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THE TURÁN NUMBER OF THE FANO PLANE

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Let $PG_2(2)$ be the Fano plane, i.e., the unique hypergraph with 7 triples on 7 vertices in which every pair of vertices is contained in a unique triple. In this paper we prove that for sufficiently large n, the maximum number of edges in a 3-uniform hypergraph on n vertices not containing a Fano plane is

$$ex\big(n, PG_2(2)\big) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}.$$

Moreover, the only extremal configuration can be obtained by partitioning an n-element set into two almost equal parts, and taking all the triples that intersect both of them. This extends an earlier result of de Caen and Füredi, and proves an old conjecture of V. Sós. In addition, we also prove a stability result for the Fano plane, which says that a 3-uniform hypergraph with density close to 3/4 and no Fano plane is approximately 2-colorable.

1. Introduction

Given an r-uniform hypergraph $\mathcal{F}$, the Turán number of $\mathcal{F}$ is the maximum number of edges in an r-uniform hypergraph on n vertices that does not contain a copy of $\mathcal{F}$. We denote this number by $ex(n,\mathcal{F})$. Determining these numbers is one of the central problems in Extremal Combinatorics, and it is well understood for ordinary graphs (the case $r=2$). It is completely solved for many instances, including all complete graphs. Moreover, asymptotic results are known for all non-bipartite graphs. In contrast, for nearly any

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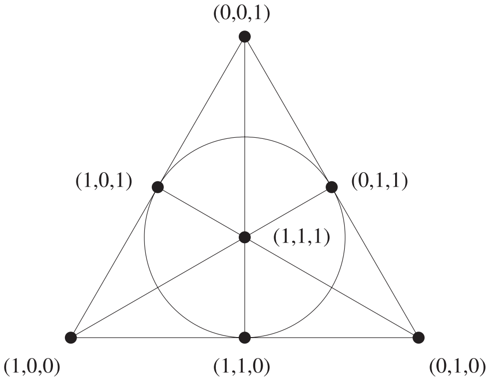

Figure 1. The Fano Plane

r-uniform hypergraph $\mathcal{F}$ with $r > 2$, the problem of finding the numbers $ex(n, \mathcal{F})$ is notoriously difficult. Exact results on hypergraph Turán numbers are very rare (see, e.g., the excellent survey of Füredi [4]). In this paper we obtain such a result which determines the Turán number of the Fano plane.

The Fano plane (see figure 1) is the projective plane over the field with 2 elements. It has 7 vertices, which can be identified with the non-zero vectors of length 3. It has 7 edges, corresponding to the lines of the plane. A triple xyz is an edge if $x + y = z$. A hypergraph is *2-colorable* if its vertices can be labeled as red or blue so that no edge is monochromatic. It is easy to check that the Fano plane is not 2-colorable, and therefore any 2-colorable hypergraph cannot contain the Fano plane. Partition an n-element set into two almost equal parts, and take all the triples that intersect both of them. This is clearly the largest 2-colorable 3-uniform hypergraph on n vertices. In 1976 V. Sós [10] conjectured that this construction gives the exact value of $ex(n, PG_2(2))$. We will prove the following theorem, which confirms this conjecture.

Theorem 1.1. *Let H be a 3-uniform hypergraph on n vertices that does not contain a copy of the Fano plane and let n be sufficiently large. Then the number of edges in H is at most*

$$e(H) \le \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3},$$

with equality only when H is obtained by partitioning an n-element set into two almost equal parts, and taking all the triples that intersect both of them.

The asymptotics of hypergraph Turán numbers are poorly understood. It is not hard to show that the limit $\pi(F) = \lim_{n\to\infty} ex(n, F)/\binom{n}{r}$ exists. It is usually called the *Turán density*. Until very recently, even the question of finding the Turán density of the Fano plane was open. Thus it was a significant breakthrough, when de Caen and Füredi [2] determined that the Turán density of the Fano plane is 3/4.

Our methods build on those of de Caen and Füredi in a manner that we hope may provide a general framework for proving exact results for Turán numbers, once the density result is known. The idea is to use the density result to prove an approximate structure theorem for hypergraphs with density close to the maximum possible, and then to find the exact structure that has maximum size among the approximate structures.

