



Shadows and intersections: Stability and new proofs [☆]

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Abstract

We give a short new proof of a version of the Kruskal–Katona theorem due to Lovász. Our method can be extended to a stability result, describing the approximate structure of configurations that are close to being extremal, which answers a question of Mubayi. This in turn leads to another combinatorial proof of a stability theorem for intersecting families, which was originally obtained by Friedgut using spectral techniques and then sharpened by Keevash and Mubayi by means of a purely combinatorial result of Frankl. We also give an algebraic perspective on these problems, giving yet another proof of intersection stability that relies on expansion of a certain Cayley graph of the symmetric group, and an algebraic generalisation of Lovász’s theorem that answers a question of Frankl and Tokushige.

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1. Introduction

The Kruskal–Katona theorem [19,22] is a classical result in Extremal Combinatorics that gives a tight lower bound on the size of the shadow of a k -graph.¹ It states that $|\partial G| \geq |\partial G_0|$, where G_0 is the initial segment of length $|G|$ in the colexicographic order² on k -tuples of some ordered set. The quantitative form of this statement is a bit technical, and it is often

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¹ A k -graph G consists of a vertex set $V(G)$ and an edge set $E(G)$, each edge being some k -tuple of vertices. Its shadow ∂G is the $(k-1)$ -graph consisting of all $(k-1)$ -tuples that are contained in some edge of G . We write $|G| = |E(G)|$ for the number of edges in G .

² If $(X, <)$ is an ordered set we order subsets of X by $A < B$ iff the largest element of $(A \cup B) \setminus (A \cap B)$ lies in B .

more convenient to use the following slightly weaker form due to Lovász [24, Ex. 13.31(b)]: if $|G| = \binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$ for some real number $x \geq k$ then $|\partial G| \geq \binom{x}{k-1}$. He also showed that equality occurs if and only if x is an integer and $G = K_x^k$ is the complete k -graph on x vertices.

This result has many consequences in Extremal Combinatorics (see [12]). Also, its isoperimetric nature leads to broader applications, such as the proof of the existence of threshold functions for monotone properties by Bollobás and Thomason [5]. It can also be interpreted as giving an upper bound on the number of copies of K_{r+1}^r in an r -graph G in terms of $|G|$ (setting $r = k - 1$). The general question of estimating the number of copies of one hypergraph in another was studied in [1] and [14]. The latter paper gives two general bounds, one using Shearer's entropy inequality and another using the Bonami–Beckner hypercontractive estimate. These bounds give the correct order of magnitude in many interesting cases, but fall short of giving the correct constant of proportionality for complete r -graphs.

There are many known proofs of the Kruskal–Katona theorem (see [7,10,16,24]) relying on compression techniques and/or induction arguments. We start by giving a new proof (not using either of these methods) of an upper bound on $K_{r+1}^r(G)$, the number of copies of K_{r+1}^r in an r -graph G , in terms of $|G|$. This can be easily translated into Lovász's result by noting that if G is a k -graph then $G \subset K_{k-1}^k(\partial G)$. Our proof has the advantages that it is very simple, and the idea can be used to obtain certain structural information not available with other arguments.

Theorem 1. (See Lovász [24].) *Suppose $r \geq 1$ and G is an r -graph with $\binom{x}{r}$ edges, for some real number $x \geq r$. Then $K_{r+1}^r(G) \leq \binom{x}{r+1}$, with equality if and only if x is an integer and $G = K_x^r$.*

Building on the idea in our proof of Theorem 1, we can describe the approximate structure of an r -graph G that is close to being extremal. We show that shadows have ‘stability,’ a phenomenon which was originally discovered by Erdős and Simonovits in the 60s in the context of graphs with excluded subgraphs, but has only been systematically explored relatively recently, as researchers have realised the importance and applications of such results in hypergraph Turán theory, enumeration of discrete and extremal set theory (see [20] as an example and for further references). Answering a question of Mubayi (personal communication) we prove the following stability version of the Kruskal–Katona theorem.

