2_w = F_w \setminus F^1_w$). In Case K we choose F^1_w with $|F^1_w| - n/2 \in [0, 2K]$ to consist of some cycles of length at least K and at most one path of length at least K. To see that this is possible, consider any induced subgraph F'_w of F_w with $|F'_w| = n/2 + K$ obtained by greedily adding cycles of length at least K until the size is at least $n/2 + K$, and then deleting a (possibly empty) path from one cycle. Let P_1 and P_2 denote the two paths of the (possibly) split cycle, where $P_1 \in F'_w$. If $|P_1|, |P_2| \geq K$ we let $F^1_w = F'_w$. If $|P_1| < K$ we let $F^1_w = F'_w \setminus P_1$. If $|P_2| < K$ we let $F^1_w = F'_w \cup P_2$. In all cases, F^1_w is as required.

The algorithm is randomised, so we start by defining probability parameters. The graphs G_1 and G_2 are binomial random subdigraphs of G of sizes that are slightly less than one would expect (we leave space for a greedy embedding that will occur between the approximate decomposition step and the exact decomposition step). For each $w \in W$ we let $p^g_w = (1-\eta)n^{-1}|F^g_w| + n^{-.2}$ (so $1-\eta \leq p^1_w + p^2_w \leq 1 - L^{-3}\eta$). We let $p_g = |W|^{-1}\sum_{w \in W} p^g_w$ (so $1-\eta \leq p_1 + p_2 \leq 1 - L^{-3}\eta$). For each arc e of G independently we will let $\mathbb{P}(e \in G_g) = p_g$ for $g \in [2]$.

We introduce further probabilities corresponding to the cycle distributions of each F^g_w. For $c < K$ we write $q^g_{w,c}n$ for the number of cycles of length c in F^g_w and let $p^g_{w,c} = (1-\eta)q^g_{w,c}$. We define $p^g_{w,K}$ so that F^g_w has about $8p^g_{w,K}n$ vertices not contained in cycles of length $< K$ (for technical reasons, we also ensure that each $p^g_{w,K} \geq n^{-.2}$, which explains the term $n^{-.2}$ in the definition of p^g_w). Averaging over W gives the corresponding probabilities that describe the uses of arcs in each G_g: we let $p^g_c = |W|^{-1}\sum_{w \in W} p^g_{w,c}$ so that for each $c < K$, the number of edges in G_g allocated to cycles of length c will be roughly $\sum_{w \in W} cp^g_{w,c}n = |W|cp^g_c n = \alpha c p^g_c n^2 = cp^g_c|G| + O(n)$, and similarly, roughly $8p^g_K|G| + O(n)$ arcs in G_g will be allocated to long cycles.

The remainder of the algorithm is concerned with the auxiliary digraphs J_g. For any colour c, we let J^c_g denote the arcs of colour c in J_g. We also write $J^*_g = \cup_{c \neq K} J^c_g$. First we consider arcs within $J_g[V]$. Throughout the paper, we fix a cyclic order on V, which we choose uniformly at random. For $v \in V$, let v^+ denote the successor of v and v^- denote the predecessor of V. Arcs of the special colour K should correspond to 1/8 of the factor arcs that are not in short cycles, so should form a graph of density about p^g_K. For each arc $\overrightarrow{xy} \in G_g$ not of the form $\overrightarrow{zz}^+$ (to avoid loops, we don't mind double edges) independently we assign $\overrightarrow{xy}$ to colour K with probability p^g_K/p_g or colour 0 with probability p^g_*/p_g (where $p^g_K + p^g_*$ is slightly less than p_g). If $\overrightarrow{xy}$ has colour K we add $\overrightarrow{xy}^-$ to J^K_g.

Now we consider $J_g[V, W]$. These arcs are all directed from V to W. For each $w \in W$ and cycle length $c < K$, there should be about $cp^g_{w,c}n$ vertices available for the c-cycles in F^g_w. The colouring of $\overrightarrow{W}_c$ requires $1/c$-fraction of these to be joined to w in colour c, so we should have $N^-_{J^c_g}(w) \approx p^g_{w,c}n$. Similarly, there should be about $8p^g_{w,K}n$ vertices available for vertices of F^g_w not in short cycles, and the colouring of $\overrightarrow{W}^K_8$ requires 1/8 of these to be joined to w in colour c, so we should have $N^-_{J^K_g}(w) \approx p^g_{w,K}n$. These arcs are chosen randomly, not independently, but according to a random collection of intervals, of sizes $d_i = d/(2s)^{i-1}$ with $i \in [2s+1]$, where d is small enough that the resulting graph is roughly typical, but large enough to give a good upper bound on the number of vertices in long cycles that become unused when they are chopped up into paths, and so need to be embedded greedily.

These intervals must be chosen quite carefully, because of the following somewhat subtle constraint. Recall that in Case K we will reduce to a path factor problem in some subdigraph H of G. This can only have a solution if each vertex x has degree $d^{\pm}_H(x) = d_2(x) - d_{\pm}(x)$, where $d_2(x)$ is the number of path factors that will use x and $d_-(x)$ (respectively $d_+(x)$) is the number of these in which x is the start (respectively end). The path factors will be obtained from a set of arc-disjoint $\overrightarrow{W}_8^K$'s, where for each $w \in W$, its colour K neighbourhood is given by a set of intervals $([x_i^w, (y_i^w)^-] : i \in I_w)$, so its $\overrightarrow{W}_8^K$'s will define paths from x_i^w to y_i^w. Thus in the auxiliary digraph J, the degree of x into W must be $d^+_J(x, W) = d_2(x) - d'_1(x)$, where $d'_1(x)$ is the number of path factors in which x is some successor $(y_i^w)^+$. To relate these two formulae, we note that a wheel decomposition of J requires $d^+_J(x, W) = d^+_J(x, V) = d^-_J(x, V)$ and $d^+_{J^K}(x, W) = d^-_{J^K}(x, V)$, and that in the twisting construction, $d^-_{J^K}(x^-, V)$ arcs of H at x are not counted by $d^-_J(x, V)$, whereas $d^-_{J^K}(x, V)$ arcs of H not at x are counted by $d^-_J(x, V)$. Writing $\Delta(x) = d^-_{J^K}(x^-, V) - d^-_{J^K}(x, V) = d^+_{J^K}(x^-, W) - d^+_{J^K}(x, W)$, we deduce $d^+_H(x) = d^+_J(x, V)$ and $d^-_H(x) = d^-_J(x, V) + \Delta(x)$, so we need $\Delta(x) = d'_1(x) - d_+(x)$ and $d'_1(x) = d_-(x)$. So $\Delta(x) = d_-(x) - d_+(x)$. We will ensure that both sides are always 0 (taking H equal to the digraph G'_1 in which we need to solve the path factor problem), i.e.

i. every vertex is used equally often as a startpoint or as a successor of an interval, and
ii. all vertices appear in some interval for the same number of factors.

(The successor of an interval is the successor of its largest member.) To achieve this, we identify V with $[n]$ under the natural cyclic order, and select our intervals from canonical sets $\mathcal{I}^i_j$, $i \in [2s+1]$, $j \in [d_i]$, where each $\mathcal{I}^i_j$ is a partition of $[n]$ into $n/d_i \pm 1$ intervals of length at most d_i, we have $\mathcal{I}^i_j \cap \mathcal{I}^i_{j'} = \emptyset$ for $j \neq j'$, and for each i, every $v \in [n]$ occurs exactly once as a startpoint of some interval in $\mathcal{I}^i = \cup_j \mathcal{I}^i_j$, and also exactly once as a successor of some interval in $\mathcal{I}^i$. The two conditions discussed in the previous paragraph will then be satisfied if there are numbers t_i, $i \in [2s+1]$ such that every interval in $\mathcal{I}^i$ is used by exactly t_i factors. Each w will select intervals from some $\mathcal{I}^{i(w)}_{j(w)}$, and these intervals must be non-consecutive, so that the paths do not join up into longer paths. This explains why we use several different interval sizes: if we only used one size d then a pair of vertices in V at cyclic distance d could never be both used for the same factor, and so we would be unable to satisfy the conditions of the wheel decomposition results in section 3.

Now we describe how factors choose intervals. For each $w \in W$, we start by independently choosing $i = i(w) \in [2s+1]$ and $j = j(w) \in [d_i]$ uniformly at random. Given i and j, we activate each interval in $\mathcal{I}^i_j$ independently with probability $1/2$, and select any interval I such that I is activated, and its two neighbouring intervals $I^{\pm}$ are not activated. We thus obtain a random set of non-consecutive intervals where each interval appears with probability $1/8$ (not independently). We form random sets of intervals $\mathcal{X}^g_w$ where each interval selected for w is included in $\mathcal{X}^g_w$ independently with probability $8p^g_{w,K}$ (and is included in at most one of $\mathcal{X}^1_w$ or $\mathcal{X}^2_w$). Thus, given $w \in W_i := \{w' : i(w') = i\}$, any interval $I \in \mathcal{I}^i$ appears in $\mathcal{X}^g_w$ with probability $p^g_{w,K}/d_i$. The events $\{I \in \mathcal{X}^g_w\}$ for $w \in W_i$ are independent, so whp about $\sum_{w \in W_i} p^g_{w,K}/d_i$ factors use I.

Our final sets of intervals $\mathcal{Y}^g_w$ are obtained from $\mathcal{X}^g_w$ by removing a small number of intervals so that every interval in $\mathcal{I}^i$ is used exactly t^g_i times, where t^g_i is about $\sum_{w\in W_i} p^g_{w,K}/d_i$. (We only need this property when $g = 1$, but for uniformity of the presentation we do the same thing for $g = 2$.) These intervals determine $J^K_g[V,W]$: we let $N^-_{J^K_g}(w) = Y^g_w := \bigcup \mathcal{Y}^g_w$, i.e. the subset of V which is the union of the intervals in $\mathcal{Y}^g_w$. As each x is the startpoint of exactly one interval in $\mathcal{I}^i$ it occurs as the startpoint of an interval for exactly $t_g := \sum_i t^g_i$ factors; the same statement holds for successors of intervals. As each $x \in V$ appears in exactly one interval in each $\mathcal{I}^i_j$ we deduce $d^+_{J^K_g}(x,W) = \sum_{i=1}^{2s+1}\sum_{j=1}^{d_i} t^g_i \approx \sum_{w\in W} p^g_{w,K} = |W|p^g_K$.

The other arcs of J incident to w will come from $\overline{Y}_w := V \setminus \big(Y^1_w \cup Y^2_w \cup (Y^1_w)^+ \cup (Y^2_w)^+\big)$, where $(Y^g_w)^+$ is the set of successors of intervals in $\mathcal{Y}^g_w$ (these vertices are endpoints of paths so should be avoided by the short cycles, and also by the 7/8 of the paths not specified by the intervals). We define $\overline{J}[V,W]$ by $N^-_{\overline{J}}(w) = \overline{Y}_w$. For any $x \in V$ we will have $\mathbb{P}(x \in Y^g_w) \approx \mathbb{P}(x \in X^g_w) = p^g_{w,K}$ and $\mathbb{P}(x \in Y^g_w \mid w \in W_i) \approx \mathbb{P}(x \in X^g_w \mid w \in W_i) = p^g_{w,K}/d_i$, so $|\overline{Y}_w| \approx \overline{p}_w n$, where $\overline{p}_w = 1 - \frac{d_i+1}{d_i}(p^1_{w,K} + p^2_{w,K})$.

In $J^*_g = J_g \setminus J^K_g$ we require about $p^g_{w,*}n$ such arcs, where $p^g_{w,*} := p^g_w - p^g_{w,K}$, and of these, for each cycle length $c < K$ we require about $p^g_{w,c}n$ arcs of colour c. For each $x \in \overline{Y}_w$ independently we include the arc xw in at most one of the J^*_g with probability $p^g_{w,*}/\overline{p}_w$, which is a valid probability as $p^1_{w,*} + p^2_{w,*} = 1 - L^{-3}\eta - p^1_{w,K} - p^2_{w,K} < \overline{p}_w$. Then we give each $xw \in J^*_g[V,W]$ colour c with probability $p^g_{w,c}/p^g_{w,*}$. In particular, xw in J^*_g is coloured 0 with probability $p^g_{w,0}/p^g_{w,*}$, where $p^g_{w,0} := p^g_{w,*} - \sum_{c=3}^{K-1} p^g_{w,c}$.

4.1. Formal statement of the algorithm

The input to the algorithm consists of an αn-regular digraph G on V, a family $(F_w : w \in W)$ of αn oriented one-factors, each partitioned as $F_w = F^1_w \cup F^2_w$, and parameters satisfying $n^{-1} \ll \varepsilon \ll t^{-1} \ll K^{-1} \ll d^{-1} \ll \eta \ll s^{-1} \ll L^{-1} \ll \alpha$. We identify V with $[n]$ according to a uniformly random bijection and adopt the natural cyclic order on $[n]$: each $x \in [n]$ has successor $x^+ = x + 1$ (where $n + 1$ means 1) and predecessor $x^- = x - 1$ (where 0 means n). Let $d_i = d/(2s)^{i-1}$ for $i \in [2s + 1]$. We write $n = r_i d_i + s_i$ with $r_i \in \mathbb{N}$ and $0 \le s_i < d_i$, and let

$$P^i_j = \begin{cases} \{kd_i + j : 0 \le k \le r_i\} & \text{if } j \in [s_i], \\ \{kd_i + j : 0 \le k \le r_i - 1\} & \text{if } j \in [d_i] \setminus [s_i]. \end{cases}$$

For each $i \in [s+1]$ and $j \in [d_i]$ we define a partition of $[n]$ into a family of cyclic intervals $\mathcal{I}^i_j$ defined as all $[a, b^-]$ where $a \in P^i_j$ and b is the next element of P^i_j in the cyclic order. (So $|\mathcal{I}^i_j| = n/d_i \pm 1$, each $I \in \mathcal{I}^i_j$ has $|I| \le d_i$, and $\mathcal{I}^i_j \cap \mathcal{I}^i_{j'} = \emptyset$ for $j \ne j'$.) We let $\mathcal{I}^i = \cup_{j\in[d_i]}\mathcal{I}^i_j$. (So for every $v \in [n]$, exactly one $[a, b^-] \in \mathcal{I}^i$ has $a = v$, and exactly one $[a, b^-] \in \mathcal{I}^i$ has $b = v$.) Each $w \in W$ will be assigned $i(w) \in [2s + 1]$. For $c < K$ write $q^g_{w,c}n$ for the number of cycles of length c in F^g_w. Let

$$p^g_w = (1-\eta)n^{-1}|F^g_w| + n^{-.2}, \qquad p^g_{w,c} = (1-\eta)q^g_{w,c} \text{ for } 3 \le c < K,$$
$$p^g_{w,K} = \tfrac{1}{8}\left(p^g_w - \Sigma_{c=3}^{K-1} c p^g_{w,c}\right),$$
$$p^g_{w,*} = p^g_w - p^g_{w,K}, \qquad p^g_{w,0} = p^g_{w,*} - \Sigma_{c=3}^{K-1} p^g_{w,c}, \qquad p_{w,K} = p^1_{w,K} + p^2_{w,K},$$
$$\overline{p}_w = 1 - \tfrac{d_{i(w)}+1}{d_{i(w)}} p_{w,K}, \qquad p_g = |W|^{-1}\Sigma_{w\in W} p^g_w,$$
$$p^g_c = |W|^{-1}\Sigma_{w\in W} p^g_{w,c} \text{ for } c \in [0,K] \cup \{*\}.$$

We complete the algorithm by applying the following subroutines INTERVALS and DIGRAPH.