To be more concrete, we will describe an analogous situation for ordinary graphs. Let $T_r(n)$ be the complete r-partite graph on n vertices with parts as equal as possible. It is usually called the *Turán graph* and we write $t_r(n)$ for the number of edges in $T_r(n)$. Turán's theorem states that any K_{r+1}-free graph on n vertices can contain at most $t_r(n)$ edges, and equality only occurs for $T_r(n)$. Furthermore, we know the structure of K_{r+1}-free graphs with nearly $t_r(n)$ edges. This so-called *stability* theorem was proved by Simonovits [9]. It states that for all $\epsilon > 0$ there is $\delta > 0$ such that if G is a K_{r+1}-free graph with at least $(1-\delta)t_r(n)$ edges then there is a partition of the vertices of G as $V_1 \cup \cdots \cup V_r$ with $\sum_i e(V_i) < \epsilon n^2$.

The key to our proof of Theorem 1.1 is the following stability result for the Fano plane, which roughly speaking says that a 3-uniform hypergraph with density close to 3/4 and no Fano plane is approximately 2-colorable.

Theorem 1.2. *For all $\epsilon > 0$ there is $\delta > 0$ such that such that if H is a 3-uniform hypergraph with $(1-\delta)\frac{3}{4}\binom{n}{3}$ edges and no Fano plane then we can partition $V(H) = A \cup B$ so that $e(A) + e(B) < \epsilon n^3$.*

An analogous stability result for $\mathcal{D}_3 = \{123, 124, 345\}$, was recently obtained by Keevash and Mubayi [7]. The Turán number of $\mathcal{D}_3$ had been previously determined by Frankl and Füredi [5].

The rest of this paper is organized as follows. The next section contains some definitions and preliminary lemmas. In Section 3 we prove the stability theorem and in Section 4 we use it to deduce Theorem 1.1. The last section is devoted to some concluding remarks.

We will normally use the letter G to denote an ordinary graph and H to denote a 3-uniform hypergraph. The vertex and edge sets are respectively $V(G)$,$E(G)$ for G, and similarly $V(H)$,$E(H)$ for H. If X,Y are sets of vertices then $e(X)$ is the number of edges contained within the set X, and $e(X,Y)$ is the number of edges incident to both X and Y. This notation applies to both graphs and 3-uniform hypergraphs, but when there is a possibility of confusion we will introduce the graph/hypergraph as a subscript, e.g. $e_G(X)$. We will let $H_2(n)$ denote the maximum 2-colorable 3-uniform hypergraph on n vertices, which is obtained by partitioning an n-element set into two almost equal parts, and taking all the triples that intersect both of them. It is easy to see that it contains $h_2(n)=\binom{n}{3}-\binom{\lfloor n/2\rfloor}{3}-\binom{\lceil n/2\rceil}{3}$ edges.

2. Preliminaries

Let H be a 3-uniform hypergraph. The *link* of a vertex $x\in V(H)$ is $L(x)=\{(a,b) : abx \in E(H)\}$. We can think of the link as a graph on $V(H)$. The number of edges in the link graph $L(x)$ is *the degree* $d_H(x)$ of x, i.e., the number of hyperedges containing x. We will often consider several links simultaneously, regarding them as a multigraph. This is a loopless graph in which each edge has some non-negative integral multiplicity. If $S\subset V(H)$ is a set of vertices of H then the *link multigraph* of S is the multigraph sum of the links of each vertex in S.

We will need the following two simple ways of recognizing a copy of a Fano plane in a 3-uniform hypergraph.

Observation 2.1. *Let H be a 3-uniform hypergraph.*
(i) Let $x_1x_2x_3$ be an edge of H and let L_i be the link graph of x_i. Suppose $L_1\cup L_2\cup L_3$ contains four vertices $abcd$ spanning a K_4, such that the edges of this K_4 can be partitioned into three matchings M_1, M_2, M_3 with $M_i\subset L_i$. Then H contains a Fano plane.
(ii) Let x be a vertex of H with link graph $L(x)$. Suppose $L(x)$ contains three disjoint edges e_1, e_2, e_3 such that all the triples $x_1x_2x_3$ with $x_i\in e_i$ are edges of H. Then H contains a Fano plane.

Proof. (i) Without loss of generality ab, cd are in L_1, ac, bd are in L_2 and ad, bc are in L_3. Then by assigning $x_1=(1,0,0)$, $x_2=(0,1,0)$, $x_3=(1,1,0)$, $a=(0,0,1)$, $b=(1,0,1)$, $c=(0,1,1)$ and $d=(1,1,1)$ we obtain a copy of the Fano plane.

(ii) Let $e_i=a_ib_i$. Then by assigning $x=(1,0,0)$, $a_1=(0,1,0)$, $b_1=(1,1,0)$, $a_2=(0,0,1)$, $b_2=(1,0,1)$, $a_3=(0,1,1)$ and $b_3=(1,1,1)$ we again obtain a copy of the Fano plane. ∎

The result of de Caen and Füredi, that gives the Turán density of the Fano plane, is based on the following three lemmas which we will also use in our proofs.