Theorem 2. *For any $\epsilon > 0$ and $r \geq 1$ there is $\delta > 0$ so that if G is an r -graph with $\binom{x}{r}$ edges and $K_{r+1}^r(G) > (1 - \delta)\binom{x}{r+1}$ then there is a set S of $\lceil x \rceil$ vertices so that all but at most $\epsilon\binom{x}{r}$ edges of G are contained in S .*

In fact, we can obtain further structural information and quantify the dependence of δ on r and ϵ to sufficient precision to deduce a stability theorem for intersecting r -graphs. An r -graph is said to be intersecting if every two of its edges have at least one common vertex. A classical theorem of Erdős, Ko and Rado [9] states that an intersecting r -graph G on $n \geq 2r$ vertices³ has at most $\binom{n-1}{r-1}$ edges, and for $n > 2r$ equality holds only when there is some vertex x that belongs to every edge of G . Using spectral techniques, Friedgut [13] obtained a stability version, namely that, given $\zeta > 0$ there is $c > 0$ so that if $\zeta n < r < (1/2 - \zeta)n$ and G is an intersecting r -graph on n vertices with $|G| > (1 - \delta)\binom{n-1}{r-1}$, for some $\delta > 0$, then there is some vertex x

³ The case $n < 2r$ is trivial, as then K_n^r is intersecting.

that belongs to all but at most $c\delta \binom{n}{r}$ edges of G . The assumption that $r > \zeta n$ was eliminated by Dinur and Friedgut [8]. With $r = n/2 - t$ and $0 < t = o(n)$ one needs a lower bound of $|G| > (1 - O(t/n)) \binom{n-1}{r-1}$ for a stability result to hold, and such a result was obtained by Keevash and Mubayi [20] using a purely combinatorial result of Frankl [11]. Frankl’s argument relies heavily on compression techniques, but our methods give a direct proof of the following theorem, which although weaker than that in [20] gives structural information for all $r < n/2$.

Theorem 3. *Suppose $1 \leq r < n/2$, $\delta < 10^{-3}n^{-4}$ and G is an intersecting r -graph on n vertices with $|G| > (1 - \delta) \binom{n-1}{r-1}$. Then there is some vertex v so that all but at most $25n\delta^{1/2} \binom{n-1}{r-1}$ edges of G contain v .*

Next we take an algebraic perspective on the problem and give yet another proof of stability, this time using expansion of a certain Cayley graph of the symmetric group S_{n-1} .⁴ Here we need to assume a stronger lower bound on $|G|$, but the method seems interesting in its own right, and has potential applications to other problems.

Theorem 4. *Suppose $1 \leq r < n/2$, $\delta < \frac{1}{2rn^4}$ and G is an intersecting r -graph on n vertices with $|G| > (1 - \delta) \binom{n-1}{r-1}$. Then there is some vertex v so that all but at most $\delta r \binom{n-1}{r-1}$ edges of G contain v .*

Given an r -graph G there are some naturally associated algebraic objects called (higher) inclusion matrices. For $s \leq r$ we define $M_s^r(G)$ as a $\{0, 1\}$ matrix with rows indexed by edges of G and columns indexed by subsets of $V(G)$ of size s : the entry corresponding to an edge e and a set S is 1 if $S \subset e$ and 0 otherwise. Frankl and Tokushige [12] posed the problem of determining the minimum rank $\text{rk } M_s^r(G)$ of $M_s^r(G)$ in terms of $|G|$. We obtain the following result.

Theorem 5. *For every $r \geq s \geq 0$ there is a number $n_{r,s}$ so that if G is an r -graph with $|G| = \binom{x}{r} \geq n_{r,s}$ then $\text{rk } M_s^r(G) \geq \binom{x}{s}$. If $r > s > 0$ then equality holds only if x is an integer and $G = K_x^r$.*

Note that this generalises the result of Lovász, and also its iterated version, i.e. that if G is an r -graph, $|G| = \binom{x}{r}$ and $s \leq r$ then $|\partial_s^r G| \geq \binom{x}{s}$, where the s -shadow $\partial_s^r G$ consists of all s -sets that are contained in some edge of G . This is immediate from Theorem 5 (for large x) since the rank of $M_s^r(G)$ is at most the number of non-zero columns, which is the size of the s -shadow. Keevash and Sudakov [21] obtained a non-uniform version of this inequality, and our proof uses elements of that approach, but requires a number of new ideas. We highlight one lemma that we think is of independent interest, as it expresses a certain rigidity property of the complete inclusion matrices $M_s^r(K_n^r)$.

