Addendum to *Shadows and intersections: stability and new proofs*

Peter Keevash, January 2010

I list here various corrections, comments, further developments and open problems.

1. The calculations at the end of the proof of Theorem 4 are incorrect: I accidently ignored a factor of n^3 in the error term! However, this can be eliminated by other arguments, and the point of this section is not the actual bounds, which are weak, but the idea of the proof and the possibility of applying a similar method for a range of extremal problems.

2. (i) The proof of Lemma 6 is not correct as written, because the assumption $r < n/2$ is not preserved by the induction, which reduces the case (n, r, s) to the cases $(n-1, r-1, s)$ and $(n-1, r, s-1)$. This can be easily remedied by instead using the assumption $n > 2r + s$, which is preserved by the induction. This does not affect the rest of the paper, as I only use the lemma when r, s are fixed and n is very large.

(ii) The proof of Lemma 6 can be improved by noting that there is no need to use Gottlieb's theorem: the case $r = s$ is trivial, as then $G = K_n^r$ and $M_r^r(G)$ is the identity matrix, which has full rank.

(iii) An open problem: is Lemma 6 true under the assumption $|F| < \binom{n-s}{r-s}$? This would be best possible, because of the example of all r-sets containing some fixed s-set. Even if there is a small counterexample, is it at least true for r, s fixed and n large?

3. Bollobás has remarked on some similarities between my proof of Theorem 1 and the argument given in Lovász's *Combinatorial Problems and Exercises* (which he says was somewhat 'folklore' at the time). Both arguments proceed by induction and analyse vertex degrees in the hypergraph. However, Lovász considers a vertex of minimum degree and employs a technical lemma on binomial coefficients, whereas I average over the vertices and find that the bound drops out with very little calculation. My approach also has the advantage of leading to a stability version. Actually, it seems plausible that stability can also be obtained from the Lovász argument, although the calculational difficulties seem formidable (they would be somewhat similar to the technical Lemma 8 in my paper).

4. Chowdhury and Patkós have obtained a vector space analogue of the Lovász form of Kruskal-Katona (Theorem 1). Besides other ideas, they employ the averaging approach (see remark 3), and it seems that a minimum degree approach would not work. It is a long-standing open problem to obtain a vector space analogue of (the exact form of) Kruskal-Katona.

5. Samorodnitsky has used stability for Kruskal-Katona to obtain stability for the edge isoperimetric inequality in the cube. It is an open problem to obtain stability for the vertex isoperimetric inequality.

6. It is an open problem is to determine the correct dependence of the parameters in the stability version of the Kruskal-Katona theorem. One might initially hope for the parameters δ and ϵ (which measure proximity in size and structure to the extremum) to be constants independent of the uniformity. However, this is not the case! Bukh has a class of examples showing that very different structures can appear for this range of parameters. For example, fix a number t and consider the family $\mathcal{F}$ of subsets A of size $n/2$ of the numbers $1..n$ such that the number of elements of A in the range $1...n/2$ lies in $1...t/2 \bmod t$. One can verify that $\mathcal{F}$ is within $1 - O(1/t)$ of optimal, but far from being a clique. O'Donnell and Wimmer have established some structure for families in which $\delta = O\left(\frac{\log n}{n}\right)$: there must be some vertex i of $1..n$ which is 'negatively correlated' with the family. This means that the proportion of sets not containing i exceeds the proportion containing i. The negative correlation (difference between these proportions) they obtain is n^{-c}, for any $c > 0$. This information is rather weak when compared to the correlations of vertices in the extremal families (cliques), where some vertices are in no sets. However, I think it makes it plausible that more substantial structure should start to appear when δ is roughly of order $1/n$, as opposed to the $1/n^2$ assumption that I make in my paper.