Lemma 2.2. *Let H be a 3-uniform hypergraph on n vertices with at least $\left(\frac{3}{4}-\epsilon\right)\binom{n}{3}$ edges. If ϵ is sufficiently small, then H contains a copy of $K_4^{(3)}$, i.e., a complete 3-uniform hypergraph on four vertices.*

Lemma 2.3. *A multigraph of order n in which every 4 vertices span at most 20 edges has at most $3\binom{n}{2}+O(n)$ edges.*

Lemma 2.4. *Let H be a 3-uniform hypergraph and let S be a $K_4^{(3)}$ subhypergraph of H with link multigraph G. If $G-S$ has a set of 4 vertices spanning 21 edges then H contains a Fano plane.*

The first lemma holds for any $\epsilon < 1/12$ by an old bound of de Caen [1] on the Turán density of $K_4^{(3)}$. This is not the strongest bound known but it is sufficient for our purposes. The second lemma is a special case of a result of Füredi and Kündgen [6] (see also [2] for a short proof). The third result is also from [2] and can be easily obtained from part (i) of Observation 2.1.

Our next lemma describes a few additional forbidden configurations in the link multigraph of a 3-uniform hypergraph which contains no Fano plane.

Lemma 2.5. *Let H be a 3-uniform hypergraph and let $S \subset V(H)$ span a copy of $K_4^{(3)}$. Let G be its link multigraph restricted to $V-S$ and let $G' \subset G$ be the edges of G with multiplicity ≥ 3.*
(i) If G' contains a copy of K_4 then H contains a Fano plane, unless all edges of this K_4 have multiplicity exactly 3.
(ii) Suppose G contains a copy of K_4 with vertex set $y_1y_2y_3y_4$ such that y_1y_2, y_1y_3 have multiplicity 4, y_2y_3 has multiplicity 3, and the other edges have multiplicities 4,3,2 in some order. Then H contains a Fano plane.
(iii) If G' contains a copy of K_5 then H contains a Fano plane.

Proof. Let $x_1x_2x_3x_4$ be the vertex set of S and let L_i be the link of x_i.
(i) Suppose $y_1y_2y_3y_4$ are the vertices of a K_4 in G'. Partition the edges of this K_4 into 3 disjoint matchings M_i for $1 \leq i \leq 3$ and let $I_i \subset \{1,2,3,4\}$ index all the link graphs which contain both edges of M_i. Each edge belongs to at least 3 link graphs, so $|I_i| \geq 3+3-4 = 2$ for each i. By part (i) of Observation 2.1, a system of distinct representatives for the sets I_i gives a Fano plane. So by Hall's theorem, if there is no Fano plane we must have $I_1 = I_2 = I_3$ equal to a set of size 2. It is easy to check that in this case all edges of the K_4 have multiplicity exactly 3.

(ii) Next suppose that $y_1y_2y_3y_4$ are the vertices of a K_4 which satisfies all the conditions of part (ii) of the lemma. Define M_i and I_i as in the proof of part (i). It is possible that one of the I_i has only one element. This can only occur when the edge y_1y_4 has multiplicity 2. In this case, the remaining two I_j each have size at least 3, and there is a Fano plane by part (i) of Observation 2.1 and Hall's theorem. Otherwise, the edge with multiplicity 2 is opposite an edge of multiplicity 4, so all the I_i have size at least 2. Also, it is easy to see that in this case there is always an edge with multiplicity ≥ 3 opposite an edge of multiplicity 4. Thus one of the I_i has size at least 3, and so Hall's condition always holds. This implies that H contains a Fano plane.

(iii) Suppose there is no Fano plane in H and G' contains a copy of K_5 with vertex set $y_1y_2y_3y_4y_5$. By part (i) all edges in this copy have multiplicity exactly 3. Also, by the proof of (i), for every K_4 subgraph of K_5 there are two link graphs that contain all the edges of K_4, and the other two link graphs partition the edge set such that neither contains two opposite edges, i.e. one is a star and the other is a triangle. Therefore, without loss of generality, we can assume that L_1, L_2 contain all six edges on $y_1y_2y_3y_4$, L_3 contains y_1y_2, y_1y_3, y_1y_4 and L_4 contains all the edges of triangle $y_2y_3y_4$. Note that the set $y_1y_3y_4$ is complete in the link graphs L_1 and L_2, spans the edges y_1y_3, y_1y_4 in L_3 and the edge y_3y_4 in L_4. Consider the set of vertices $y_1y_3y_4y_5$. This set cannot be complete in L_3 or L_4. Thus it must be complete in L_1 and L_2, span the star y_1y_3, y_1y_4, y_1y_5 in L_3 and the triangle $y_3y_4y_5$ in L_4. Similarly, considering the vertices $y_1y_2y_4y_5$ we see that they span a complete subgraph in L_1 and L_2, a star in L_3 and a triangle $y_2y_4y_5$ in L_4. This implies that the vertices $y_2y_3y_4y_5$ span a complete subgraph in L_1, L_2 and L_4. Then by part (i) of Observation 2.1, H contains a Fano plane. This contradiction completes the proof of (iii) and the proof of the lemma. ∎