INTERVALS

i. For each $w \in W$ independently choose $i(w) \in [2s+1]$ and $j(w) \in [d_{i(w)}]$ uniformly at random. Let $W_i = \{w : i(w) = i\}$.
ii. For each $w \in W$, let $\mathcal{A}_w$ include each interval of $\mathcal{I}^{i(w)}_{j(w)}$ independently with probability 1/2.
Let $\mathcal{S}_w$ consist of all $I \in \mathcal{A}_w$ such that both neighbouring intervals $I^\pm$ of I are not in $\mathcal{A}_w$.
iii. Let $\mathcal{X}^g_w$, $g \in [2]$ be disjoint and chosen with $\mathbb{P}(I \in \mathcal{X}^g_w) = 8p^g_{w,K}$ independently for each $I \in \mathcal{S}_w$.
iv. Let $t^g_i = \min\{|\mathcal{X}^g(I)| : I \in \mathcal{I}^i\}$, where $\mathcal{X}^g(I) := \{w \in W_i : I \in \mathcal{X}^g_w\}$, and obtain $\mathcal{Y}^g_w \subseteq \mathcal{X}^g_w$ by deleting each $I \in \mathcal{I}^i$, $i \in [2s+1]$ from $|\mathcal{X}^g(I)| - t^g_i$ sets $\mathcal{X}^g_w$ with $w \in \mathcal{X}^g(I)$, independently uniformly at random. Write $\mathcal{Y}^g(I) := \{w \in W_i : I \in \mathcal{Y}^g_w\}$ (so $|\mathcal{Y}^g(I)| = t^g_i$ for $I \in \mathcal{I}^i$).

DIGRAPH

i. Let G_1 and G_2 be arc-disjoint with $\mathbb{P}(\overrightarrow{e} \in G_g) = p_g$ independently for each arc $\overrightarrow{e}$ of G.
ii. For each $g \in [2]$ and $\overrightarrow{xy} \in G_g$ independently, if $\overrightarrow{xy}$ is $\overrightarrow{zz^-}$ or $\overrightarrow{zz^+}$ for some z add $\overrightarrow{xy}$ to J^0_g, otherwise choose exactly one of $\mathbb{P}(\overrightarrow{xy} \in J^0_g) = p^g_*/p_g$ or $\mathbb{P}(\overrightarrow{xy^-} \in J^K_g) = p^g_K/p_g$.
iii. For each $w \in W$, add $\overrightarrow{xw}$ to J^K_g for each $x \in Y^g_w := \bigcup \mathcal{Y}^g_w$, and add $\overrightarrow{xw}$ to $\overline{J}$ for each $x \in \overline{Y}_w := V \setminus (Y^1_w \cup Y^2_w \cup (Y^1_w)^+ \cup (Y^2_w)^+)$.
iv. For each arc $\overrightarrow{xw}$ of $\overline{J}[V,W]$ independently, add $\overrightarrow{xw}$ to $J^*_g[V,W]$ with probability $p^g_{w,*}/\overline{p}_w$, and give it exactly one colour $c \neq K$ (including 0) with probability $p^g_{w,c}/p^g_{w,*}$.

We conclude this section by recording some estimates on the algorithm parameters used throughout the paper.

$$\text{In Case } K, \text{ all } |F^g_w| = n/2 \pm 2K, \quad p^1_w, p^2_w > .49, \quad p^1_{w,K} = p^1_w/8 > 1/17,$$
$$p^1_{w,c} = 0 \text{ for } c \in [3, K-1], \quad p^1_{w,*} = p^1_{w,0} = 7p^1_w/8 > 1/3$$

$$\text{and } p^2_{w,*} \geq p^2_{w,0} \geq 2p^2_w/3 > 1/4.$$

$$\text{In Case } \ell^*, \text{ all } |F^1_w| = \ell^* L^{-3}n, \quad |F^2_w| = n - \ell^* L^{-3}n, \quad p^1_w > (1-\eta)\ell^* L^{-3} > 2L^{-3},$$

$$p^2_w > 1 - 2L^{-2} > .9, \quad p^1_{w,\ell^*} = p^1_w/\ell^* > .9L^{-3}, \quad p^1_{w,K} = n^{-.2}/8,$$

$$p^1_{w,c} = 0 \text{ for } c \in [3, K-1] \setminus \{\ell^*\},$$

$$p^1_{w,*} > 2L^{-3}, \quad p^1_{w,0} \geq 2p^1_{w,*}/3 > L^{-3} \quad \text{ and } p^2_{w,*} \geq p^2_{w,0} \geq 2p^2_w/3 > .6.$$

$$\text{In both cases, } p^2_{w,K} \geq n^{-.2}/8.$$

5. Analysis I: intervals

In this section we analyse the families of intervals chosen by the INTERVALS subroutine in section 4; our goal is to establish various regularity and extendability properties of $J^K_g[V, W]$ and $\overline{J}_g[V, W]$ (which are defined in step (iii) of DIGRAPH but are completely determined by INTERVALS). We also deduce some corresponding properties that follow from these under the random choices in DIGRAPH. Before starting the analysis, we state some standard results on concentration of probability that will be used throughout the remainder of the paper. We use the following classical inequality of Bernstein (see e.g. [4, (2.10)]) on sums of bounded independent random variables. (In the special case of a sum of independent indicator variables we will simply refer to the 'Chernoff bound'.)

Lemma 5.1. *Let $X = \sum_{i=1}^n X_i$ be a sum of independent random variables with each $|X_i| < b$.*
Let $v = \sum_{i=1}^n \mathbb{E}(X_i^2)$. Then $\mathbb{P}(|X - \mathbb{E}X| > t) < 2e^{-t^2/2(v+bt/3)}$.

We also use McDiarmid's bounded differences inequality, which follows from Azuma's martingale inequality (see [4, Theorem 6.2]).

Definition 5.2. Suppose $f : S \to \mathbb{R}$ where $S = \prod_{i=1}^n S_i$ and $b = (b_1, \dots, b_n) \in \mathbb{R}^n$. We say that f is b-*Lipschitz* if for any $s, s' \in S$ that differ only in the ith coordinate we have $|f(s) - f(s')| \leq b_i$. We also say that f is v-*varying* where $v = \sum_{i=1}^n b_i^2/4$.

Lemma 5.3. *Suppose $Z = (Z_1, \dots, Z_n)$ is a sequence of independent random variables, and $X = f(Z)$, where f is v-varying. Then $\mathbb{P}(|X - \mathbb{E}X| > t) \leq 2e^{-t^2/2v}$.*

The next lemma records various regularity and extendability properties of $J^K_g[V, W]$ and $\overline{J}_g[V, W]$. We recall that each $N^-_{J^K_g}(w) = Y^g_w$ and $N^-_{\overline{J}_g}(w) = \overline{Y}_w$, and also our notation for common neighbourhoods, e.g. $N^-_{J^K_g}(R) = \bigcap_{w \in R} N^-_{J^K_g}(w)$ in statement (iv). Statements (iv) and (v) will be applied to $n^{O(1)}$ choices of set U or function h, so their conclusions apply whp simultaneously to all these choices (recalling our convention that 'whp' refers to events with exponentially small failure probability). For $x \in V$ we write $t^-_g(x)$ or $t^+_g(x)$ for the number of w such that x is the startpoint or successor of an interval in $\mathcal{Y}^g_w$. We also use the separation property from Definition 3.2.

Lemma 5.4. *Let $g \in [2]$, $U \subseteq V$ and $h : W \to \mathbb{R}$ with each $|h(w)| < n^{.01}$. Then whp:*

i. $|\mathcal{Y}^g(I)| = t_i^g = \frac{|W|p_K^g}{(2s+1)d_i} \pm n^{.51}$ *for all* $I \in \mathcal{I}^i$, $i \in [2s+1]$.

ii. $d^+_{J_g^K}(x, W) = |W|p_K^g \pm n^{.52}$ *and* $t_g^\pm(x) = t_g := \sum_i t_i^g$ *for each* $x \in V$.

iii. $d^-_{J_g^K}(w) = |Y_w^g| = p_{w,K}^g n \pm n^{3/4}$ *and* $d^-_{\overline{J}}(w) = |\overline{Y}_w| = \overline{p}_w n \pm n^{3/4}$ *for all* $w \in W$.

iv. For any disjoint $R, R' \subseteq W$ *of sizes* $\leq s$ *we have*

$$\left|U \cap N^-_{J_g^K}(R) \cap N^-_{\overline{J}}(R')\right| = |U| \prod_{w\in R} p_{w,K}^g \prod_{w\in R'} \overline{p}_w \pm 3sn^{3/4}.$$

v. Consider $H := \sum \left\{h(w) : w \in N^+_{J_g^K}(S) \cap N^+_{\overline{J}}(S')\right\}$ *for disjoint* $S, S' \subseteq V$ *of sizes* $\leq s$.

If $S \cup S'$ *is* $3d$*-separated then* $H = \sum_{w\in W} (p_{w,K}^g)^{|S|}\overline{p}_w^{|S'|}h(w) \pm 5sn^{3/4}$.

If (S, S') *is* $3d$*-separated then* $H \geq 2^{-2s} \sum_{w\in W} (p_{w,K}^g)^{|S|}h(w)$.

Write $X_w^g = \bigcup \mathcal{X}_w^g$ and $\overline{X}_w = V \setminus (X_w^1 \cup X_w^2 \cup (X_w^1)^+ \cup (X_w^2)^+)$. In the proof we repeatedly use the observation that if $S \cup S' \subseteq V$ is $3d$-separated and $w \in W$, given $i(w)$ and $j(w)$, the events $\{\{x \in X_w^g\} : x \in S\} \cup \{\{x \in \overline{X}_w\} : x \in S'\}$ are independent, as they are determined by disjoint sets of random decisions in INTERVALS. The weaker assumption that (S, S') is $3d$-separated only implies independence of $\{S \subseteq X_w^g\}$ and $\{S' \subseteq \overline{X}_w\}$. We also note that for any S, S' the events $\{S \subseteq X_w^g\} \cap \{S' \subseteq \overline{X}_w\}$ are independent over $w \in W$.

Proof. For (i), consider any $I \in \mathcal{I}_j^i$ with $i \in [2s+1]$, $j \in [d_i]$. For each $w \in W_i$ independently we have $\mathbb{P}(j(w) = j) = 1/d_i$, $\mathbb{P}(I \in \mathcal{S}_w \mid j(w) = j) = 1/8$, $\mathbb{P}(I \in \mathcal{X}_w^g \mid I \in \mathcal{S}_w) = 8p_{w,K}^g$, so $\mathbb{P}(I \in \mathcal{X}_w^g) = p_{w,K}^g/d_i$. As $\mathbb{P}(w \in W_i) = 1/(2s+1)$ for each $w \in W$ and $\sum_{w\in W} p_{w,K}^g = |W|p_K^g$, by a Chernoff bound, whp $|\mathcal{X}^g(I)| = \frac{|W|p_K^g}{(2s+1)d_i} \pm n^{.51}$. This estimate holds for all such I, and so for $t_i^g = \min\{|\mathcal{X}^g(I)| : I \in \mathcal{I}^i\}$; thus (i) holds.

For (ii), note that each $x \in V$ appears in exactly one interval in each $\mathcal{I}_j^i$, so

$$d^+_{J_g^K}(x, W) = \sum_{i=1}^{2s+1} \sum_{j=1}^{d_i} \left(\tfrac{|W|p_K^g}{(2s+1)d_i} \pm n^{.51}\right) = |W|p_K^g \pm n^{.52}.$$

Next we recall that INTERVALS chooses uniformly at random $\mathcal{Y}^g(I) \subseteq \mathcal{X}^g(I)$ of size t_i^g. The statements on $t_g^\pm(x)$ hold as for each i there is exactly one $[a, b] \in \mathcal{I}^i$ with $a = x$ and exactly one $[a, b] \in \mathcal{I}^i$ with $b^+ = x$. For future reference, we note that each $|\mathcal{X}^g(I) \setminus \mathcal{Y}^g(I)| < 2n^{.51}$.

For (iii), consider any $w \in W$. We start INTERVALS by choosing $i = i(w) \in [2s+1]$ and $j = j(w) \in [d_i]$ uniformly at random. Given these choices, any $I \in \mathcal{I}_j^i$ appears in $\mathcal{S}_w$

if $I \in \mathcal{A}_w$ and $I^\pm \notin \mathcal{A}_w$; this occurs with probability $1/8$, so $\mathbb{E}|\mathcal{S}_w| = |\mathcal{I}^i_j|/8 = n/8d_i \pm 1$. As $|\mathcal{S}_w|$ is a 3-Lipschitz function of the events $\{I \in \mathcal{A}_w\}$, $I \in \mathcal{I}^i_j$, by Lemma 5.3 whp $|\mathcal{S}_w| = n/8d_i \pm n^{.51}$. Each $I \in \mathcal{S}_w$ is included in $\mathcal{X}^g_w$ independently with probability $8p^g_{w,K}$, so by a Chernoff bound whp $|\mathcal{X}^g_w| = p^g_{w,K} n/d_i \pm 2n^{.51}$. For each $I \in \mathcal{X}^g_w$ independently we have $I \in \mathcal{Y}^g_w$ with probability $t^g_i/|\mathcal{X}^g(I)| = 1 \pm n^{-.27}$, as $p^g_K \geq n^{-.2}$. Thus $d_i\mathbb{E}|\mathcal{Y}^g_w| = p^g_{w,K} n \pm n^{.73}$, so by a Chernoff bound whp $d^-_{J^K_g}(w) = |Y^g_w| = d_i|\mathcal{Y}^g_w| \pm d_i = p^g_{w,K} n \pm 2n^{.73}$. We deduce $d^-_{\overline{J}}(w) = n - \frac{d_i+1}{d_i}(|Y^1_w| + |Y^2_w|) = \overline{p}_w n \pm n^{3/4}$, so (ii) holds. We note that each $|Y^g_w| = |X^g_w| \pm n^{3/4}$ and $|\overline{Y}_w| = |\overline{X}_w| \pm n^{3/4}$.