Lemma 6. *Suppose $0 \leq s \leq r < n/2$ and $G = K_n^r \setminus F$ is an r -graph on $[n]$ with $|F| < \binom{r}{s}^{-1} \binom{n}{r-s}$. Then $\text{rk } M_s^r(G) = \binom{n}{s}$.*

The rest of this paper is organised as follows. The next section gives a very short proof of Theorem 1. In Section 3 we collect some facts about binomial coefficients and other inequalities

⁴ There is no similarity with the methods in [13] and [8] which use Fourier analysis on \mathbb{Z}_2^n .

that will be subsequently useful. In Section 4 we extend the ideas from our proof of Theorem 1 to prove a generalised form of Theorem 2. This is then combined with an idea of Daykin in the following section to obtain our first proof of stability for intersecting families. Section 6 contains our second proof, based on expansion in the symmetric group. In Section 7 we prove our bound on the rank of inclusion matrices, Theorem 5, and the final section contains some concluding remarks.

Notation. We write $[n] = \{1, \dots, n\}$. Suppose G is an r -graph. Let $K_{r+1}^r(G)$ be the number of copies of K_{r+1}^r in G . For a vertex $v \in V(G)$ let $K_{r+1}^r(v)$ be the number of K_{r+1}^r 's that contain v . The link $(r - 1)$ -graph is $L(v) = \{A \subset V(G) : |A| = r - 1, A \cup \{v\} \in E(G)\}$. The degree $d(v) = |L(v)|$ is the number of edges containing v .

2. Proof of Theorem 1

We argue by induction on r . The base case $r = 1$ is trivial. We can assume that the degree $d(v)$ is non-zero for every vertex v . Note that $S \cup \{v\}$ spans a K_{r+1}^r in G if and only if S is an edge of G and spans a K_r^{r-1} in the link $L(v)$. The first condition gives the estimate $K_{r+1}^r(v) \leq |G| - d(v)$ and the second $K_{r+1}^r(v) \leq K_r^{r-1}(L(v))$. We claim that $K_{r+1}^r(v) \leq (x/r - 1)d(v)$ for every v , and equality is only possible when $d(v) = \binom{x-1}{r-1}$. To see this, suppose first that $d(v) \geq \binom{x-1}{r-1}$. Then by the first condition it suffices to observe that $\binom{x}{r} - d(v) \leq (x/r - 1)d(v)$. On the other hand, if $d(v) \leq \binom{x-1}{r-1}$ then define the real number $x_v \geq r$ by $d(v) = \binom{x_v-1}{r-1}$. Then by induction hypothesis $K_r^{r-1}(L(v)) \leq \binom{x_v-1}{r-1} = (x_v/r - 1)d(v) \leq (x/r - 1)d(v)$. The equality conditions are clear, so the claim holds in either case. Now

$$\begin{aligned} (r + 1)K_{r+1}^r(G) &= \sum_v K_{r+1}^r(v) \leq (x/r - 1) \sum_v d(v) \\ &= (x/r - 1)r|G| = (x - r) \binom{x}{r} = (r + 1) \binom{x}{r + 1}. \end{aligned}$$

Therefore $K_{r+1}^r(G) \leq \binom{x}{r+1}$, as required. Equality holds only when all vertices have degree $\binom{x-1}{r-1}$. Then if G has n vertices we have $n \binom{x-1}{r-1} = \sum_v d(v) = r \binom{x}{r}$, so $n = x$ and $G = K_x^r$.

3. Technical estimates

We pause to collect some technical estimates that will be helpful in the following sections. The first two concern binomial coefficients, and we will prove them in Appendix A to the paper. The others are straightforward, so we omit the proofs. We consider the binomial coefficient $\binom{x}{r}$ to be the polynomial $x(x - 1) \cdots (x - r + 1)/r!$ defined for every real number x . It is positive and increasing for $x > r - 1$.