Let $K^{(r)}(t_1,\ldots,t_r)$ be a complete r-partite r-uniform hypergraph with parts of size $t_1,\ldots,t_r$, whose edges are all possible r-tuples that contain one vertex from each part. The following well-known lemma first appears implicitly in Erdős [3] (see also, e.g., [4]).

Lemma 2.6. *Let H be a r-uniform hypergraph on n vertices with m edges. For every set of positive integers $t_1 \leq \cdots \leq t_r$, there exists a constant c such that if $m \gg n^{r-1/(t_1\cdots t_{r-1})}$ then H contains at least*

$$c\frac{m^{t_1\cdots t_r}}{n^{rt_1\cdots t_r - t_1 - \cdots - t_r}}$$

copies of $K^{(r)}(t_1,\ldots,t_r)$. In particular, for $r=3$ and $t_1=t_2=t_3=2$ there exists a constant c' such that any 3-uniform hypergraph on n vertices with αn^3 edges contains at least $c'\alpha^8 n^6$ copies of $K^{(3)}(2,2,2)$.

3. A stability result

In this section we prove that every 3-uniform hypergraph with edge density $3/4+o(1)$ which does not contain the Fano plane is essentially 2-colorable.

Proof of Theorem 1.2. Let H be a 3-uniform hypergraph of order n with at least $(1-\delta)\frac{3}{4}\binom{n}{3}$ edges and no Fano plane. Throughout the proof we assume that n is sufficiently large and δ is sufficiently small. We let $\delta_1, \delta_2, \ldots$ denote positive functions of δ that tend to zero as δ tends to zero and n tends to infinity. These functions could be explicitly computed, but we prefer not to do so for the sake of clarity of presentation. We will show that we can delete a set of vertices U of size at most $\delta_i n$ and partition the rest as $A \cup B$ such that $e_H(A) + e_H(B) < \delta_j n^3$, for some j. Then, clearly $e_H(A \cup U) + e_H(B) < (\delta_j + \delta_i)n^3 = \delta_k n^3$ and by choosing δ sufficiently small we can ensure that $\delta_k < \epsilon$.

First, suppose that H contains a vertex of degree less than $(1-\delta_1)\frac{3}{4}\binom{n}{2}$, for some δ_1 which we will define shortly. Then delete this vertex and continue. If in this process we deleted $\delta_2 n$ vertices, then we arrive at a hypergraph on $(1-\delta_2)n$ vertices with at least $\big(1-\delta-3\delta_2(1-\delta_1)\big)\frac{3}{4}\binom{n}{3} - O(n^2)$ edges. Choosing $\delta_1 = \delta^{1/4}$ and $\delta_2 = \delta^{1/2}$ we see that this is larger than $(1+\delta)\frac{3}{4}\binom{(1-\delta_2)n}{3}$. For n sufficiently large this contradicts the de Caen–Füredi result that the Turán density of the Fano plane is 3/4. Therefore we deleted at most $\delta_2 n$ vertices, which we can put in the set U and ignore. Thus we may and will assume in the rest of the proof that all degrees in H are at least $(1-\delta_1)\frac{3}{4}\binom{n}{2}$.

Since the density of H is at least $3/4 - \delta_1$, by Lemma 2.2 it contains a copy $S = abcd$ of $K_4^{(3)}$. Let $L(a)$, $L(b)$, $L(c)$ and $L(d)$ be the link graphs of the vertices in S. Then, by the degree condition, the link multigraph G of S has at least $|L(a)| + |L(b)| + |L(c)| + |L(d)| \geq (1-\delta_1)3\binom{n}{2}$ edges. The main part of our proof is to obtain a structural result for the multigraph G.