For (iv), we first estimate the number N of $u \in U$ such that $u \in X^g_w$ for all $w \in R$ and $u \in \overline{X}_w$ for all $w \in R'$. The actual quantity we need to estimate is obtained by replacing 'X' with 'Y', and so differs in size by at most $2sn^{3/4}$. For each $u \in U$, we have independently $\mathbb{P}(u \in X^g_w) = p^g_{w,K}$ for all $w \in R$ and $\mathbb{P}(u \in \overline{X}_w) = \overline{p}_w$ for all $w \in R'$, so $\mathbb{E}N = |U| \prod_{w \in R} p^g_{w,K} \prod_{w \in R'} \overline{p}_w$. Indeed, given choices of $i = i(w)$ and $j = j(w)$, letting I be the unique interval in $\mathcal{I}^i_j$ whose successor is u, we have $\mathbb{P}(u \in \overline{X}_w) = 1 - \sum_{g=1}^2 (\mathbb{P}(u \in X^g_w) + \mathbb{P}(I \in \mathcal{X}^g_w)) = \overline{p}_w$. Now (iv) follows from Lemma 5.3, as N is a $3d$-Lipschitz function of $\leq 2n$ independent random decisions in INTERVALS.

For (v), we will estimate $H' = \sum\{h(w) : S \subseteq X^g_w, S' \subseteq \overline{X}_w\}$. The actual quantity H we need to estimate is obtained from H' by replacing 'X' with 'Y'. We have $|H - H'| < 4sn^{3/4}$, as for each i, j there are $\leq 2s$ intervals $I \in \mathcal{I}^i_j$ with $I \cap (S \cup S') \neq \emptyset$ each with $< 2n^{.51}$ choices of $w \in \mathcal{X}^g(I) \setminus \mathcal{Y}^g(I)$ each with $|h(w)| < n^{.01}$. If $S \cup S'$ is $3d$-separated then independently for all $w \in W$ we have $\mathbb{P}(x \in X^g_w) = p^g_{w,K}$ for all $x \in S$ and $\mathbb{P}(x \in \overline{X}_w) = \overline{p}_w$ for all $x \in S'$; the required estimates on H' and so H follow whp from Lemma 5.1.

Finally, we consider (v) when (S, S') is $3d$-separated. We fix $w \in W$, condition on $i(w) = i$ and $j(w) = j$, and recall $\mathbb{P}(S \subseteq X^g_w, S' \subseteq \overline{X}_w) = \mathbb{P}(S \subseteq X^g_w)\mathbb{P}(S' \subseteq \overline{X}_w)$. We have the bound $\mathbb{P}(S' \subseteq \overline{X}_w) \geq 2^{-s}$ from the event $I \notin \mathcal{A}_w$ for all $I \in \mathcal{I}^i_j$ with $I \cap S' \neq \emptyset$. We claim that $\mathbb{P}(S \subseteq X^g_w) > (5s)^{-1}(p^g_{w,K})^{|S|}$, which by Lemma 5.1 suffices to complete the proof.

To prove the claim, we first note that if for some $\mathcal{I}^i_j$ no two vertices of S lie in consecutive intervals then $\mathbb{P}(S \subseteq X^g_w \mid i(w) = i, j(w) = j) \geq (p^g_{w,K})^{|S|}$: indeed, the events $\{I \in \mathcal{X}^g_w\}$ for $I \in \mathcal{I}^i_j$ with $I \cap S \neq \emptyset$ are positively correlated. For $i \in [2s+1]$ let J^i_s be the set of $j \in [d_i]$ for which some pair x, x' of S lie in consecutive intervals of $\mathcal{I}^i_j$: we say j is i-bad for x, x'. We note that if j is i-bad for some pair in S then it is i-bad for some consecutive pair x, x' in S (i.e. $\{x, x'\} \cap S = \emptyset$). It suffices to show that some $|J^i_s| < d_i/2$. For this, we note that as $|S| \leq s$ we can fix $i \in [2s+1]$ so that the cyclic distance between any pair of vertices in S is either $< d_{i+1}$ or $\geq d_{i-1}$. There are no i-bad j for any pair x, x' with $d(x, x') \geq d_{i-1} = 2sd_i$. Also, if $d(x, x') < d_{i+1}$ then j is i-bad for x, x' only if $\mathcal{I}^i_j$ contains an interval with an endpoint in the cyclic interval $[x, x']$, so there are at most d_{i+1} such j. We deduce $|J^i_s| < sd_{i+1} = d_i/2$, which completes the proof of the claim, and so of the lemma. □

The next lemma contains similar statements to those in the previous one concerning the colours and directions introduced in DIGRAPH. In (iii) we define $J_g^{K'}$ by $J_g^{K'}[V,W] = J_g^K[V,W]$ and $\overrightarrow{uv} \in J_g^{K'}[V] \Leftrightarrow \overrightarrow{uv}^- \in J_g^K[V]$, thus removing the twist: if for some arc $\overrightarrow{uv}$ of G_g we add $\overrightarrow{uv}^-$ to J_g^K then we add $\overrightarrow{uv}$ to $J_g^{K'}$.

Lemma 5.5. *Let $g \in [2]$. Write $q_0^g = p_*^g$, $q_{K'}^g = p_K^g$ and $q_c^g = 0$ otherwise. Then whp:*

i. For every $v \in V$ and $c \in [3,K] \cup \{0\}$ we have $d^{\pm}_{J_g}(v,V) = p_g(1 \pm \varepsilon)\alpha n \pm n^{.6}$, $d^{\pm}_{J_g^c}(v,V) = p_c^g(1 \pm \varepsilon)\alpha n \pm n^{.6}$, $d^{+}_{J_g^c}(v,W) = p_c^g \alpha n \pm 2n^{3/4}$.

ii. For every $w \in W$ and $c \in [3,K] \cup \{0\}$ we have $d^{-}_{J_g^c}(w,V) = p_{w,c}^g n \pm 2n^{3/4}$.

iii. For any mutually disjoint sets $R_c \subseteq W$ and $S_c^+, S_c^- \subseteq V$ for $c \in [3,K-1] \cup \{0,K'\}$ with $\sum_c |R_c| \le s$ and $\sum_c |S_c^{\pm}| \le s$ we have

$$
\begin{aligned}
&\Big|\bigcap_c \big(N^-_{J_g^c}(R_c) \cap N^+_{J_g^c}(S_c^+) \cap N^-_{J_g^c}(S_c^-)\big)\Big| \\
&= |N_G^+(\cup_c S_c^+) \cap N_G^-(\cup_c S_c^-)| \prod_c \Big((q_c^g)^{|S_c^+|+|S_c^-|} \prod_{w \in R_c} p_{w,c}^g\Big) \pm 4sn^{3/4}.
\end{aligned}
$$

iv. Consider $H' := \big|W \cap N^+_{J_g^K}(S) \cap \bigcap_c N^+_{J_g^c}(S_c)\big|$ for disjoint $S, S' \subseteq V$ of sizes $\le s$ with S' partitioned as $(S_c : c \in [3,K-1] \cup \{0\})$.

$$\text{If } S \cup S' \text{ is } 3d\text{-separated then } H' = \sum_{w \in W} (p_{w,K}^g)^{|S|} \prod_c (p_{w,c}^g)^{|S_c|} \pm 6sn^{3/4}.$$

$$\text{If } (S,S') \text{ is } 3d\text{-separated then } H' + n^{.6} \ge 2^{-2s} \sum_{w \in W} (p_{w,K}^g)^{|S|} \prod_c (p_{w,c}^g)^{|S_c|}.$$

Proof. All quantities considered are 1-Lipschitz functions of the random choices in DIGRAPH, so by Lemma 5.3 it suffices to estimate the expectations. For (i), we recall that G has vertex in- and outdegrees $(1 \pm \varepsilon)\alpha n$, and for each $\overrightarrow{xy}$ in G we have $\mathbb{P}(\overrightarrow{xy} \in J_g) = p_g$, so $\mathbb{E} d^+_{J_g}(v,V) = p_g(1 \pm \varepsilon)\alpha n$. The other expectations are similar, with slightly modified calculations due to the twisting in colour K and avoiding loops; for example, $\mathbb{E} d^-_{J_g^K}(v,V) = p_K^g(d_G^-(v^+) \pm 1) = p_K^g(1 \pm \varepsilon)\alpha n \pm 1$. For (ii), we recall $d^-_{\overline{J}}(w) = \overline{p}_w n \pm n^{3/4}$ from Lemma 5.4.iii, so for $c \ne K$ we have $\mathbb{E} d^-_{J_c^K}(w) = p_{w,c}^g \overline{p}_w^{-1} d^-_{\overline{J}}(w) = p_{w,c}^g n \pm n^{3/4}$. (The estimate for $c = K$ was already given in Lemma 5.4.iii.) For (iii), we first apply Lemma 5.4.iv with $U = N_G^+(\cup_c S_c^+) \cap N_G^-(\cup_c S_c^-)$, $R = R_K$ and $R' = \cup_{c \ne K} R_c$ to obtain

$$
\begin{aligned}
&\Big|N_G^+(\cup_c S_c^+) \cap N_G^-(\cup_c S_c^-) \cap N^-_{J_g^K}(R_K) \cap N^-_{\overline{J}}(\cup_{c \ne K} R_c)\Big| \\
&= |N_G^+(\cup_c S_c^+) \cap N_G^-(\cup_c S_c^-)| \prod_{w \in R_K} p_{w,K}^g \prod_{w \in \cup_{c \ne K} R_c} \overline{p}_w \pm 3sn^{3/4}.
\end{aligned}
$$

For each vertex v counted here independently we have $\mathbb{P}(\overrightarrow{vw} \in J_g^c \mid \overrightarrow{vw} \in \overline{J}) = p_{w,c}^g/\overline{p}_w$ for all $w \in R_c$, $\mathbb{P}(\overrightarrow{vx} \in J_g^c \mid \overrightarrow{vx} \in G) = q_c^g$ for all $x \in S_c^-$ and $\mathbb{P}(\overrightarrow{xv} \in J_g^c \mid \overrightarrow{xv} \in$

$G) = q_c^g$ for all $x \in S_c^+$, so whp the stated bound for (iii) holds. For (iv) we first consider $H := |N^+_{J_g^K}(S) \cap N^+_{\overline{J}}(S')|$. By Lemma 5.4.v with $h(w) = 1$, if $S \cup S'$ is $3d$-separated then $H = \sum_{w \in W} (p^g_{w,K})^{|S|} \overline{p}_w^{|S'|} \pm 5sn^{3/4}$, and if (S, S') is $3d$-separated then $H \geq 2^{-2s} \sum_{w \in W} (p^g_{w,K})^{|S|}$. For each vertex w counted here independently we have $\mathbb{P}(\overrightarrow{vw} \in J_g^c \mid \overrightarrow{vw} \in \overline{J}) = p^g_{w,c}/\overline{p}_w$ for all $v \in S_c$, so whp the stated bound for (iv) holds. □

6. Analysis II: wheel regularity

In this section we show how to assign weights to wheels in each J_g so that for any arc $\overrightarrow{e}$ there is total weight about 1 on wheels containing $\overrightarrow{e}$, and furthermore all weights on wheels with $c+1$ vertices are of order n^{1-c}. This regularity property is an assumption in the wheel decomposition results of section 3, and is also sufficient in its own right for approximate decompositions by a result of Kahn [10]. The estimate for the total weight of wheels on an arc will hold even if we add any new arc to J_g, which is useful as we will need to consider small perturbations of J_1 due to arcs of G not allocated to G_1 or G_2 or not covered in the approximate decomposition of G_2.

We start by considering wheels $\overrightarrow{W}_c$ with $c < K$. Let

$$W^g_{w,c} = n^c p^g_{w,c} (p^g_{w,0})^{c-1} (\alpha p^g_*)^c.$$

The motivation for this formula is that it is about the expected number of $\overrightarrow{W}_c$'s in J_g using w. For any arc $\overrightarrow{e}$ let $W^g_c(\overrightarrow{e})$ be the set of copies of $\overrightarrow{W}_c$ in J_g with hub in W using $\overrightarrow{e}$. Let

$$\hat{W}^g_c(\overrightarrow{e}) = \sum \{ p^g_{w,c} n (W^g_{w,c})^{-1} : \mathcal{W} \in W^g_c(\overrightarrow{e}), w \in V(\mathcal{W}) \}.$$

(If $p^g_{w,c} = 0$ there are no such $\mathcal{W}$, so $(W^g_{w,c})^{-1}$ is always defined when used.) In the following lemma we calculate the total weights on arcs due to copies of $\overrightarrow{W}_c$, although we note that we do not have a good estimate for $\overrightarrow{xy} \in J^0_g[V]$ if $d(x,y) < 3d$. In J_2 we can ignore such arcs, as we only need an approximate decomposition, whereas in J_1 we will cover these by wheels greedily before finding the exact decomposition – this forms part of the perturbation referred to above.

Lemma 6.1. *Let $c' \in \{0, c\}$, $N_c = 1$ and $N_0 = c - 1$. Then whp:*

i. If $p^g_{w,c'} \neq 0$ and we add $\overrightarrow{xw}$ to $J_g^{c'}[V, W]$ then $\hat{W}^g_c(\overrightarrow{xw}) = (1 \pm 4\varepsilon) N_{c'} p^g_{w,c}/p^g_{w,c'} \pm n^{-.2}$.

ii. If $d(x,y) \geq 3d$ and we add $\overrightarrow{xy}$ to $J^0_g[V]$ then $\hat{W}^g_c(\overrightarrow{xy}) = (1 \pm 4\varepsilon) c p^g_c / p^g_* \pm n^{-.2}$.

Proof. As a preliminary step for counting copies of $\overrightarrow{W}_c$ we count c-prewheels, which we define to consist of a wheel with oriented rim cycle in G and all spokes in $\overline{J}$. For any arc $\overrightarrow{e}$ we let $P_c(\overrightarrow{e})$ be the set of c-prewheels using $\overrightarrow{e}$; we will estimate $|P_c(\overrightarrow{e})|$ using the analysis of INTERVALS in Lemma 5.4.