Lemma 7. *If $x > y \geq r - 1$ then $(x - y) \binom{y-1}{r-1} < \binom{x}{r} - \binom{y}{r} < (x - y) \binom{x}{r-1}$.*

Lemma 8. *Suppose $r \geq 2, 1 \leq s \leq r - 1, \binom{u}{r} = \binom{v}{r} + \binom{w}{r-1}$ with $1 \leq \binom{w}{r-1} < \binom{u-1}{r-1} - \frac{1}{2r!}u^{r-s-1}$, and $u > u_0(r, s)$ is sufficiently large. Then $\binom{u}{s} < \binom{v}{s} + \binom{w}{s-1} - (3r)^{-r}u^{-1}$.*

There follow some assorted easy facts. Throughout n is a natural number, other parameters are real.

$$\text{If } 0 < \theta < 1 \text{ and } \theta x > n \text{ then } \binom{\theta x}{n} < \theta^n \binom{x}{n}. \tag{1}$$

$$\text{If } a > b > 0 \text{ then } \left(\frac{a}{b}\right)^n < \binom{a}{n} / \binom{b}{n} = \prod_{i=0}^{n-1} \frac{a-i}{b-i} < \left(\frac{a-n+1}{b-n+1}\right)^n. \tag{2}$$

$$\text{If } 0 < \theta < 2/3n \text{ then } (1 + \theta)^n < 1 + 2n\theta. \tag{3}$$

$$\text{If } 0 < \theta < 1/2n \text{ then } (1 + \theta)^n < 2. \tag{4}$$

$$\text{If } 0 < \theta < 1/2 \text{ then } (1 - \theta)^{1/n} > 1 - 2\theta/n. \tag{5}$$

4. Stability for shadows

Building on the idea in our proof of Theorem 1, we can describe the approximate structure of an r -graph G that is close to being extremal. Answering a question of Mubayi (personal communication) we obtain a quantitative stability version of the Kruskal–Katona theorem: statement (1) in the following theorem.

Theorem 9. *Suppose $0 < \epsilon < 1/2$, $r \geq 2$, $x \geq (1 + \epsilon)(r + 1)$, $\delta < (\epsilon/6r)^2$ and G is an r -graph with $\binom{x}{r}$ edges and $K_{r+1}^r(G) > (1 - \delta)\binom{x}{r+1}$. Then:*

- (1) *There is a set S of $\lceil x \rceil$ vertices so that all but at most $10r(\epsilon^{-1} + 1)\delta^{1/2}\binom{x}{r}$ edges of G are contained in S .*
- (2) *There are at most $\delta^{1/2}x$ vertices v with $d(v) > (1 + \delta^{1/2})\binom{x-1}{r-1}$.*
- (3) *The vertices of degree less than $\binom{(1-\delta^{1/2})(x-1)}{r-1}$ are incident to at most $\delta^{1/2}x\binom{x-1}{r-1}$ edges.*
- (4) *There is a set C of size $|C| < (1 + 3r\delta^{1/2})x$ that contains at least $\binom{(1-4\delta^{1/2})x}{r+1}$ copies of K_{r+1}^r .*

Proof. Note that our assumption $K_{r+1}^r(G) > 0$ implies that $x \geq r + 1$. For each vertex v we recall the bounds $K_{r+1}^r(v) \leq \binom{x}{r} - d(v)$, and $K_{r+1}^r(v) \leq (x/r - 1)d(v)$ proved above, and the bound $K_{r+1}^r(v) \leq \binom{x-1}{r-1}$, which follows by combining the first two bounds. Also, if $d(v) = \binom{x_v-1}{r-1}$ we recall that $K_{r+1}^r(v) \leq (x_v/r - 1)d(v)$. Let

$$A = \left\{ v: d(v) > \binom{x-1}{r-1} \right\}, \quad A_0 = \left\{ v: d(v) > (1 + \delta^{1/2})\binom{x-1}{r-1} \right\},$$

$$B = V(G) \setminus A = \left\{ v: d(v) \leq \binom{x-1}{r-1} \right\}, \quad \text{and} \quad B_0 = \left\{ v: d(v) < \binom{x-1-y}{r-1} \right\},$$

where $y = \delta^{1/2}(x - r)$. For a set S write $d_S = \sum_{v \in S} d(v)$. We have

$$\begin{aligned}
 (1 - \delta)(r + 1) \binom{x}{r + 1} &< (r + 1)K_{r+1}^r(G) = \sum_v K_{r+1}^r(v) = \sum_{v \in A} K_{r+1}^r(v) + \sum_{v \in B} K_{r+1}^r(v) \\
 &\leq |A| \binom{x}{r} - d_A + (x/r - 1)d_B \\
 &= |A| \binom{x}{r} + (x/r - 1)(d_A + d_B) - (x/r)d_A
 \end{aligned}$$