Claim. *There exists a partition of the vertices as $A \cup B$ such that all but at most $\delta_9 n^2$ edges of G satisfy the following conditions:*
(i) Every edge within A belongs to $L(a)$ and $L(b)$ but not $L(c)$ or $L(d)$.
(ii) Every edge within B belongs to $L(c)$ and $L(d)$ but not $L(a)$ or $L(b)$.
(iii) Every cross edge from A to B belongs to all of $L(a)$, $L(b)$, $L(c)$ and $L(d)$.

Proof of Claim. First, suppose that G contains a vertex of degree less than $(1-\delta_3)3n$, for some δ_3 which we will define shortly. Then delete this vertex and continue. If in this process we deleted $\delta_4 n$ vertices, then we arrive at a multigraph on $(1-\delta_4)n$ vertices with at least $\big(1-\delta_1-2\delta_4(1-\delta_3)\big)3\binom{n}{2} - O(n)$ edges. Choosing $\delta_3 = \delta_1^{1/4}$ and $\delta_4 = \delta_1^{1/2}$ we see that this this is larger than $(1+\delta_1)3\binom{(1-\delta_4)n}{2}$. Then by Lemma 2.3 we find 4 vertices spanning 21 edges, and by Lemma 2.4 this is a contradiction. Thus we deleted at most $\delta_4 n$ vertices, which can be incident with at most $4\delta_4 n^2$ edges of G, so we can ignore them. Therefore we may and will assume in the rest of the proof that all degrees in G are at least $(1-\delta_3)3n$.

Following the method of proof of Lemma 2.3, we distinguish two cases according to whether G contains a set of 3 vertices spanning ≥ 11 edges.

Case 1. We suppose first that every 3 vertices of G span at most 10 edges. Note that there must be some edge of multiplicity 4, since otherwise G contains at least $(1-3\delta_1)\binom{n}{2}$ edges of multiplicity 3. Then by Turán's theorem, for sufficiently small δ_1 ($<1/12$) these edges contain a K_5, which contradicts Lemma 2.5. Let edge pq have multiplicity 4. By our assumption for this case, any vertex r in $V-\{p,q\}$ has degree at most 6 in $\{p,q\}$. By the degree condition, there are at least $(1-\delta_3)6n$ edges from $\{p,q\}$ to $V-\{p,q\}$. Then only at most $\delta_5 n$ vertices can have degree less than 6 in $\{p,q\}$. These vertices are incident to at most $4\delta_5 n^2$ edges, so we can delete and ignore them. Therefore we can assume that $V-\{p,q\}$ can be partitioned as $A\cup B\cup C$, where for each x in A edge xp has multiplicity 4 and xq has multiplicity 2, for each x in B edge xp has multiplicity 2 and xq has multiplicity 4, and for each x in C edges xp and xq both have multiplicity 3.

All edges within A and B have multiplicity at most 2, or with either p or q they will form 3 vertices spanning 11 edges. So the maximum possible number of edges in $A\cup B$ is achieved when all edges within A and B have multiplicity 2 and all cross edges from A to B have multiplicity 4, the total being at most

$$2\binom{|A|}{2}+2\binom{|B|}{2}+4|A||B| \leq 3\binom{|A|+|B|}{2}+O(n)=3\binom{|A\cup B|}{2}+O(n).$$

Also, the edges from $A\cup B$ to C have weight at most 3. Indeed, if edge xy has multiplicity 4 with say, x in A and y in C, then pxy has 11 edges, a contradiction. Therefore

$$\begin{aligned} e(C) &= e(G)-e(A\cup B)-e(A\cup B,C) \\ &> (1-\delta_1)3\binom{n}{2}-3\binom{|A\cup B|}{2}-O(n)-3|A\cup B||C| \\ &= 3\binom{|C|}{2}-3\delta_1\binom{n}{2}-O(n). \end{aligned}$$

Let $\delta_6=2\delta_1^{1/2}$. If $|C|\geq\delta_6 n$ then $\binom{|C|}{2}\geq 2\delta_1 n^2-O(n)\gg 3\delta_1\binom{n}{2}$. This gives

$$e(C)>3\binom{|C|}{2}-3\delta_1\binom{n}{2}-O(n)\gg 2\binom{|C|}{2},$$

and therefore there is an edge xy within C of multiplicity ≥ 3. But then $pqxy$ form a K_4 which contradicts Lemma 2.5. Thus there are at most $\delta_6 n$

vertices in C which can be incident only to at most $4\delta_6 n^2$ edges, so we delete and ignore these vertices.