For (i), we estimate $|P_c(\overrightarrow{xw})|$ as follows. We let $x = x_c$ and choose the other rim vertices $x_1, \dots, x_{c-1}$ sequentially in cyclic order. At $c-2$ steps we choose $x_{i+1} \in N^+_G(x_i) \cap N^-_{\overline{J}}(w)$: each has $\alpha n \overline{p}_w \pm 3sn^{3/4}$ options by Lemma 5.4.iv with $U = N^+_G(x_i)$, $R = \emptyset$, $R' = \{w\}$, using $|N^+_G(x_i)| = \alpha n$ (G is αn-regular). At the last step we choose $x_{c-1} \in N^+_G(x_{c-2}) \cap N^-_G(x_c) \cap N^-_{\overline{J}}(w)$, so similarly there are $|N^+_G(x_{c-2}) \cap N^-_G(x_c)|\overline{p}_w \pm 3sn^{3/4}$ options, where $|N^+_G(x_{c-2}) \cap N^-_G(x_c)| = ((1 \pm \varepsilon)\alpha)^2 n$ by typicality of G. Thus $|P_c(\overrightarrow{xw})| = (1 \pm 3\varepsilon)\alpha^c(\overline{p}_w n)^{c-1}$.

Now consider the case $c' = c$, i.e. $\overrightarrow{xw}$ is added to $J^c[V, W]$. For any c-prewheel containing $\overrightarrow{xw}$, independently we include the cycle arcs in J^0_g with probability p^g_* and give each $\overrightarrow{x_i w}$ with $i \neq c$ colour 0 with probability $p^g_{w,0}/\overline{p}_w$, so $\mathbb{E}|W^g_c(\overrightarrow{xw})| = (1 \pm 3\varepsilon)(\alpha p^g_*)^c(p^g_{w,0}n)^{c-1} = (1 \pm 3\varepsilon)W^g_{w,c}/p^g_{w,c}n$. Of these random decisions, $\leq 2n$ concern an arc containing one of x, w, which affect $|W^g_c(\overrightarrow{xw})|$ by $O(n^{c-2})$, and the others have effect $O(n^{c-3})$. Thus $|W^g_c(\overrightarrow{xw})|$ is $O(n^{2c-3})$-varying, so by Lemma 5.3 whp $|W^g_c(\overrightarrow{xw})| = (1 \pm 4\varepsilon)W^g_{w,c}/p^g_{w,c}n$, i.e. $\hat{W}^g_c(\overrightarrow{xw}) = 1 \pm 4\varepsilon$. When $c' = 0$ we argue similarly. Now x can be any x_i with $i \neq c$, for which we have $c-1$ choices. The probability factors are the same as in the previous calculation, except that for $\overrightarrow{x_c w}$ we replace $p^g_{w,0}/\overline{p}_w$ by $p^g_{w,c}/\overline{p}_w$. Again, the stated estimate holds whp by Lemma 5.3, so (i) holds.

For (ii), we write $\hat{W}^g_c(\overrightarrow{xy}) = \sum_{w \in W} \hat{W}^g_c(xyw)$, where $\hat{W}^g_c(xyw)$ is the sum of $(W^g_{w,c})^{-1}$ over the set $W^g_c(xyw)$ of copies of $\overrightarrow{W}_c$ in J_g using $\overrightarrow{xy}$, $\overrightarrow{xw}$ and $\overrightarrow{yw}$. Fix $w \in N^+_{\overline{J}}(x) \cap N^+_{\overline{J}}(y)$ and consider the number $|P_c(xyw)|$ of c-prewheels using $\{\overrightarrow{xy}, \overrightarrow{xw}, \overrightarrow{yw}\}$. Choosing rim vertices sequentially as in (i), now there are $c-3$ steps with $\alpha n \overline{p}_w \pm 3sn^{3/4}$ options and again $((1 \pm \varepsilon)\alpha)^2\overline{p}_w n \pm 3sn^{3/4}$ options at the last step, so $|P_c(xyw)| = (1 \pm 3\varepsilon)\alpha^{c-1}(\overline{p}_w n)^{c-2}$.

Now we consider which of these c-prewheels extend to wheels in $W^g_c(xyw)$: there are c choices for the position of $\overrightarrow{xy}$ on the rim, then some probabilities determined by independent random decisions: the $c-1$ rim edges are each correct with probability p^g_*, the spoke of colour c with probability $p^g_{w,c}/\overline{p}_w$, and the other $c-1$ spokes each with probability $p^g_{w,0}/\overline{p}_w$. Therefore

$$\begin{aligned}\mathbb{E}\hat{W}^g_c(xyw) &= (1 \pm 3\varepsilon)c(\alpha p^g_*)^{c-1}p^g_{w,c}(p^g_{w,0})^{c-1}\overline{p}_w^{-2}n^{c-2}p^g_{w,c}n(W^g_{w,c})^{-1} \\ &= (1 \pm 3\varepsilon)c(\alpha p^g_*)^{-1}p^g_{w,c}n(\overline{p}_w n)^{-2}.\end{aligned}$$

By Lemma 5.3 whp $\hat{W}^g_c(\overrightarrow{xy}) = (1 \pm 3.1\varepsilon)c(\alpha p^g_* n)^{-1}H$, with $H = \sum\{p^g_{w,c}\overline{p}_w^{-2} : w \in N^+_{\overline{J}}(x) \cap N^+_{\overline{J}}(y)\}$.

We estimate H by Lemma 5.4.v with $S = \emptyset$, $S' = \{x, y\}$ and $h(w) = p^g_{w,c}\overline{p}_w^{-2}$ (each $7/8 \leq \overline{p}_w \leq 1$). As $S \cup S'$ is $3d$-separated, whp $H = |W|p^g_c \pm 5sn^{3/4}$, giving $\hat{W}^g_c(\overrightarrow{xy}) = (1 \pm 4\varepsilon)cp^g_c/p^g_* \pm n^{-.2}$. □

Now we apply a similar analysis for $\overrightarrow{W}^K_8$. Let

$$W^g_{w,K} = n^8 \alpha p^g_K p^g_{w,K}(\alpha p^g_* p^g_{w,0})^7.$$

For any arc $\overrightarrow{e}$ let $W^g_K(\overrightarrow{e})$ be the set of copies of $\overrightarrow{W}^K_8$ in J_g using $\overrightarrow{e}$. We define $\hat{W}^g_K(\overrightarrow{e})$ by setting $c = K$ in $\hat{W}^g_c(\overrightarrow{e})$. Now we calculate the total weights on arcs due to copies of $\overrightarrow{W}^K_8$. Note that we cannot give a good estimate for $\overrightarrow{xy} \in J^K_g[V]$ if $d(x,y) < 3d$. We can ignore such arcs in J_2 (as mentioned above), but in J_1 we will replace such arcs by arcs of colour 0 (modified by twisting) – this also forms part of the perturbation.

Lemma 6.2. *Let $c' \in \{0, K\}$, $N_K = 1$, $N_0 = 7$, $q^g_K = p^g_K$, $q^g_0 = p^g_*$. Then whp:*

i. If we add $\overrightarrow{xw}$ to $J^{c'}_g[V,W]$ then $\hat{W}^g_K(\overrightarrow{xw}) = (1 \pm 4\varepsilon)N_{c'}p^g_{w,K}/p^g_{w,c'}$.

ii. Suppose we add $\overrightarrow{xy}$ to $J^{c'}_g[V]$. If $d(x,y) \geq 3d$ then $\hat{W}^g_c(\overrightarrow{xy}) = (1 \pm 4\varepsilon)N_{c'}q^g_K/q^g_{c'}$. If $c' = 0$ then $\hat{W}^g_K(\overrightarrow{xy}) > 2^{-2s-1}p^g_K/p^g_*$.

Proof. For (i), we start by counting (K,g)-prewheels, which we define to consist of a hub $w \in W$ and an oriented 8-path in G between z and z^+ for some z such that $\overrightarrow{zw} \in J^K_g$ and $\overrightarrow{z'w} \in \overline{J}$ for all internal vertices z' of the path. For any arc $\overrightarrow{e}$ we let $P^g_K(\overrightarrow{e})$ be the set of (K,g)-prewheels using $\overrightarrow{e}$.

To estimate $|P^g_K(\overrightarrow{xw})|$, suppose first that $c' = K$. We require $z = x$. We choose the vertices of the path one by one. At 6 steps there are $\alpha n\overline{p}_w \pm 3sn^{3/4}$ options, and at the last step $((1 \pm \varepsilon)\alpha)^2\overline{p}_w n \pm 3sn^{3/4}$ options of a common outneighbour of some vertex and z^+, so $|P^g_K(\overrightarrow{xw})| = (1 \pm 3\varepsilon)\alpha^8(\overline{p}_w n)^7$. On the other hand, if $c' = 0$ then there are 7 choices for the position of x as an internal vertex, dividing the path into two segments. We construct one segment by choosing its vertices one by one, and then do the same for the other segment, starting with one of length ≤ 4 so that $\{z, z^+\}$ is not the last choice. At the step where we choose $\{z, z^+\}$, there is some vertex v on the path for which we need the arc $\overrightarrow{vz}$ or $\overrightarrow{vz}^+$. We also require $z \in N^-_{J^K_g}(w)$. The number of options is $\alpha np^g_{w,K} \pm 3sn^{3/4}$ by Lemma 5.4.iv, with $R = \{w\}$, $R = \emptyset$ and $U = N^+_G(v)$ or $U = N^+_G(v)^- = \{z : \overrightarrow{vz}^+ \in G\}$. There are also 5 steps with $\alpha n\overline{p}_w \pm 3sn^{3/4}$ options, and at the last step $((1 \pm \varepsilon)\alpha)^2\overline{p}_w n \pm 3sn^{3/4}$ options, so $|P^g_K(\overrightarrow{xw})| = (1 \pm 3\varepsilon)7\alpha^8 p^g_{w,K}(\overline{p}_w)^6 n^7$ (as $p^g_{w,K} \geq n^{-.2}/8$).

To estimate $|W^g_K(\overrightarrow{xw})|$, we first consider $c' = K$. For any (K,g)-prewheel containing $\overrightarrow{xw}$, independently we include the last path arc (to z^+) in J^K_g with probability p^g_K, the other 7 path arcs in J^0_g with probability p^g_*, and give $\overleftarrow{wz}'$ for each internal vertex z' colour 0 with probability $p^g_{w,0}/\overline{p}_w$, so

$$\mathbb{E}|W^g_K(\overrightarrow{xw})| = (1 \pm 3\varepsilon)\alpha p^g_K(\alpha p^g_* p^g_{w,0}n)^7 = (1 \pm 3\varepsilon)W^g_{w,K}/p^g_{w,K}n.$$

As $|W^g_K(\overrightarrow{xw})|$ is $O(n^{13})$-varying, by Lemma 5.3 whp $|W^g_K(\overrightarrow{xw})| = (1 \pm 3.1\varepsilon)W^g_{w,K}/p^g_{w,K}n \pm n^{6.51}$, so $\hat{W}^g_K(\overrightarrow{xw}) = 1 \pm 4\varepsilon$ (using $p^g_K > n^{-.2}$).

For $c' = 0$ we have a similar calculation. Indeed, the path arcs are again correct with probability $(p^g_*)^7 p^g_K$, and the arcs $\overleftarrow{wz}'$ (now excluding $z' = x$) are correct with probability $(p^g_{w,0}/\overline{p}_w)^6$, so

$$\mathbb{E}|W^g_K(\overrightarrow{xw})| = (1 \pm 3\varepsilon)7\alpha p^g_K p^g_{w,K}(p^g_{w,0})^6(\alpha p^g_* n)^7 = (1 \pm 3\varepsilon)7W^g_{w,K}/p^g_{w,0}n.$$

By Lemma 5.3 whp $|W^g_K(\overrightarrow{xw})| = (1 \pm 4\varepsilon)7W^g_{w,K}/p^g_{w,0}n \pm n^{6.51}$, so $\hat{W}^g_K(\overrightarrow{xw}) = (1 \pm 4\varepsilon)7p^g_{w,K}/p^g_{w,0}$.

For (ii), we write $\hat{W}^g_K(\overrightarrow{xy}) = \sum_{w \in W} |\hat{W}^g_K(xyw)|$, where $\hat{W}^g_K(xyw)$ is the sum of $(W^g_{w,K})^{-1}$ over the set $W^g_K(xyw)$ of copies of $\overrightarrow{W}^K_8$ in J_g using $\overrightarrow{xy}$, $\overrightarrow{xw}$ and $\overrightarrow{yw}$. For each w we consider the set $P^g_K(xyw)$ of (K,g)-prewheels using $\{\overrightarrow{xy}, \overrightarrow{xw}, \overrightarrow{yw}\}$

Suppose first that $\overrightarrow{xy}$ has colour $c' = K$. We assume $d(x,y) \geq 3d$ (or there is nothing to prove). We must have $y = z$ and in our prewheels the oriented 8-paths from z to z^+ must end with the arc $\overrightarrow{xz^+}$, corresponding to $\overrightarrow{xy} \in J^K$ under twisting. We need $w \in N^+_J(x) \cap N^+_{J^K_g}(y)$ so that $\overrightarrow{yw}$ has colour K and $\overrightarrow{xw}$ can receive colour 0. Choosing rim vertices sequentially, now $\{z, z'\}$ is already fixed, there are 5 steps with $\alpha n \overline{p}_w \pm 3sn^{3/4}$ options, and at the last step $((1 \pm \varepsilon)\alpha)^2 \overline{p}_w n \pm 3sn^{3/4}$ options, so $|P^g_K(\overrightarrow{xw})| = (1 \pm 3\varepsilon)\alpha^7(\overline{p}_w)^6 n^6$.

Now consider which of these prewheels extend to wheels in $W^g_K(xyw)$, according to the following independent random decisions: the other 7 arcs of the oriented 8-path excluding $\overrightarrow{xy}$ are each correct with probability p^g_*, we already have $\overrightarrow{yw} \in J^K_g$, and for each of the 7 internal vertices z' we have $\overrightarrow{z'w}$ correct with probability $p^g_{w,0}/\overline{p}_w$. Therefore

$$\mathbb{E}\hat{W}^g_K(xyw) = (1 \pm 3\varepsilon)(\alpha p^g_*)^7 (p^g_{w,0})^7 \overline{p}_w^{-1} n^6 p^g_{w,K} n (W^g_{w,K})^{-1} = (1 \pm 3\varepsilon)(\alpha p^g_K \overline{p}_w n)^{-1}.$$

By Lemma 5.3 whp $\hat{W}^g_K(\overrightarrow{xy}) = (1 \pm 3.1\varepsilon)(\alpha p^g_K n)^{-1} H \pm n^{-.2}$, with $H = \sum\{\overline{p}_w^{-1} : w \in N^+_J(x) \cap N^+_{J^K_g}(y)\}$. We estimate H by Lemma 5.4.v with $S = \{y\}$ and $S' = \{x\}$. As $d(x,y) \geq 3d$, whp $H = |W|p^g_K \pm 5sn^{3/4}$, giving $\hat{W}^g_K(\overrightarrow{xy}) = 1 \pm 4\varepsilon$.