and $(x/r - 1)(d_A + d_B) = (x - r) \binom{x}{r} = (r + 1) \binom{x}{r+1}$, so $|A| > d_A \binom{x-1}{r-1}^{-1} - \delta(x - r)$. Now

$$|A_0| \delta^{1/2} \binom{x-1}{r-1} + |A| \binom{x-1}{r-1} < d_A < \binom{x-1}{r-1} (|A| + \delta(x - r)),$$

so $|A_0| < \delta^{1/2}(x - r)$. This implies (2). We also deduce

$$\sum_{v \in A_0} K_{r+1}^r(v) \leq |A_0| \binom{x-1}{r} < \delta^{1/2}(x - r) \binom{x-1}{r} < \delta^{1/2}(r + 1) \binom{x}{r+1}. \tag{6}$$

Next we have

$$\begin{aligned}
 (1 - \delta)(r + 1) \binom{x}{r + 1} &< (r + 1)K_{r+1}^r(G) = \sum_v K_{r+1}^r(v) \leq (x/r - 1)d_A + \sum_{v \in B} (x_v/r - 1)d(v) \\
 &< (x/r - 1)(d_A + d_B) - (y/r)d_{B_0} = (r + 1) \binom{x}{r + 1} - (y/r)d_{B_0},
 \end{aligned}$$

so $d_{B_0} < \delta^{1/2}r \binom{x}{r}$. This implies (3). We also deduce

$$\sum_{v \in B_0} K_{r+1}^r(v) \leq (x/r - 1)d_{B_0} < \delta^{1/2}(r + 1) \binom{x}{r+1}. \tag{7}$$

Define an $(r + 1)$ -graph H on the same vertex set of G where an $(r + 1)$ -tuple is an edge exactly when it spans a K_{r+1}^r in G . Let $C = V(G) \setminus (A_0 \cup B_0)$, $H_0 \subset H$ consist of all $(r + 1)$ -tuples of H that are contained in C and $H_1 = H \setminus H_0$. Using Eqs. (6) and (7) we have

$$\begin{aligned}
 (1 - 3\delta^{1/2})(r + 1) \binom{x}{r + 1} &< \sum_{v \in C} K_{r+1}^r(v) = \sum_{v \in C} d_H(v) < (r + 1)|H_0| + r|H_1| \\
 &= (r + 1)|E(H)| - |H_1| < (r + 1) \binom{x}{r + 1} - |H_1|,
 \end{aligned}$$

where in the last step we use Theorem 1. Therefore $|H_1| < 3\delta^{1/2}(r + 1) \binom{x}{r+1}$ and

$$\begin{aligned}
 |H_0| &> (r + 1)^{-1} \left((1 - 3\delta^{1/2})(r + 1) \binom{x}{r + 1} - r|H_1| \right) \\
 &> (1 - 3(r + 1)\delta^{1/2}) \binom{x}{r + 1} > \binom{(1 - 4\delta^{1/2})x}{r + 1},
 \end{aligned} \tag{8}$$

where in the last step we use fact (1) to obtain the inequality

$$\binom{x}{r+1}^{-1} \binom{(1-4\delta^{1/2})x}{r+1} < (1-4\delta^{1/2})^{r+1} < 1 - (r+1) \cdot 4\delta^{1/2} + \binom{r+1}{2} (4\delta^{1/2})^2 < 1 - 3(r+1)\delta^{1/2},$$

which is valid since $\delta < (\epsilon/6r)^2 < 1/64r^2$. Now we can apply Theorem 1 to H_0 to deduce that at least $\binom{(1-4\delta^{1/2})x}{r}$ edges of G are contained in C .

Next, since $r \binom{x}{r} > \sum_{v \in C} d(v) \geq |C| \binom{x-1-y}{r-1}$ by fact (2) we have

$$|C|/x \leq \binom{x-1}{r-1} \binom{x-y-1}{r-1}^{-1} < \left(\frac{x-r+1}{x-y-r+1} \right)^{r-1} < 1 + \frac{2(r-1)\delta^{1/2}}{1-\delta^{1/2}},$$

where we use fact (3). Thus we can write $|C| = x + t$ with $t/x < 2(r-1)\delta^{1/2}(1-\delta^{1/2})^{-1} < 3r\delta^{1/2}$, which proves (4).