We will now show that $A \cup B$ is the required partition. Denote by E' the set of edges with multiplicity less than the maximum allowed by the above, i.e. pairs within A or B with multiplicity at most 1 and pairs with one vertex in each of A and B with multiplicity at most 3. Note that

$$\begin{aligned}(1-\delta_1)3\binom{n}{2} \le e(G) &\le 2\binom{|A|}{2} + 2\binom{|B|}{2} + 4|A||B| - |E'| \\ &\le 3\binom{|A|+|B|}{2} - \frac{(|A|-|B|)^2}{2} + O(n) - |E'| \\ &\le 3\binom{n}{2} + O(n) - |E'|\end{aligned}$$

and therefore $|E'| < 2\delta_1 n^2$ and also $||A| - n/2|$ and $||B| - n/2|$ are at most $\delta_7 n$.

For x in A, let $B(x)$ be the set of vertices in B joined to x by an edge of multiplicity 4. Since $|A|$ and $|B|$ are at most $(\frac{1}{2} + \delta_7)n$ we have

$$(1-\delta_3)3n < d(x) < 2|A| + 4|B| - (|B| - |B(x)|) < (1+2\delta_7)3n - (|B| - |B(x)|),$$

so $|B - B(x)| < \delta_8 n$. Now let xy be an edge of multiplicity 2 inside A. Without loss of generality assume that it belongs to $L_A(a) \cap L_A(b)$, where set subscripts indicate restriction of the link to that set. Since $|B| - |B(x) \cap B(y)| < 2\delta_8 n$ we can delete all the vertices in $B - B(x) \cap B(y)$ and assume $B = B(x) \cap B(y)$. Now no edge wz of B can be in $L_B(a)$, as then partitioning the edges of $wxyz$ as $M_a = \{xy, wz\}$, $M_b = \{wx, yz\}$, $M_c = \{wy, xz\}$ gives a Fano plane by part (i) of Observation 2.1. Similarly no edge in B can be in $L_B(b)$. Now let uv be an edge of multiplicity 2 in B. Then, as we just proved, it belongs to $L_B(c) \cap L_B(d)$. So arguing as above, we can assume that all vertices in A are adjacent to both u and v by edges of multiplicity 4, and therefore no edge in A can be in $L(c)$ or $L(d)$. Now distribute all the vertices we deleted arbitrarily between A and B. Then for all but at most $\delta_9 n^2$ edges of G, those in A belong to $L(a) \cap L(b)$, those in B belong to $L(c) \cap L(d)$ and those between A and B have multiplicity 4. This shows that $A \cup B$ is the required partition, and completes the analysis of this case.

Case 2. Now suppose that there are 3 vertices p, q, r in G that span at least 11 edges. Without loss of generality we can assume that pq and pr have multiplicity 4 and qr has multiplicity ≥ 3. By Lemma 2.4 each vertex s in $V(G) - \{p, q, r\}$ has degree at most 9 in $\{p, q, r\}$. By the degree condition, there are at least $d_G(p) + d_G(q) + d_G(r) - 12 \ge (1-\delta_3)9n - 12$ edges from

$\{p,q,r\}$ to $V-\{p,q,r\}$. Therefore at most $\delta_7 n$ vertices of G can have degree less than 9 in $\{p,q,r\}$. These vertices are incident to at most $4\delta_7 n^2$ edges, so we can delete and ignore them. Note that by Lemma 2.5 no vertex of $V(G)-\{p,q,r\}$ can be adjacent to vertices in $\{p,q,r\}$ by edges with multiplicities $3,3,3$ or $4,3,2$. Therefore the degree pattern in $\{p,q,r\}$ of all the vertices from $V-\{p,q,r\}$ is $(4,4,1)$ in some order. Now suppose there is some edge st in $V-\{p,q,r\}$ of multiplicity 4. Since the degree pattern of s and t in $\{p,q,r\}$ is $(4,4,1)$, there is some vertex x in $\{p,q,r\}$ to which both s and t are connected by an edge of multiplicity 4. If there are 2 such vertices in $\{p,q,r\}$ then together with s and t we find 4 vertices spanning at least 23 edges, a contradiction to Lemma 2.4. Thus there is exactly one such x. Now x certainly belongs to one of the edges pq or pr. Say x is in pq. Then all edges but one of $pqst$ have multiplicity 4, and the other has multiplicity at least 1, which again is a contradiction to Lemma 2.4. The same argument holds if x is in pr, so there are no edges in $V-\{p,q,r\}$ of multiplicity 4. On the other hand, by the degree condition $V-\{p,q,r\}$ contains at least $(1-\delta_3)3n^2/2-O(n)$ edges. Therefore at least $(1-\delta_8)\binom{n}{2}$ edges in $V-\{p,q,r\}$ have multiplicity 3. Since δ_8 is sufficiently small ($<1/4$), by Turán's theorem these edges contain a K_5, which contradicts Lemma 2.5. Therefore Case 2 always leads to a contradiction, and thus never happens. This completes the proof of the Claim. ∎