Now suppose that $\overrightarrow{xy}$ has colour $c' = 0$. For the hub w we require $\overrightarrow{yw} \in J^0$ and $\overrightarrow{xw}$ in J^K or J^0. We first consider the contribution from $\overrightarrow{xw} \in J^K$, when the first vertex of the oriented 8-path must be $z = x$. The estimate of $|P^g_K(\overrightarrow{xw})|$ is the same as when $c' = K$, and the probability factors are the same except that the factor for the last path edge (to z^+) is now p^g_K instead of p^g_*. If $d(x,y) \geq 3d$ then the same calculation with Lemma 5.3 and Lemma 5.4.v shows that the contribution to $\hat{W}^g_K(\overrightarrow{xy})$ from $w \in N^+_J(x) \cap N^+_{J^K_g}(y)\}$ is $(1 \pm 4\varepsilon)(p^g_* n)^{-1}$.

Now we consider the contribution from $\overrightarrow{xw} \in J^0$. There are 6 positions for $\overrightarrow{xy}$ on the path avoiding $\{z, z'\}$. The estimate of $|P^g_K(\overrightarrow{xw})|$ is the same as before except that one factor of $\overline{p}_w$ is replaced by $p^g_{w,K}$ (at the choice of $\{z, z'\}$). The probability factors are the same as in the previous calculation for $\overrightarrow{xw} \in J^K$, so $\mathbb{E}\hat{W}^g_K(xyw) = (1 \pm 3\varepsilon)p^g_{w,K}(\alpha p^g_* \overline{p}^2_w n)^{-1}$. By Lemma 5.3 whp the contribution to $\hat{W}^g_K(\overrightarrow{xy})$ from such w is $(1 \pm 3.1\varepsilon)6(\alpha p^g_* n)^{-1} H$, with $H = \sum\{h(w) : w \in N^+_J(x) \cap N^+_J(y)\}$, $h(w) = p^g_{w,K}(\overline{p}_w)^{-2}$.

We estimate H by Lemma 5.4.v with $S = \emptyset$, $S' = \{x, y\}$. As (S, S') is $3d$-separated (vacuously) whp $H \geq 2^{-2s} \sum_{w \in W} h(w) = 2^{-2s}|W|p^g_K$, so $\hat{W}^g_K(\overrightarrow{xy}) > 2^{-2s-1}p^g_K/p^g_*$. Now suppose $d(x,y) \geq 3d$. Then $S \cup S'$ is $3d$-separated, so whp $H = |W|p^g_K \pm 5sn^{3/4}$. The contribution here to $\hat{W}^g_K(\overrightarrow{xy})$ is $(1 \pm 4\varepsilon)6p^g_K/p^g_*$, so altogether $\hat{W}^g_K(\overrightarrow{xy}) = (1 \pm 4\varepsilon)7p^g_K/p^g_*$. $\square$

We combine the above estimates to deduce the main lemma of this section, establishing wheel regularity. Let

$$\hat{W}^g(\overrightarrow{e}) = \sum\{\hat{W}^g_c(\overrightarrow{e}) : c \in [3, K]\}.$$

Lemma 6.3. *Suppose we add $\overrightarrow{e}$ to J in any colour, such that if $\overrightarrow{e} \in J[V]$ then $\overrightarrow{e} = \overrightarrow{xy}$ with $d(x, y) \geq 3d$, and if $\overrightarrow{e}$ has a vertex in W then it is an endvertex. Then $\hat{W}^g(\overrightarrow{e}) = 1 \pm 5\varepsilon$.*

Proof. By Lemmas 6.1 and 6.2 we can analyse the various cases as follows.

- If $\overrightarrow{e} \in J^c_g[V, W]$ with $c \neq 0$ then $\hat{W}^g(\overrightarrow{e}) = \hat{W}^g_c(\overrightarrow{e}) = 1 \pm 5\varepsilon$.
- If $\overrightarrow{xy} \in J^K_g[V]$ with $d(x, y) \geq 3d$ then $\hat{W}^g(\overrightarrow{e}) = \hat{W}^g_K(\overrightarrow{e}) = 1 \pm 5\varepsilon$.
- If $\overrightarrow{e} \in J^0_g[V, W]$ then

$$\hat{W}^g(\overrightarrow{e}) = (1 \pm 4\varepsilon)7p^g_{w,K}/p^g_{w,0} + \textstyle\sum_{c=3}^{K-1}\left((1 \pm 4\varepsilon)(c-1)p^g_{w,c}/p^g_{w,0} \pm n^{-.2}\right) = 1 \pm 5\varepsilon,$$

  as $p^g_{w,0} = 7p^g_{w,K} + \sum_{c=3}^{K-1}(c-1)p^g_{w,c}$.
- If $\overrightarrow{xy} \in J^0_g[V]$ with $d(x, y) \geq 3d$ then

$$\hat{W}^g(\overrightarrow{e}) = (1 \pm 4\varepsilon)7p^g_K/p^g_* + \textstyle\sum_{c=3}^{K-1}\left((1 \pm 4\varepsilon)cp^g_c/p^g_* \pm n^{-.2}\right) = 1 \pm 5\varepsilon,$$

  as $p^g_* = p_g - p^g_K = 7p^g_K + \sum_{c=3}^{K-1} cp^g_c$. □

7. Approximate decomposition

Here we describe the approximate decomposition of G_2. Recall that at the start of section 4 we partitioned each factor F_w into subfactors F^1_w and F^2_w, that each F^g_w has $q^g_{w,c}n$ cycles of length $c \in [3, K-1]$, and $p^g_{w,c} = (1-\eta)q^g_{w,c}$. We will embed almost all of each F^2_w in G_2. We say $F'_w \subseteq F^2_w$ is valid if $F^2_w \setminus F'_w$ does not have any independent arcs (i.e. arcs $\overrightarrow{xy}$ such that both x and y have total degree 1 in $F^2_w \setminus F'_w$) and if F^2_w contains a path then $F^2_w \setminus F'_w$ contains the arcs incident to each of its ends.

Lemma 7.1. *There are arc-disjoint digraphs $G^2_w \subseteq G_2$ for $w \in W$, where each G^2_w is a copy of some valid $F'_w \subseteq F^2_w$ with $V(G^2_w) \subseteq N^-_{J_2}(w)$, such that*

i. $G^-_2 = G_2 \setminus \bigcup_{w\in W} G^2_w$ has maximum degree at most $5d^{-1/3}n$,

ii. the digraph J^-_2 obtained from $J_2[V, W]$ by deleting all $\overrightarrow{xw}$ with $x \in V(G^2_w)$ has maximum degree at most $5d^{-1/3}n$, and

iii. any $x \in V$ has degree 1 in F'_w for at most $n/\sqrt{d}$ choices of w.

Proof. Say that an arc $\overrightarrow{vw}$ with $v \in V$ and $w \in W$ is *bad* there is some $c \in [3, K-1]$ such that $\overrightarrow{vw} \in J^c$ and $p^2_{w,c} < n^{-.1}$, or $\overrightarrow{vw} \in J^K$ and $p^2_{w,K} < d^{-1/3}$. The expected bad degree of $v \in V$ is at most $(Kn^{-.1} + d^{-1/3})n$ so by Chernoff bounds we can assume that every $v \in V$ has bad degree at most $2d^{-1/3}n$. Let J_2' be obtained from J_2 by deleting all bad arcs and all $\overrightarrow{xy} \in J_2^K[V]$ with $d(x,y) < 3d$. We consider the auxiliary hypergraph $\mathcal{H}$ whose vertices are all arcs of J_2' and whose edges correspond to all copies of the coloured wheels $\overrightarrow{W}_8^K$ or $\overrightarrow{W}_c$ with $c \in [3, K-1]$. We recall that $W_{w,c}^g = n^c p_{w,c}^g (p_{w,0}^g)^{c-1} (\alpha p_*^g)^c$ and $W_{w,K}^g = n^8 \alpha p_K^g p_{w,K}^g (\alpha p_*^g p_{w,0}^g)^7$. We assign weights $(1-5\varepsilon)p_{w,c}^g n/(W_{w,c}^g)^{-1}$ to each copy of any $\overrightarrow{W}_c$ (and to $\overrightarrow{W}_8^K$ for $c = K$). By Lemma 6.3, the total weight of wheels in J_2 on any arc $\overrightarrow{e}$ satisfies $1 - 10\varepsilon < \hat{W}^g(\overrightarrow{e}) < 1$. Thus the total weight of wheels in J_2' on any arc $\overrightarrow{e}$ satisfies $1 - d^{-1/4} < \hat{W}^g(\overrightarrow{e}) < 1$, as we deleted at most $2d^{-1/3}n^7$ (say) copies of $\overrightarrow{W}_8^K$ on $\overrightarrow{e}$ using a deleted arc. Note also that for any two arcs the total weight of wheels containing both is at most $n^{-.7}$ (as $p_K^g \geq n^{-1/4}$).

Thus $\mathcal{H}$ satisfies the hypotheses of a result of Kahn [10] on almost perfect matchings in weighted hypergraphs that are approximately vertex regular and have small codegrees. A special case of this result (slightly modified) implies that for any collection $\mathcal{F}$ of at most n^{100} (say) subsets of $V(\mathcal{H}) = J$ each of size at least $\sqrt{n}$ (say) we can find a matching M in $\mathcal{H}$ such that $|F \setminus \bigcup M| < d^{-1/5}|F|$ for all $F \in \mathcal{F}$. (This is immediate from [10] if $\mathcal{F}$ has constant size, and a slight modification using better concentration inequalities implies the stated version. Alternatively, one can reduce to the problem to an unweighted version via a suitable random selection of edges and then apply a result of Alon and Yuster [2].) This is also implied by a recent result of Ehard, Glock and Joos [7].

We choose such a matching M for the family $\mathcal{F}$ where for each $v \in V \cup W$ we include sets $F_v = \{\overrightarrow{e} \in J_2[V,W] : v \in \overrightarrow{e}\}$, $F_v^K = \{\overrightarrow{e} \in J_2^K[V,W] : v \in \overrightarrow{e}\}$, and $F_v' = \{\overrightarrow{e} \in J_2[V] : v \in \overrightarrow{e}\}$ (the last just for $v \in V$). This $\mathcal{F}$ is valid as all $|F| > \sqrt{n}$ by Lemma 5.5. By construction for all $c \in [3, K-1]$ every copy of $\overrightarrow{W}_c$ in M with hub w has $p_{w,c}^2 \geq n^{-.1}$ and every copy of $\overrightarrow{W}_8^K$ in M with hub w has $p_{w,K}^2 \geq nd^{-1/3}$.

For each w we define G_w^2 to be the subgraph of G corresponding to the wheels in M containing w, where we take account of the twisting in colour K. Thus G_w^2 contains the rim c-cycle of any c-wheel in M containing w, and for any copy of $\overrightarrow{W}_8^K$ in M containing $\overrightarrow{xw} \in J^K[V,W]$ we obtain an oriented path of length 8 from x to x^+. The maximum degree bounds in (i) and (ii) clearly hold.

Recalling that $N_{J_2}^-(w)$ is disjoint from the set of interval successors $(Y_w^2)^+$, we see that these cycles and paths are vertex-disjoint, except that some paths may connect up to form longer paths, which can be described as follows. Let $\mathcal{Y}_w'$ be the set of maximal cyclic intervals I such that for every $x \in I$ there is a copy of $\overrightarrow{W}_8^K$ in M containing $\overrightarrow{xw} \in J^K[V,W]$. Then for each $[a,b] \in \mathcal{Y}_w'$ we have a component of G_w^2 that is a path of length $8d(a,b)$ from a to b^+. All these paths have length at most $8d$, as each such I is contained within an interval of $\mathcal{Y}_w^2$. Furthermore, if $x \in V$ is an endpoint of some path in G_w^2 then either x is a startpoint or successor of some interval in $\mathcal{Y}_w^2$, for which there are at most $2t_2$ choices of w by Lemma 5.4, or $x^+w \in F_{x^+}^K \setminus \bigcup M$, or $x^-w \in F_{x^-}^K \setminus \bigcup M$, giving at most $2n/K$ more choices of w, for a total of at most $n/\sqrt{d}$ (say).

It remains to show that each G_w^2 is isomorphic to some valid $F_w' \subseteq F_w$. First we show for any $c \in [3, K-1]$ that whp each G_w^2 has at most $q_{w,c}^2 n$ cycles of length c. The number of c-cycles is in G_w^2 is at most $|N_{J_2^c}^-(w)|$, which by Chernoff bounds is whp

$< p^2_{w,c}n + n^{.6} = (1-\eta)q^2_{w,c}n + n^{.6} < q^2_{w,c}n$, recalling that $p^2_{w,c} \geq n^{-.1}$. Next we bound the total length L_w of paths in G^2_w. By Lemma 5.4 we have $L_w \leq 8|Y^2_w| < 8p^2_{w,K}n + 8n^{3/4}$. Writing L'_w for the total length of long (length $\geq K$) cycles and paths in F^2_w, we recall that $8p^2_{w,K}n = p^2_w n - \sum_{c=3}^{K-1} cp^2_{w,c}n = (1-\eta)(L'_w + n^{.8})$. So since $p^2_{w,K} \geq d^{-1/3}n$, we have $L'_w > 8d^{-1/3}n$ and $L_w < (1-\eta/2)L'_w$.

We embed the paths of G^2_w into the long cycles and paths in F^2_w according to a greedy algorithm, where in each step that we embed some path P of G^2_w we delete a path of length $|P|+4$ from F^2_w, which we allocate to a copy of P surrounded on both sides by paths of length 2 that we will not include in F'_w (so that F'_w will be valid). We choose such a path (if it exists) within a remaining cycle or path of G^2_w, using an endpoint if it is a path (so that we preserve the number of components). Recalling that there are at most $n/\sqrt{d}$ endpoints of paths in G^2_w, we thus allocate a total of at most $2n/\sqrt{d}$ edges to the surrounding paths of length 2. Suppose for a contradiction that the algorithm gets stuck, trying to embed some path P in some remainder R. Then all components of R have size $\leq |P|+5 \leq 8d+5$. All components of G^2_w have size $\geq K$, so $|R| \leq (8d+5)|L'_w|/K$. However, we also have $|R| \geq |L'_w| - |L_w| - 2n/\sqrt{d} \geq \eta|L'_w|/2 - 2n/\sqrt{d}$, which is a contradiction, as $K^{-1} \ll d^{-1} \ll \eta$ and $L'_w > 8d^{-1/3}n$. Thus the algorithm succeeds in constructing a valid copy F'_w of G^2_w in F^2_w. $\square$

8. Exact decomposition

This section contains the two exact decomposition results that will conclude the proof in both Case K and Case ℓ^*. We start by giving a common setting for both cases. We say that G'_1 is a γ-perturbation of G_1 if $|N^{\pm}_{G_1}(x) \triangle N^{\pm}_{G'_1}(x)| < \gamma n$ for any $x \in V$. We say that J'_1 is a γ-perturbation of J_1 if J'_1 is obtained from J_1 by adding, deleting or recolouring at most γn arcs at each vertex. We will only consider perturbations which are compatible in the sense that arcs added between V and W will point towards W, and existing colours will be used.