Choose any set $S \subset V(G)$ with $|S| = \lceil x \rceil$ so that either $S \subset C$ if $|C| \geq \lceil x \rceil$ or $S \supset C$ if $|C| \leq \lceil x \rceil$. We can estimate the number of edges of G that are not contained in S as follows. Either such an edge is not contained in C , of which there are at most

$$\begin{aligned} |G| - |H_0| &< \binom{x}{r} - \binom{(1-4\delta^{1/2})x}{r} < 4\delta^{1/2}x \binom{x}{r-1} \\ &= \frac{4r\delta^{1/2}}{1-(r+1)/x} \binom{x}{r} < 4(\epsilon^{-1} + 1)r\delta^{1/2} \binom{x}{r}, \end{aligned}$$

or it is contained in C but not in S , of which there are at most

$$\begin{aligned} \binom{|C|}{r} - \binom{|S|}{r} &< t \binom{x+t}{r-1} < t \left(\frac{x+t-r+1}{x-r+1} \right)^{r-1} \binom{x}{r-1} \\ &= \frac{rt}{x-r+1} \left(1 + \frac{t}{x-r+1} \right)^{r-1} \binom{x}{r} < 6r(\epsilon^{-1} + 1)\delta^{1/2} \binom{x}{r}. \end{aligned}$$

(In both estimates we use Lemma 7. In the last step we use the estimate $\frac{t}{x-r+1} < \frac{3r\delta^{1/2}}{\epsilon(r+1)} < \frac{3r \cdot \epsilon/6r}{\epsilon(r+1)} < 1/2(r-1)$ and so by fact (4) we have $(1 + \frac{t}{x-r+1})^{r-1} < 2$.) In total we have at most $10r(\epsilon^{-1} + 1)\delta^{1/2} \binom{x}{r}$ edges of G not contained in S , as required. \square

Remark. We have tried to give good estimates in this proof so that we obtain stability results for a large range of r and x , but some price has been paid for obtaining a universal bound, and improvements can be made for particular values of the parameters. The bounds get worse for smaller x to the point where we lose an exponential factor in r if $x = r + c$ and $c \ll r$. The proof breaks down as x approaches $r + 1$, but in this range the weak Kruskal–Katona bound compares poorly to the full theorem, and in any case it is not too hard to analyse the situation by ad hoc methods.

5. Stability for intersecting families, I

Next we show how to derive a stability result for intersecting families. The proof involves combining the methods above with an idea of Daykin [6]. First we remark that if $r \leq l \leq m$ and G is an r -graph with $K_l^r(G) = \binom{x}{l}$ then $K_m^r(G) \leq \binom{x}{m}$. This follows by repeatedly applying Theorem 1 and noting that a set M of size m spans a K_m^r in G exactly when it spans a K_m^{m-1} in the $(m - 1)$ -graph of all copies of K_{m-1}^r in G .

Now we prove Theorem 3, which is as follows: Suppose $1 \leq r < n/2$, $\delta < 10^{-3}n^{-4}$ and G is an intersecting r -graph on n vertices with $|G| > (1 - \delta)\binom{n-1}{r-1}$. Then there is some vertex v so that all but at most $25n\delta^{1/2}\binom{n-1}{r-1}$ edges of G contain v .

Proof of Theorem 3. Consider the complementary r -graph $H = \{A \subset V(G) : |A| = r, A \notin E(G)\}$ and the $(n - r)$ -graph of complements $J = \{A \subset V(G) : V(G) \setminus A \in E(G)\}$. Write $|H| = \binom{n-\theta}{r}$. By the Erdős–Ko–Rado theorem we have $|G| \leq \binom{n-1}{r-1}$, so $|H| \geq \binom{n-1}{r-1}$, i.e. $0 \leq \theta \leq 1$. Write $\phi = 1 - \theta$. By Lemma 7 we have $\binom{n-\theta}{r} - \binom{n-1}{r} > \phi\binom{n-2}{r-1}$, so

$$\binom{n-1}{r} + \phi\binom{n-2}{r-1} < |H| = \binom{n}{r} - |G| < \binom{n-1}{r} + \delta\binom{n-1}{r-1},$$