Now we can complete the proof of the theorem. Let $c'\delta_{10}^8=2\delta_9$, where c' is defined in Lemma 2.6, and note that δ_{10} tends to zero with δ_9. Suppose that A contains at least $\delta_{10}n^3$ edges of H. Then by the above mentioned Lemma 2.6, A contains at least $2\delta_9 n^6$ copies of $K^{(3)}(2,2,2)$. If we find 3 disjoint edges e_1,e_2,e_3 in $L(a)$ such that they form three classes of the partition of some copy of $K^{(3)}(2,2,2)$, then by part (ii) of Observation 2.1 we see that H contains a Fano plane, a contradiction. On the other hand, only at most $\delta_9 n^2$ pairs of vertices in A are not edges of $L(a)$. Therefore these pairs can be contained in the vertex sets of most $\delta_9 n^6$ copies of $K^{(3)}(2,2,2)$. In particular, if A contains at least $\delta_{10}n^3$ edges of H then there is a copy of $K^{(3)}(2,2,2)$ in A such that all three pairs in the partition of its vertex set belong to $L(a)$, and this is a contradiction. Hence A can contain at most $\delta_{10}n^3$ edges, and a similar bound holds for B. Then $e(A)+e(B)<2\delta_{10}n^3=\delta_{11}n^3$ and by choosing a sufficiently small δ we can make δ_{11} tend to zero with δ, as required. ∎

4. The Turán number of the Fano plane

In this section we show how to use the stability theorem to prove an exact Turán result for the Fano plane. Throughout the proof we will assume that n is sufficiently large.

Proof of Theorem 1.1. Let H be a 3-uniform hypergraph on n vertices, which has at least $e(H) \geq h_2(n) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$ edges and contains no Fano plane. Let $d(n) = 3n^2/8 - 2n$ and note that $h_2(n) - h_2(n-1) > d(n) + 1$. First we claim that we can assume that H has minimum degree greater than $d(n)$. If not, we remove a vertex of minimum degree to get H_{n-1} which has $e(H_{n-1}) \geq h_2(n) - d(n) \geq h_2(n-1) + 1$. Repeating this process if possible we obtain a sequence of hypergraphs H_m on m vertices with at least $h_2(m) + n - m$ edges, where H_m is obtained from H_{m+1} by deleting a vertex of degree at most $d(m+1)$. Clearly we cannot continue this process to reach a hypergraph on $n_0 = n^{1/3}$ vertices, as then $e(H_{n_0}) > n - n_0 > \binom{n_0}{3}$, which is impossible. Therefore, we must obtain a hypergraph $H_{n'}$, where $n \geq n' > n_0$, with minimal degree at least $d(n')$, and $e(H_{n'}) \geq h_2(n')$ (with strict inequality if $n > n'$). This shows that it suffices to prove the theorem under the assumption that H has minimum degree greater than $d(n)$.

By Theorem 1.2 there is a partition of the vertices of H into two disjoint sets $A \cup B$ so that $e(A) + e(B) < 10^{-9} n^3$. Choose this partition to minimize $e(A) + e(B)$. For every vertex $x \in V(H)$ denote by $L_A(x)$ and $L_B(x)$ the subgraphs of the link graph of x induced by the sets A and B. Then for every $x \in A$ we have $|L_A(x)| \leq |L_B(x)|$, since otherwise moving x to the other side of the partition will decrease $e(A) + e(B)$. Similarly, $|L_B(x)| \leq |L_A(x)|$ holds for every $x \in B$. Also note that the partition is nearly balanced, more precisely

$$\left| |A| - \frac{n}{2} \right| < 10^{-4} n, \qquad \left| |B| - \frac{n}{2} \right| < 10^{-4} n.$$

Indeed, if this is not the case, then the number of edges that intersect both A and B is at most

$$\begin{aligned} |A|\binom{|B|}{2} + |B|\binom{|A|}{2} &= |A||B|\frac{|A| + |B| - 2}{2} \\ &\leq \left(\frac{1}{2} - 10^{-4}\right)\left(\frac{1}{2} + 10^{-4}\right) n^2 \frac{n-2}{2} \\ &< \left(\frac{1}{4} - 10^{-8}\right)\frac{n^3}{2}. \end{aligned}$$

Then, the total number of edges in H is at most

$$e(H) < \left(\frac{1}{4} - 10^{-8}\right) n^3/2 + 10^{-9} n^3 < h_2(n),$$

which is a contradiction.