Setting 8.1. Let G'_1 be an $\eta^{.9}$-perturbation of G_1. Suppose for each $w \in W$ that $Z_w \subseteq V$ with $|Z_w \triangle (V \setminus N^-_{J^1}(w))| < 5\eta n$. For $x \in V$ we write $Z(x) = \{w \in W : x \in Z_w\}$.

We start with the exact result for Case ℓ^*, where we recall that F^1_w consists of exactly $L^{-3}n$ cycles of length ℓ^*, so $p^1_w = (1-\eta)\ell^* L^{-3} + n^{-.2}$, $p^1_{w,\ell^*} = (1-\eta)L^{-3}$, $p^1_{w,K} = n^{-.2}/8$ and $p^1_{w,c} = 0$ for $c \in [3, K-1]$.

Lemma 8.2. *Suppose in Setting 8.1 and Case ℓ^* that $d^{\pm}_{G'_1}(x) = |W| - |Z(x)|$ for all $x \in V$ and ℓ^* divides $n - |Z_w|$ for all $w \in W$. Then G'_1 can be partitioned into graphs $(G^1_w : w \in W)$, where each G^1_w is an oriented C_{ℓ^*}-factor with $V(G^1_w) = V \setminus Z_w$.*

Proof. We will show that there is a perturbation J'_1 of J_1 such that $J'_1[V] = G'_1$, each $N^-_{J'_1}(w) = V \setminus Z_w$, and Theorem 3.1 applies to give a $\overrightarrow{W}_{\ell^*}$-decomposition of J'_1. This will suffice, by taking each G^1_w to consist of the rim ℓ^*-cycles of the copies of $\overrightarrow{W}_{\ell^*}$ containing w.

We construct J'_1 by starting with $J'_1 = J_1$ and applying a series of modifications as follows. First we delete all arcs of $J'_1[V]$ corresponding to arcs of $G_1 \setminus G'_1$ and add arcs of colour 0 corresponding to arcs of $G'_1 \setminus G_1$. Similarly, we delete all arcs $\overrightarrow{vw} \in J'_1[V,W]$ with $v \in N^-_{J_1}(w) \cap Z_w$ and add arcs $\overrightarrow{vw}$ of colour 0 for each $v \in (V \setminus Z_w) \setminus N^-_{J_1}(w)$. We also recolour any $\overrightarrow{vw} \in J'_1[V,W]$ of colour K to have colour 0 and replace any $\overrightarrow{xy}$ of colour K in $J'_1[V]$ by $\overrightarrow{xy}^+$ of colour 0. As each $p^1_{w,K} = n^{-.2}/8$ in this case, whp this affects at most $n^{.8}$ arcs at any vertex. Now $J'_1[V] = G'_1$, each $N^-_{J'_1}(w) = V \setminus Z_w$ and J'_1 is a $\eta^{.8}$-perturbation of J_1. We note for each $x \in V$ that $d^{\pm}_{J'_1}(x,V) = d^{\pm}_{G'_1}(x) = |W| - |Z(x)| = d^+_{J'_1}(x,W)$, so the divisibility conditions for $x \in V$ are satisfied.

Finally, to satisfy the divisibility conditions for all $w \in W$ we recolour so that $d^-_{(J'_1)^{\ell^*}}(w) = d^-_{J'_1}(w)/\ell^*$, which is an integer, as ℓ^* divides $d^-_{J'_1}(w) = n - |Z_w|$. By Lemma 5.5 each $d^-_{J_1}(w) = p^1_w n \pm 2n^{3/4}$ and $d^-_{J_1^{\ell^*}}(w) = p^1_{w,\ell^*} n \pm 2n^{3/4}$, where $p^1_w = \ell^* p^1_{w,\ell^*} + n^{-.2}$ in this case. As J'_1 is an $\eta^{.8}$-perturbation of J_1, we only need to recolour at most $2\eta^{.8}n$ arcs at any vertex, so our final digraph J'_1 is a $3\eta^{.8}$-perturbation of J_1.

Next we consider the regularity condition of Theorem 3.3. To each copy of $\overrightarrow{W}_{\ell^*}$ in J'_1 with hub w we assign weight $p^1_{w,\ell^*} n / W^g_{w,\ell^*} = p^1_{w,0} n (\alpha p^1_{w,0} p^1_* n)^{-\ell^*}$, which lies in $[n^{1-\ell^*}, L^L n^{1-\ell^*}]$. We claim that for any arc $\overrightarrow{e}$ of P' there is total weight $1 \pm \eta^{.6}$ on wheels containing $\overrightarrow{e}$. To see this, we compare the weight to $\hat{W}^1_{\ell^*}(\overrightarrow{e})$ as defined in section 6, which is $1 \pm 4\varepsilon$ by Lemma 6.1 (as $p^1_{w,0} = (\ell^* - 1)p^1_{w,\ell^*}$ and $p^1_* = (\ell^* - 1)p^1_{\ell^*}$). The actual weight on $\overrightarrow{e}$ differs from this estimate only due to wheels containing $\overrightarrow{e}$ that have another arc in $J'_1 \triangle J_1$. There are at most $40\eta^{.7} n^{\ell^*-1}$ such wheels, each affecting the weight by at most $L^L n^{\ell^*-1}$, so the claim holds. Thus regularity holds with $\delta = \eta^{.6}$ and $\omega = L^{-L}$.

It remains to show that J'_1 satisfies the extendability condition of Theorem 3.1. Consider any disjoint $A, B \subseteq V$ and $C \subseteq W$ each of size $\le h$, where $h = 2^{50(\ell^*)^3}$. By Lemma 5.5.iii, for $c \in \{0, \ell^*\}$ we have

$$
\begin{aligned}
|N^+_{J^0_1}(A) \cap N^-_{J^0_1}(B) \cap N^-_{J^c_1}(C)| &= |N^+_G(A) \cap N^-_G(B)|(p^1_*)^{|A|}(p^1_*)^{|B|} \prod_{w \in C} p^1_{w,c} \pm 4sn^{3/4} \\
&> (L^{-5}\alpha)^{2h} n,
\end{aligned}
$$

by typicality of G. Also, by Lemma 5.5.iv (with $S = \emptyset$ and $S' = A \cup B$) we have $|N^+_{J^0_1}(A) \cap N^+_{J^{\ell^*}_1}(B) \cap W| \ge 2^{-2s} L^{-7h} |W|$, say. The perturbation from J_1 to J'_1 affects these estimates by at most $6h\eta^{.7}n < \eta^{.6}n$, so J'_1 satisfies extendability with $\omega = L^{-L}$ as above. Now Theorem 3.1 applies to give a $\overrightarrow{W}_{\ell^*}$-decomposition of J'_1, which completes the proof. $\square$

Our second exact decomposition result concerns the path factors with prescribed ends required for Case K. We recall that each F^1_w consists of cycles of length $\ge K$ and at most one path of of length $\geq K$ with $|F^1_w| - n/2 \in [0, 2K]$, and that $(Y^1_w)^-$ and $(Y^1_w)^+$ are the sets of startpoints and successors of intervals in $\mathcal{Y}^1_w$. We also recall from Lemma 5.4 that for each $x \in V$, letting $t^{\pm}_1(x) = |\{w : x \in (Y^1_w)^{\pm}\}|$, we have $t^+_1(x) = t^-_1(x) = t_1$. After embedding F^2_w, and a greedy embedding connecting the paths to $(Y^1_w)^-$ and $(Y^1_w)^+$, we will need path factors G^1_w as follows.

Lemma 8.3. *Suppose in Setting 8.1 and Case K that Z_w is disjoint from $Y^1_w \cup (Y^1_w)^+$ and $8|Y^1_w| = n - |Z_w| - |(Y^1_w)^+|$ for all $w \in W$, and $d^{\pm}_{G'_1}(x) = |W| - t_1 - |Z(x)|$ for all $x \in V$. Then G'_1 can be partitioned into graphs $(G^1_w : w \in W)$, such that each G^1_w is a vertex-disjoint union of oriented paths with $V(G^1_w) = V \setminus Z_w$, where for each $[a, b] \in \mathcal{Y}^1_w$ there is an ab^+-path of length $8d(a, b)$.*

Proof. We will show that there is a perturbation P of J_1 such that each $N^-_P(w) = V \setminus Z_w$ and $P[V]$ corresponds to G'_1 under twisting, and a set E of arc-disjoint copies of $\overrightarrow{W}^K_8$ in P, such that Theorem 3.3 applies to give a $\overrightarrow{W}^K_8$-decomposition of $P' := P \setminus \bigcup E$. This will suffice, by taking each G^1_w to consist of the union of the oriented 8-paths that correspond under twisting to the rim 8-cycles of the copies of $\overrightarrow{W}^K_8$ containing w.

We construct P by starting with $P = J_1$ and applying a series of modifications as follows. First we delete all arcs of $P[V]$ corresponding to arcs of $G_1 \setminus G'_1$ and add arcs of colour 0 corresponding to arcs of $G'_1 \setminus G_1$. Similarly, we delete all arcs $\overrightarrow{vw} \in P[V, W]$ with $v \in N^-_{J_1}(w) \cap Z_w$ and add arcs $\overrightarrow{vw}$ of colour 0 for each $v \in V \setminus (Z_w \cup (Y^1_w)^+ \cup N^-_{J_1}(w))$. We also replace any $\overrightarrow{xy}$ of colour K with $d(x, y) < 3d$ by an arc $\overrightarrow{xy}^+$ of colour 0; this affects at most $6d$ arcs at each vertex. Now $P[V]$ corresponds to G'_1 under twisting, each $N^-_P(w) = V \setminus (Z_w \cup (Y^1_w)^+)$ and P is a $2\eta^{.9}$-perturbation of J_1.

We note that P now satisfies the divisibility condition $d^-_P(w) = 8|Y^1_w| = 8d^-_{P^K}(w)$, and for each $v \in V$ that $d^+_P(v, W) = |W| - t_1 - |Z(x)| = d_P(v, V)/2$, so $|P[V, W]| = |P[V]|$. We continue to modify P to obtain $|P^0[V, W]| = |P^0[V]|$ and $|P^K[V, W]| = |P^K[V]|$. To do so, we will recolour arcs of $P[V]$ according to a greedy algorithm, where if $|P^0[V]| > |P^0[V, W]|$ we replace some $\overrightarrow{xy} \in P^0[V]$ by $\overrightarrow{xy}^- \in P^K[V]$, or if $|P^0[V]| < |P^0[V, W]|$ we replace some $\overrightarrow{xy} \in P^K[V]$ by $\overrightarrow{xy}^+ \in P^0[V]$. This preserves $P[V]$ corresponding to G'_1 under twisting and $|P[V]| = |P[V, W]|$, so if we ensure $|P^0[V, W]| = |P^0[V]|$, we will also have $|P^K[V, W]| = |P^K[V]|$. During the greedy algorithm, we choose the arc to recolour arbitrarily, subject to avoiding the set S of vertices at which we have recoloured more than $\eta^{.8}n/2$ arcs. The total number of recoloured arcs is at most $||P[V, W]| - |P[V]|| \leq ||J_1[V, W]| - |J_1[V]|| + 2\eta^{.9}n^2 < 3\eta^{.9}n^2$ (by Lemma 5.5), so $|S| < 12\eta^{.1}n$. Thus the algorithm can be completed, giving P that is an $\eta^{.8}$-perturbation of J_1 with $|P^0[V, W]| = |P^0[V]|$ and $|P^K[V, W]| = |P^K[V]|$.

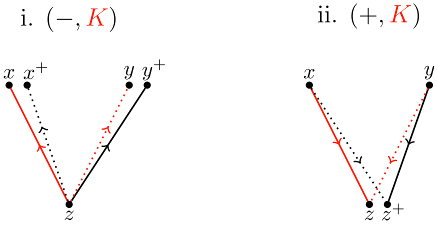
We will continue modifying $P[V]$ until it satisfies the remaining degree divisibility conditions for each $v \in V$, i.e. $d^+_P(v, V) = d^-_P(v, V) = d^+_P(v, W)$ and $d^-_{P^K}(v, V) = d^+_{P^K}(v, W)$. To do so, we will reduce to 0 the imbalance $\Delta' = \sum_{v \in V} \Delta'(v)$ with each $\Delta'(v) = |d^+_{P^K}(v, V) - d^+_{P^K}(v, W)| + |d^-_{P^K}(v, V) - d^+_{P^K}(v, W)|$. We do not attempt to control any $d^{\pm}_{P^0}(v, V)$, but nevertheless the divisibility conditions will be satisfied when $\Delta' = 0$.

To see this, note that if $\Delta' = 0$ then clearly all $d^+_{P^K}(v,V) = d^-_{P^K}(v,V) = d^+_{P^K}(v,W)$, so it remains to show that $d^-_P(v,V) = d^+_P(v,V) = d^+_P(v,W)$. Here we recall the discussion in section 4 relating the choice of intervals to degree divisibility, where (setting $H = G'_1$ and $J = P$) we noted that $d^+_{G'_1}(v) = d^+_P(v,V)$ and $d^-_{G'_1}(v) = d^-_P(v,V) + \Delta(v)$, with $\Delta(v) = d^-_{P^K}(v^-,V) - d^-_{P^K}(v,V) = d^+_{P^K}(v^-,W) - d^+_{P^K}(v,W)$. By our choice of intervals all $d^+_{P^K}(v,W)$ are equal to t_1, so $\Delta(v) = 0$ and $d^\pm_P(v,V) = d^\pm_{G'_1}(v) = |W| - t_1 - |Z(x)| = d^+_P(v,W)$, as required.

We have two types of reduction according to the two types of term in the definition of $\Delta'(v)$:

i. If $\sum_v |d^-_{P^K}(v,V) - d^+_{P^K}(v,W)| > 0$ then we can choose x,y in V with $d^-_{P^K}(x,V) > d^+_{P^K}(x,W)$ and $d^-_{P^K}(y,V) < d^+_{P^K}(y,W)$. We will find $z \in V$ such that $\overrightarrow{zx} \in P^K$, $\overrightarrow{zy}^+ \in P^0$ and replace these arcs by $\overrightarrow{zx}^+ \in P^0$, $\overrightarrow{zy} \in P^K$.

ii. If $\sum_v |d^+_{P^K}(v,V) - d^+_{P^K}(v,W)| > 0$ then we can choose x,y in V with $d^+_{P^K}(x,V) > d^+_{P^K}(x,W)$ and $d^+_{P^K}(y,V) < d^+_{P^K}(y,W)$. We will find $z \in V$ such that $\overrightarrow{xz} \in P^K$, $\overrightarrow{yz}^+ \in P^0$ and replace these arcs by $\overrightarrow{yz} \in P^K$, $\overrightarrow{xz}^+ \in P^0$.