and $\phi < \delta\frac{n-1}{n-r} < 2\delta$ (since $r < n/2$). The condition that G is intersecting may be rephrased as saying that every edge of J spans a K_{n-r}^r in H . Therefore

$$\begin{aligned} K_{n-r}^r(H) &\geq |J| = \binom{n}{r} - \binom{n-\theta}{r} = \left(\binom{n}{r} - \binom{n-1}{r}\right) - \left(\binom{n-\theta}{r} - \binom{n-1}{r}\right) \\ &> \binom{n-1}{r-1} - \phi\binom{n}{r-1} = \left(1 - \frac{n}{n-r+1}\phi\right)\binom{n-1}{n-r} \\ &> \left(1 - \frac{n}{n-r+1}\phi\right)\left(\binom{n-\theta}{n-r} - \phi\binom{n-\theta}{n-r-1}\right) \\ &= \left(1 - \frac{n}{n-r+1}\phi\right)\left(1 - \frac{n-r}{r+\phi}\phi\right)\binom{n-\theta}{n-r} \\ &> \left(1 - \frac{n}{n-r+1}\phi - \frac{n-r}{r+\phi}\phi\right)\binom{n-\theta}{n-r} \\ &> (1 - 4(n/r - 1)\phi)\binom{n-\theta}{n-r}. \end{aligned}$$

Write $c = 1 - 4(n/r - 1)\phi$. Then there must be some m with $r \leq m < n - r$ for which

$$K_m^r(H) \leq c\frac{m-r}{n-2r}\binom{n-\theta}{m} \quad \text{and} \quad K_{m+1}^r(H) \geq c\frac{m+1-r}{n-2r}\binom{n-\theta}{m+1}.$$

Write $K_m^r(H) = \binom{n-\psi}{m}$, where $\psi \geq \theta$ by the remark before the proof. Also, by the same remark we have

$$\binom{n-\psi}{n-r} \geq K_{n-r}^r(H) > c\binom{n-\theta}{n-r} > \binom{n-2}{n-r},$$

as $c\binom{n-\theta}{n-r} - \binom{n-2}{n-r} > c\binom{n-1}{n-r} - \binom{n-2}{n-r} = (1 - 2\phi\frac{n}{r+1} - \frac{r-1}{n-1}) > 0$, since $\phi < 2\delta < 2^{-9}n^{-3}$. This gives $\psi < 2$. Now we have

$$\begin{aligned} K_{m+1}^r(H) &\geq c\frac{m+1-r}{n-2r}\binom{n-\theta}{m+1} = c\frac{m-r}{n-2r}\binom{n-\theta}{m} \cdot c\frac{1}{n-2r}\frac{n-\theta-m}{m+1} \geq K_m^r(H)c\frac{1}{n-2r}\frac{n-\theta-m}{m+1} \\ &= c\frac{1}{n-2r}\frac{n-\theta-m}{n-\psi-m}\binom{n-\psi}{m+1} \geq \left(1 - \frac{8(n/r-1)\phi}{n-2r}\right)\binom{n-\psi}{m+1} \\ &> (1 - 25\delta)\binom{n-\psi}{m+1}, \end{aligned}$$

where we apply fact (5) in the penultimate inequality and then estimate $\frac{8(n/r-1)\phi}{n-2r} = 8\phi(r^{-1} + (n-2r)^{-1}) < 10\phi < 25\delta$.⁵ By part (4) of Theorem 9, we can find a set C with $|C| = n - \psi + t$ and $t < 3m(25\delta)^{1/2}(n - \psi) = 15\delta^{1/2}m(n - \psi)$ so that H has at least $\binom{(1-20\delta^{1/2})(n-\psi)}{m+1}$ copies of K_{m+1}^r contained in C . Note that $|C| < n - \psi + 15mn\delta^{1/2} < n$, since $15mn\delta^{1/2} < 1/2 < 1 - 2\delta < 1 - \phi = \theta \leq \psi$. By arbitrarily adding vertices if necessary we may assume that $|C| = n - 1$.