Next, suppose that there is a vertex $x \in A$ with $|L_A(x)| > n^2/100$. Then, by the above discussion, also $|L_B(x)| > n^2/100$. Note that any graph G

of order n with at least $n^2/100$ edges contains a matching of size at least $n/300$. To show this, observe that if M is a maximum matching in G, then $V(G)-M$ is an independent set. Hence, if M has less than $n/300$ edges then it spans at most $n/150$ vertices, and therefore G contains at most $\binom{n/150}{2}+n(n/150)<n^2/100$ edges.

Let M_A and M_B be matchings each having $n/300$ edges in $L_A(x)$ and $L_B(x)$ respectively. Since H contains no Fano plane, by part (ii) of Observation 2.1 we have that for every triple of edges e in M_A and f_1,f_2 in M_B, there is some triple of vertices of H with one point in each of e,f_1,f_2 that is not a hyperedge of H. Also, note that for different triples of edges we get different triples of vertices that are not hyperedges. This implies that at least $\frac{n}{300}\binom{n/300}{2}$ triples of vertices of H with one point in A and two points in B are not hyperedges. Then

$$\begin{aligned}
e(H) &= e(A)+e(B)+e(A,B)\\
&< 10^{-9}n^3+|A|\binom{|B|}{2}+|B|\binom{|A|}{2}-\frac{n}{300}\binom{n/300}{2}\\
&< 10^{-9}n^3+n^3/8-10^{-8}n^3<h_2(n),
\end{aligned}$$

again a contradiction.

The same argument works for all $x\in B$. Thus we can assume that for every vertex $x\in V(H)$ the link graph $L(x)$ has at most $n^2/100$ edges within the same class of the partition as x. Now suppose there is an edge xyz in A, the case when there is an edge in B can be treated similarly. By the minimum degree assumption on the vertices of H

$$\begin{aligned}
|L_B(x)| &> d_H(x)-|A||B|-n^2/100\\
&> 3n^2/8-2n-n^2/4-n^2/100 = n^2/8-2n-n^2/100.
\end{aligned}$$

The same inequality holds for $L_B(y)$ and $L_B(z)$. Since $|B|<(1/2+10^{-4})n$ we have $\binom{|B|}{2}<(1/8+10^{-4})n^2$, and so

$$\begin{aligned}
|L_B(x)\cap L_B(y)\cap L_B(z)| &\geq |L_B(x)|+|L_B(y)|+|L_B(z)|-2\binom{|B|}{2}\\
&> 3n^2/8-6n-3n^2/100-2\binom{|B|}{2} > \frac{2}{3}\binom{|B|}{2}.
\end{aligned}$$

Then by Turán's theorem $L_B(x)\cap L_B(y)\cap L_B(z)$ contains a copy of K_4, and therefore by part (i) of Observation 2.1, H contains the Fano plane. This contradiction shows that there are no edges within A or B and that H is 2-colorable. Since $H_2(n)$ is the largest 2-colorable 3-uniform hypergraph we have $e(H) \le e(H_2) = h_2(n)$. Therefore $e(H) = h_2(n)$ and $H = H_2(n)$ as required. ∎

5. Concluding remarks

The proof of Theorem 1.1 can be used to show that there is some small $\delta > 0$ so that any 3-uniform hypergraph on n vertices with minimum degree at least $\left(\frac{3}{8} - \delta\right) n^2$ that does not contain a copy of the Fano plane is 2-colorable.

Another observation is that the Fano plane is a 3-*color-critical* hypergraph, in that it is not 2-colorable, but becomes so on deletion of any edge. For color-critical graphs the Turán numbers are given by a result of Simonovits [9], which says that if $\mathcal{G}$ is a $(r+1)$-color-critical graph then for n sufficiently large, any $\mathcal{G}$-free graph on n vertices can contain at most $t_r(n)$ edges, and equality only occurs for $T_r(n)$. So it is natural to check wheather at least some partial generalization of this result to hypergraphs might be true. One might think that if $\mathcal{F}$ is a 3-color-critical 3-uniform hypergraph, then for n sufficiently large, any $\mathcal{F}$-free 3-uniform hypergraph on n vertices should contain at most $h_2(n)$ edges with equality only for $H_2(n)$. Unfortunately this is not the case, as shown by a construction of Sidorenko (see, e.g., [8]). He proved that $K_5^{(3)}$, which is 3-color-critical, does not have Turán number $h_2(n)$.

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