Each of these operations preserves $P[V]$ corresponding to G'_1 under twisting and reduces Δ'.

To reduce Δ' to 0 we apply a greedy algorithm where in each step we apply one of the above operations. We do not allow z with $d(x,z) < 3d + 2$ or $d(y,z) < 3d + 2$ (to avoid creating close arcs in colour K) or z in the set S' of vertices that have played the role of z at $\eta^{.7}n/2$ previous steps. The total number of steps is at most $2\eta^{.8}n^2$, so $|S'| < 4\eta^{.1}n$. To estimate the number of choices for z at each step, we apply Lemma 5.5.iii to $|N^-_{J_1^{K'}}(x^+) \cap N^-_{J_1^0}(y^+)|$ for operation (i), $|N^+_{J_1^{K'}}(x) \cap N^+_{J_1^0}(y)|$ to find z^+ for (ii). By typicality of G this gives at least $\alpha^2 n/9$ choices, of which at most $5\eta^{.1}n$ are forbidden by lying in S or too close to x or y, or due to requiring an arc of $J_1 \setminus P$, so some choice always exists. Thus the algorithm can be completed, giving P that is an $\eta^{.7}$-perturbation of J_1, satisfies the divisibility conditions, and has $P[V]$ corresponding to G'_1 under twisting.

Next we construct E as a set of arc-disjoint copies of $\overrightarrow{W}^K_8$ that cover all $\overrightarrow{xy} \in P[V]$ with $d(x,y) < 3d$. Note that all such $\overrightarrow{xy}$ have colour 0. We apply a greedy algorithm, where in each step that we consider some $\overrightarrow{xy}$ we choose a copy of $\overrightarrow{W}^K_8$ that is arc-disjoint from all previous choices and does not use any vertex in the set S of vertices that have been used $.1d^2$ times. Then $|S|.1d^2 < 27dn$, so this forbids at most $270n^7/d$ choices of $\overrightarrow{W}^K_8$. By Lemma 6.2 we have $\hat{W}^1_K(\overrightarrow{xy}) > 2^{-2s-1}p^1_K/p^1_* > 2^{-3s}$, so the number of choices is at least $2^{-3s}\min_{w\in W} W^2_{w,K}/p^2_{w,K}n > 2^{-4s}n^7$, say. Thus there is always some choice that is not forbidden, so the algorithm can be completed. We note that $\bigcup E$ has maximum degree at most d^2 by definition of S, so $P' := P \setminus \bigcup E$ is a $2\eta^{\cdot 7}$-perturbation of J_1. Furthermore, P' satisfies the divisibility conditions, as P does and so does each $\overrightarrow{W}_8^K$ in E.

Next we consider the regularity condition of Theorem 3.3. To each $3d$-separated copy of $\overrightarrow{W}_8^K$ in P' with hub w we assign weight $p^1_{w,K}n/W^1_{w,K} = (\alpha p^1_K(\alpha p^1_* p^1_{w,0} n)^7)^{-1}$, which lies in $[n^{-7}, Ln^{-7}]$. We claim that for any arc $\overrightarrow{e}$ of P' there is total weight $1 \pm \eta^{\cdot 6}$ on wheels containing $\overrightarrow{e}$. To see this, we compare the weight to $\hat{W}^1_K(\overrightarrow{e})$ as defined in section 6, which is $1 \pm 4\varepsilon$ by Lemma 6.2 (as $\overrightarrow{e}$ is $3d$-separated, $p^1_{w,0} = 7p^1_{w,K}$ and $p^1_* = 7p^1_K$). The actual weight on $\overrightarrow{e}$ differs from this estimate only due to wheels containing $\overrightarrow{e}$ that have another arc in $P' \triangle J_1$. There are at most $40\eta^{\cdot 7} n^7$ such wheels, each affecting the weight by at most Ln^{-7}, so the claim holds. Thus regularity holds with $\delta = \eta^{\cdot 6}$ and $\omega = L^{-1}$.

It remains to show that P' satisfies the extendability condition of Theorem 3.3. Consider any disjoint $A, B \subseteq V$ and $L \subseteq W$ each of size $\leq h$ and $a, b, \ell \in \{0, K\}$. By Lemma 5.5.iii we have $|N^+_{J^a_1}(A) \cap N^-_{J^b_1}(B) \cap N^-_{J^\ell_1}(L)| = |N^+_G(A) \cap N^-_G(B)|(p^a_1)^{|A|}(p^b_1)^{|B|} \times \prod_{w \in L} p^1_{w,\ell} \pm 4sn^{3/4} > (10^{-3}\alpha)^{2h} n$, say. Also, if (A, B) is $3d$-separated then by Lemma 5.5.iv we have $|N^+_{J^0_1}(A) \cap N^+_{J^K_1}(B) \cap W| \geq 2^{-2s+10h}|W|$, say. The perturbation from J_1 to P' affects these estimates by at most $6h\eta^{\cdot 7} n < \eta^{\cdot 6} n$, so P' satisfies extendability with $\omega = L^{-1}$ as above. Now Theorem 3.3 applies to give a $\overrightarrow{W}_8^K$-decomposition of P', which completes the proof. $\square$

9. The proof

This section contains the proof of our main theorem. We give the reduction to cases in the first subsection and then the proof for both cases in the second subsection.

9.1. Reduction to cases

In this subsection we formalise the reduction to cases discussed in section 2. For Theorem 1.2, we are given an (ε, t)-typical αn-regular digraph G on n vertices, where $n^{-1} \ll \varepsilon \ll t^{-1} \ll \alpha$, and we need to decompose G into some given family $\mathcal{F}$ of αn oriented one-factors on n vertices. We prove Theorem 1.2 assuming that it holds in the following cases with $t^{-1} \ll K^{-1} \ll \alpha$:

Case K: each $F \in \mathcal{F}$ has at least $n/2$ vertices in cycles of length at least K,

Case ℓ for all $\ell \in [3, K-1]$: each $F \in \mathcal{F}$ has $\geq K^{-3}n$ cycles of length ℓ.

We will divide into subproblems via the following partitioning lemma.

Lemma 9.1. *Let $n^{-1} \ll \varepsilon \ll t^{-1} \ll \alpha_0$. Suppose G is an (ε, t)-typical αn-regular digraph on n vertices and $\alpha = \sum_{i \in I} \alpha_i$ with each $\alpha_i > \alpha_0$. Then G can be decomposed into digraphs $(G_i : i \in I)$ on $V(G)$ such that each G_i is $(2\varepsilon, t)$-typical and $\alpha_i n$-regular.*

Proof. We start by considering a random partition of G into graphs $(G'_i : i \in I)$ where for each arc $\overrightarrow{e}$ independently we have $\mathbb{P}(\overrightarrow{e} \in G'_i) = \alpha_i/\alpha$. We claim that whp each G'_i is $(1.1\varepsilon, t)$-typical. Indeed, this holds by Chernoff bounds, as $\mathbb{E}d(G'_i) = \alpha_i d(G)/\alpha$ for each i, so whp $d(G'_i) = \alpha_i \pm n^{-.4}$ (say), and for any set $S = S_- \cup S_+$ of at most t vertices, by typicality of G we have $\mathbb{E}|N^-_{G'_i}(S_-) \cap N^+_{G'_i}(S_+)| = (\alpha_i/\alpha)^{|S|}|N^-_G(S_-) \cap N^+_G(S_+)| = ((1 \pm \varepsilon)d(G)\alpha_i/\alpha)^{|S|}n$, so whp $|N^-_{G'_i}(S_-) \cap N^+_{G'_i}(S_+)| = ((1 \pm 1.1\varepsilon)d(G'_i))^{|S|}n$,

Now we modify the partition to obtain $(G_i : i \in I)$, by a greedy algorithm starting from all $G_i = G'_i$. First we ensure that all $|G_i| = \alpha_i n^2$. At any step, if this does not hold then some $|G_i| > \alpha_i n^2$ and $|G_j| < \alpha_j n^2$. We move an arc from G_i to G_j, arbitrarily subject to not moving more than $n^{.7}$ arcs at any vertex. We move at most $n^{1.6}$ arcs, so at most $2n^{.9}$ vertices become forbidden during this algorithm. Hence the algorithm can be completed to ensure that all $|G_i| = \alpha_i n^2$. Each $|N^-_{G'_i}(S_-) \cap N^+_{G'_i}(S_+)|$ changes by at most $tn^{.7}$, so each G_i is now $(1.2\varepsilon, t)$-typical.

Let $\widetilde{G_i}$ be the undirected graph of G_i (which could have parallel edges). We will continue to modify the partition until each $\widetilde{G_i}$ is $2\alpha_i n$-regular, maintaining all $|G_i| = \alpha_i n^2$. At each step we reduce the imbalance $\sum_{i,x} |d_{\widetilde{G_i}}(x) - 2\alpha_i n|$. If some $\widetilde{G_i}$ is not $2\alpha_i n$-regular we have some $d_{\widetilde{G_i}}(x) > 2\alpha_i n$ and $d_{\widetilde{G_i}}(y) < 2\alpha_i n$. Considering the total degree of x, there is some j with $d_{\widetilde{G_j}}(x) < 2\alpha_j n$. We will choose some z with $xz \in \widetilde{G_i}$ and $yz \in \widetilde{G_j}$, then move xz to $\widetilde{G_j}$ and yz to $\widetilde{G_i}$, thus reducing the imbalance by at least 2. We will not choose z in the set L of vertices that have played the role of z at $n^{.8}$ previous steps. We had all $d_{\widetilde{G_i}}(x) = 2(\alpha_i n \pm n^{.7})$ after the first algorithm, so this algorithm will have at most $2n^{1.7}$ steps, giving $|L| < n^{.9}$. By typicality, there are at least $3\alpha_i\alpha_j n$ choices of z, of which at most $2n^{.9}$ are forbidden by L or requiring an edge that has been moved, so the algorithm to make each $\widetilde{G_i}$ be $2\alpha_i n$-regular can be completed. Each $|N^-_{G_i}(S_-) \cap N^+_{G_i}(S_+)|$ changes by at most $tn^{.8}$, so each G_i is now $(1.1\varepsilon, t)$-typical.

We will continue to modify the partition until each G_i is $\alpha_i n$-regular, maintaining all $d_{\widetilde{G_i}}(x) = 2\alpha_i n$. At each step we reduce the imbalance $\sum_{i,x} |d^+_{G_i}(x) - \alpha_i n|$ (if it is 0 then since total degrees $d_{\widetilde{G_i}}(x)$ are correct, G_i is regular). If it is not 0 we have some $d^+_{G_i}(x) > \alpha_i n$ and $d^+_{G_i}(y) < \alpha_i n$. Again there is some j with $d^+_{G_j}(x) < \alpha_j n$ and we choose some z with $\overrightarrow{xz} \in G_i$ and $\overrightarrow{yz} \in G_j$, then move $\overrightarrow{xz}$ to G_j and $\overrightarrow{yz}$ to G_i, avoiding vertices z which have played this role at $n^{.9}$ previous steps. By typicality we can find such z at every step and complete the algorithm. Each $|N^-_{G_i}(S_-) \cap N^+_{G_i}(S_+)|$ changes by at most $tn^{.9}$, so each G_i is now $(2\varepsilon, t)$-typical. $\square$

Factors of a type that is too rare will be embedded greedily via the following lemma.

Lemma 9.2. *Let $n^{-1} \ll \varepsilon \ll t^{-1} \ll \alpha$. Suppose G is an (ε, t)-typical αn-regular digraph on n vertices and $\mathcal{F}$ is a family of at most εn oriented one-factors. Then we can remove from G a copy of each $F \in \mathcal{F}$ to leave a $(\sqrt{\varepsilon}, t)$-typical $(\alpha n - |\mathcal{F}|)$-regular graph.*

Proof. We embed the one-factors one by one. At each step, the remaining graph G' is obtained from G by deleting a graph that is regular of degree at most $2\varepsilon n$, so is $(\sqrt{\varepsilon}, t)$-

typical. It is a standard argument (which we omit) using the blow-up lemma of Komlós, Sárközy and Szemerédi [16] to show that any one-factor can be embedded in G', so the process can be completed. □

Now we prove Theorem 1.2 assuming that it holds in the above cases. We introduce new parameters $\alpha_1, \alpha_2, M_1', M_1, M_2, M_3$ with $\varepsilon \ll t^{-1} \ll M_3^{-1} \ll \alpha_2 \ll M_2^{-1} \ll \alpha_1 \ll (M_1')^{-1} \ll M_1^{-1} \ll \alpha$. For $\ell \in [3, M_2]$ let $\mathcal{F}_\ell$ consist of all factors $F \in \mathcal{F}$ such that F has $\geq M_2^{-3}n$ cycles of length ℓ but $< M_2^{-3}n$ cycles of each smaller length. Let $\mathcal{F}_2$ consist of all remaining factors in $\mathcal{F}$. Note that each $F \in \mathcal{F}_2$ has fewer than n/M_2 vertices in cycles of length less than M_2, so at least $(M_2-1)n/M_2$ in cycles of length at least M_2. Let B be the set of $\ell \in [3, M_2]$ such that $|\mathcal{F}_\ell| < \alpha_2 n$. Then for $\ell \in I' := [3, M_2] \setminus B$ we have $\beta_\ell := n^{-1}|\mathcal{F}_\ell| \geq \alpha_2$. Also, writing $\mathcal{F}_B = \bigcup_{\ell \in B} \mathcal{F}_\ell$, we have $\beta_B := n^{-1}|\mathcal{F}_B| < M_2\alpha_2 < \sqrt{\alpha_2}$.

Let $\mathcal{F}_1$ be the set of F in $\mathcal{F}$ with at least $n/2$ vertices in cycles of length $> M_1$. We first consider the case $\eta := n^{-1}|\mathcal{F}_1| \geq \alpha/2$. Let $B^1 = B \cap [3, M_1]$, $\mathcal{F}_{B^1} = \bigcup_{\ell \in B^1} \mathcal{F}_\ell$, and $\beta_{B^1} := n^{-1}|\mathcal{F}_{B^1}| < \beta_B < \sqrt{\alpha_2}$. We apply Lemma 9.1 with $I = (I' \cap [3, M_1]) \cup \{1\}$, letting $\alpha_\ell = \beta_\ell$ for all $\ell \in I' \cap [3, M_1]$ and $\alpha_1 = \eta + \beta_{B^1}$, thus decomposing G into $(2\varepsilon, t)$-typical $\alpha_i n$-regular digraphs G_i on $V(G)$. For each $\ell \in I' \cap [3, M_1]$ we decompose G_ℓ into $\mathcal{F}_\ell$ by Case ℓ of Theorem 1.2, where in place of the parameters $n^{-1} \ll \varepsilon \ll t^{-1} \ll K^{-1} \ll \alpha$ we use $n^{-1} \ll 2\varepsilon \ll t^{-1} \ll M_3^{-1} \ll \alpha_2$. For G_1, we first embed $\mathcal{F}_{B^1}$ via Lemma 9.2, leaving an ηn-regular digraph G_1' that is (ε', t)-typical with $\alpha_2 \ll \varepsilon' \ll t^{-1} \ll M_2^{-1}$. We then conclude the proof of this case by decomposing G_1' into $\mathcal{F}_1$ by Case K of Theorem 1.2, where in place of the parameters $n^{-1} \ll \varepsilon \ll t^{-1} \ll K^{-1} \ll \alpha$ we use $n^{-1} \ll \varepsilon' \ll t^{-1} \ll M_1^{-1} \ll \eta$.

It remains to consider the case $\eta < \alpha/2$. Here there are at least $\alpha n/2$ factors $F \in \mathcal{F}$ with at least $n/2$ vertices in cycles of length $\leq M_1$, so we can fix $\ell^* \in [M_1] \cap I'$ with $\beta_{\ell^*} > \alpha/2M_1$. We consider two subcases according to $\beta_2 := n^{-1}|\mathcal{F}_2|$.

Suppose first that $\beta_2 < \alpha_1 n$. We apply Lemma 9.1 with $I = I'$, letting $\alpha_\ell = \beta_\ell$ for all $\ell \in I \setminus \{\ell^*\}$ and $\alpha_{\ell^*} = \beta_{\ell^*} + \beta_{B^1} + \beta_2$. For each $\ell \in I \setminus \{\ell^*\}$ we decompose G_ℓ into $\mathcal{F}_\ell$ by Case ℓ of Theorem 1.2, where (as before) in place of the parameters $n^{-1} \ll \varepsilon \ll t^{-1} \ll K^{-1} \ll \alpha$ we use $n^{-1} \ll 2\varepsilon \ll t^{-1} \ll M_3^{-1} \ll \alpha_2$. For G_{ℓ^*} we first embed $\mathcal{F}_B \cup \mathcal{F}_2$ by Lemma 9.2, leaving a $\beta_{\ell^*}n$-regular digraph G_{ℓ^*}' that is (ε', t)-typical with $\alpha_1 \ll \varepsilon' \ll t^{-1} \ll M_1^{-1}$. We then complete the decomposition by decomposing G_{ℓ^*}' into $\mathcal{F}_{\ell^*}$ by Case ℓ^* of Theorem 1.2, where in place of the parameters $n^{-1} \ll \varepsilon \ll t^{-1} \ll K^{-1} \ll \alpha$ we use $n^{-1} \ll \varepsilon' \ll t^{-1} \ll (M_1')^{-1} \ll \beta_{\ell^*}$.

It remains to consider the subcase $\beta_2 \geq \alpha_1 n$. We apply Lemma 9.1 with $I = I' \cup \{2\}$, letting $\alpha_\ell = \beta_\ell$ for all $\ell \in I \setminus \{\ell^*\}$ and $\alpha_{\ell^*} = \beta_{\ell^*} + \beta_{B^1}$. The same argument as in the first subcase applies to decompose G_ℓ into $\mathcal{F}_\ell$ for all $\ell \in I' \setminus \{\ell^*\}$, and also to embed $\mathcal{F}_B$ in G_{ℓ^*} by Lemma 9.2 and decompose the leave G_{ℓ^*}' into $\mathcal{F}_{\ell^*}$. We complete the proof of this case, and so of the entire reduction, by decomposing G_2 into $\mathcal{F}_2$ by Case K of Theorem 1.2, where in place of the parameters $n^{-1} \ll \varepsilon \ll t^{-1} \ll K^{-1} \ll \alpha$ we use $n^{-1} \ll 2\varepsilon \ll t^{-1} \ll M_2^{-1} \ll \beta_2$.

9.2. Proof of Theorem 1.2

We are now ready to prove our main theorem. We are given an (ε, t)-typical αn-regular digraph G on n vertices, where $n^{-1} \ll \varepsilon \ll t^{-1} \ll \alpha$, and we need to decompose G into some given family $\mathcal{F}$ of αn oriented one-factors on n vertices. By the reductions in section 9.1, we can assume that we are in one of the following cases with $t^{-1} \ll M^{-1} \ll \alpha$:

Case K: each $F \in \mathcal{F}$ has at least $n/2$ vertices in cycles of length at least M,

Case ℓ^* with $\ell^* \in [3, M-1]$: each $F \in \mathcal{F}$ has $\geq M^{-3}n$ cycles of length ℓ^*.

Here the parameters of section 9.1 are renamed: ℓ is now ℓ^* so that 'ℓ' is free to denote generic cycle lengths; K is now M, as we want K to take different values in each case: we introduce M' with $t^{-1} \ll M'^{-1} \ll M^{-1}$ and define

$$K = \begin{cases} M & \text{in Case } K, \\ M' & \text{in Case } \ell^*. \end{cases}$$

We define a parameter L by $L = M$ in Case ℓ^* (so $K^{-1} \ll L^{-1} \ll (\ell^*)^{-1}$), or as a new parameter with $K^{-1} \ll L^{-1} \ll \alpha$ in Case K. We use these parameters to apply the algorithm of section 4 as in (1), so we can apply the conclusions of the lemmas in sections 5 to 8.

We recall that each factor F_w is partitioned as $F_w^1 \cup F_w^2$, where F_w^1 either consists of exactly $L^{-3}n$ cycles of length ℓ^* in Case ℓ^*, or in Case K we have $|F_w^1| - n/2 \in [0, 2K]$ and F_w^1 consists of cycles of length $\geq K$ and at most one path of length $\geq K$ (and then $F_w^2 = F_w \setminus F_w^1$).

By Lemma 7.1, there are arc-disjoint digraphs $G_w^2 \subseteq G_2$ for $w \in W$, where each G_w^2 is a copy of some valid $F'_w \subseteq F_w^2$ with $V(G_w^2) \subseteq N^-_{J_2}(w)$, such that

i. $G_2^- = G_2 \setminus \bigcup_{w \in W} G_w^2$ has maximum degree at most $5d^{-1/3}n$,

ii. the digraph J_2^- obtained from $J_2[V, W]$ by deleting all $\overrightarrow{xw}$ with $x \in V(G_w^2)$ has maximum degree at most $5d^{-1/3}n$,

iii. any $x \in V$ has degree 1 in F'_w for at most $n/\sqrt{d}$ choices of w.

(Recall that 'valid' means that $F_w^2 \setminus F'_w$ does not have any independent arcs, and if F_w^2 contains a path then $F_w^2 \setminus F'_w$ contains the arcs incident to each of its ends.)

Note that (ii) implies for each $w \in W$ that $|F'_w| \geq |N^-_{J_2}(w)| - 5d^{-1/3}n > p_w^2 n - 6d^{-1/3}n$ (by Lemma 5.5), so as $p_w^2 n = (1-\eta)|F_w^2| + n^{.8}$ we have $|F_w^2 \setminus F'_w| < \eta n$.

Next we will embed oriented graphs $R_w = (F_w^2 \setminus F'_w) \cup L_w$ for $w \in W$, where $L_w \subseteq F_w^1$ is defined as follows. In Case ℓ^* we let each L_w consist of $2\eta L^{-3}n$ cycles of length ℓ^*. In Case K we partition each F_w^1 as $\mathcal{P}_w \cup L_w$, where $\mathcal{P}_w$ is a valid vertex-disjoint union of paths, such that for each $[a, b] \in \mathcal{Y}_w^1$ we have an oriented path P_w^{ab} in $\mathcal{P}_w$ of length $8d(a, b)$ (which we will embed as an ab^+-path). To see that such a partition exists, we apply the same argument as at the end of the proof of Lemma 7.1. We consider a greedy algorithm, where at each step that we consider some path P_w^{ab} we delete a path of length $8d(a, b)+4$ from F_w^1, which we allocate as P_w^{ab} surrounded on both sides of paths of length 2 that we add to L_w. As $|\mathcal{Y}_w^1| < n/2d_{2s+1} = (2s)^{2s}n/2d$ we thus allocate $< (2s)^{2s}n/d$

edges to L_w. Suppose for contradiction that the algorithm gets stuck, trying to embed some path P in some remainder Q_w. Then all components of Q_w have size $\leq 8d+5$. All components of F^1_w have size $\geq K$, so $|Q_w| \leq (8d+5)|F^1_w|/K < 5dn/K$. However, we also have $|Q_w| \geq |F^1_w| - |Y^1_w| - |L_w| \geq \eta n/3$, as $|F^1_w| \geq n/2$ and $|Y^1_w| = (1-\eta)n/2 \pm 2n^{3/4}$ by Lemma 5.4. This is a contradiction, so the algorithm finds a partition $F^1_w = \mathcal{P}_w \cup L_w$ with $\mathcal{P}_w$ valid. We note that each $|R_w| < 2\eta n$.

Now we apply a greedy algorithm to construct arc-disjoint embeddings $(\phi_w(R_w) : w \in W)$ in G_1. At each step we choose some $\phi_w(x) \in N^-_{J^1}(w)$ (which is disjoint from $G^2_w \subseteq N^-_{J_2}(w)$). We require $\phi_w(x)$ to be an outneighbour of some previously embedded $\phi_w(x_1)$ or both an outneighbour of $\phi_w(x_1)$ and an inneighbour of $\phi_w(x_2)$ for some previously embedded images; the latter occurs when we finish a cycle or a path (the image under ϕ_w of the ends of the paths in R_w have already been prescribed: they are either images of endpoints of paths in F'_w or startpoints / successors of intervals in $\mathcal{Y}^1_w$). We also require $\phi_w(x)$ to be distinct from all previously embedded $\phi_w(x_1)$ and not to lie in the set S of vertices that are already in the image of $\phi_{w'}$ for at least $\eta^{.9}n/2$ choices of w'. As $\eta^{.9}n|S|/2 \leq \sum_{w\in W}|R_w| < 2\eta n^2$ we have $|S| < 4\eta^{.1}n$. To see that it is possible to choose $\phi_w(x)$, first note for any v, v' in V and $w \in W$ that $|N^+_{G_1}(v) \cap N^-_{G_1}(v') \cap N^-_{J^1}(w)| > \alpha^2 n/3$, by Lemma 5.5.iii and typicality of G. At most $|R_w| + |S| < 5\eta^{.1}n$ choices of $\phi_w(x)$ are forbidden due to using S or some previously embedded $\phi_w(x_1)$. Also, by definition of S, we have used at most $\eta^{.9}n$ arcs at each of v and v' for other embeddings $\phi_{w'}$, so this forbids at most $2\eta^{.9}n$ choices of $\phi_w(x)$. Thus the algorithm never gets stuck, so we can construct $(\phi_w(R_w) : w \in W)$ as required.

Let $G'_1 = G \setminus \bigcup_{w\in W}(G^2_w \cup R_w)$. For each $w \in W$ let Z_w be the set of vertices of in- and outdegree 1 in $G^2_w \cup R_w$. We claim that G'_1 and Z_w satisfy Setting 8.1. To see this, first note that by definition of S above each $|N^{\pm}_{G_1}(x) \setminus N^{\pm}_{G'_1}(x)| < \eta^{.9}n/2$. As $d^{\pm}_{G_2^-}(x) < 5d^{-1/3}n$ by (i) above and (by Lemma 5.5) $d^{\pm}_G(x) - d^{\pm}_{G_1}(x) - d^{\pm}_{G_2}(x) < (1-p_1-p_2)d^{\pm}_G(x) + n^{.6} < 2\eta n$ we have $|N^{\pm}_{G_1}(x) \triangle N^{\pm}_{G'_1}(x)| < \eta^{.9}n$, so G'_1 is an $\eta^{.9}$-perturbation of G_1. Also, as $|N^-_{J_2}(w) \setminus F'_w| \leq 5d^{-1/3}n$, $|R_w| < 2\eta n$ and $|V \setminus N^-_J(w)| < 2\eta n$ (the last by Lemma 5.5) we have $|Z_w \triangle (V \setminus N^-_{J^1}(w))| < 5\eta n$, as claimed.

In Case ℓ^*, every vertex has equal in- and outdegrees 0 or 1 in $G^2_w \cup R_w$ (it is a vertex-disjoint union of cycles) so $d^{\pm}_{G'_1}(x) = |W| - |Z(x)|$ for all $x \in V$ and ℓ^* divides $n - |Z_w|$ for all $w \in W$. Thus Lemma 8.2 applies to partition G'_1 into graphs $(G^1_w : w \in W)$, where each G^1_w is a C_{ℓ^*}-factor with $V(G^1_w) = V \setminus Z_w$, thus completing the proof of this case.

In Case K, a vertex x has indegree (respectively outdegree) 1 in $G^2_w \cup R_w$ exactly when $x \in (Y^1_w)^-$ (respectively $(Y^1_w)^+$), for which there are each t_1 choices of w, so $d^{\pm}_{G'_1}(x) = |W| - t_1 - |Z(x)|$ for all $x \in V$. By construction, Z_w is disjoint from $(Y^1_w)^- \cup (Y^1_w)^+$, and the total length of paths required in the remaining path factor problem satisfies $8|Y^1_w| = n - |Z_w| - |(Y^1_w)^+|$ for all $w \in W$. Thus Lemma 8.3 applies to partition G'_1 into graphs $(G^1_w : w \in W)$, such that each G^1_w is a vertex-disjoint union of oriented paths with $V(G_w^1) = V \setminus Z_w$, where for each $[a,b] \in \mathcal{Y}_w^1$ there is an ab^+-path of length $8d(a,b)$. This completes the proof of this case, and so of Theorem 1.2.

10. Concluding remarks

As mentioned in the introduction, our solution to the generalised Oberwolfach Problem is more general than the result of [9] in three respects: it applies to any typical graph (theirs is for almost complete graphs) and to any collection of two-factors (they need some fixed F to occur $\Omega(n)$ times), and it applies also to directed graphs. Although there are some common elements in both of our approaches (using [12] for the exact step and some form of twisting), the more general nature of our result reflects a greater flexibility in our approach that has further applications. One such application is our recent proof [14] that every quasirandom graph with n vertices and rn edges can be decomposed into n copies of any fixed tree with r edges. The case of the complete graph solves Ringel's tree-packing conjecture [19] (solved independently via different methods by Montgomery, Pokrovskiy and Sudakov [18]).

A natural open problem raised in [9] is whether the generalised Oberwolfach problem can be further generalised to decompositions of K_n into any family of regular graphs of bounded degree (where the total of the degrees is $n-1$).

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