By Theorem 1 there are at least $\binom{(1-20\delta^{1/2})(n-\psi)}{r}$ edges of H contained in C . Write $\{v\} = V(G) \setminus C$. Then the number of edges of H containing v is at most $Q = \binom{n-\theta}{r} - \binom{(1-20\delta^{1/2})(n-\psi)}{r}$. Since G is the complement of H , the number of edges of G containing v is at least $\binom{n-1}{r-1} - Q$, and so by Erdős–Ko–Rado the number of edges of G not containing v is at most

$$\begin{aligned} |G| - \left(\binom{n-1}{r-1} - Q\right) &\leq Q < (\phi + 20\delta^{1/2}n)\binom{n-\theta}{r-1} \\ &< (\phi + 20\delta^{1/2}n)\left(\binom{n-1}{r-1} + \phi\binom{n-\theta}{r-2}\right) \\ &< (\phi + 20\delta^{1/2}n)\left(1 + \phi\frac{n(r-1)}{(n-r+2)(n-r+1)}\right)\binom{n-1}{r-1} \\ &= \epsilon\binom{n-1}{r-1}, \end{aligned}$$

where $\epsilon < (2\delta + 20\delta^{1/2}n)(1 + 4\delta) < 25\delta^{1/2}n$. This completes the proof. \square

6. Stability for intersecting families, II

Now we give another argument using expansion properties of the symmetric group. We need to assume that G is closer to the maximum, but then the bound on bad edges improves. Also, we think that the method is interesting in itself, as it may apply to a much wider class of problems.

Our approach is based on Katona’s permutation method. We write a permutation $\sigma \in S_n$ as a sequence $(\sigma(1), \dots, \sigma(n))$. Say that σ and τ are cyclically equivalent if there is some $i \in [n]$ such that $\tau(x) = \sigma(x + i)$ for all $x \in [n]$. (Addition is mod n , i.e. $x + i$ means either $x + i$ or $x + i - n$,

⁵ If $1 \leq a \leq r \leq b < n/2$ then $r^{-1} + (n - 2r)^{-1}$ is maximised at $r = a$ or $r = b$, so we can improve our bounds with more information about r .

whichever lies in $[n]$.) Let C_n be the set of equivalence classes of this relation, which are called cyclic orders. We will abuse notation and identify a given cyclic order with the permutation σ that represents this class and has $\sigma(n) = n$. Then restricting σ to $[n - 1]$ establishes a bijection between C_n and S_{n-1} .

We consider the Cayley graph C on S_{n-1} generated by the set of adjacent transpositions $T = \{(12), (23), \dots, (n-2\ n-1)\}$, i.e. the vertex set of C is S_{n-1} and permutations σ and τ are adjacent in C if $\tau = \sigma \circ t$ for some $t \in T$. Note that we use the multiplication convention ‘first t then σ ,’ so that transpositions act by interchanging adjacent positions (rather than values) in the sequence representing a permutation, i.e. $(\tau(1), \dots, \tau(n))$ is obtained from $(\sigma(1), \dots, \sigma(n))$ by interchanging two consecutive elements. C is a regular graph with degree $d = n - 2$. The adjacency matrix of C has eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{(n-1)!}$. A theorem of Bacher [4] states that the second eigenvalue satisfies $d - \lambda_2 = 2 - 2 \cos(\pi/(n - 1))$. We will just use the estimate $d - \lambda_2 > 2/n^2$ for $n \geq 3$, which can easily be derived from this formula and the inequality $\cos x < 1 - x^2/4$ for $0 < x < 2$.

It follows that C is an α -expander, with $\alpha = (d - \lambda_2)/2d > 1/n^3$, i.e. for any set $W \subset V(G)$ with $|W| \leq (n - 1)!/2$ we have $|N(W)| \geq |W|/n^3$, where $N(W)$ is the set of vertices in $V(G) \setminus W$ that are adjacent to some vertex of W . (This value of α is given by Corollary 9.2 in Alon and Spencer [2]; it is not optimal, but suffices for our purpose.)

We need the following well-known lemma, which is the basis for Katona’s proof of the Erdős–Ko–Rado theorem. Given a cyclic order σ , the intervals of length r are the sets $I_{\sigma,r}(x) = \{\sigma(x), \sigma(x + 1), \dots, \sigma(x + r - 1)\}$ for $x \in [n]$ (addition mod n).

Lemma 10. *Suppose σ is a cyclic order of $[n]$ and F is an intersecting family of intervals of length $r < n/2$ in σ . Then $|F| \leq r$, and equality holds exactly when there is a single point x that belongs to all of the intervals.*

For the convenience of the reader we include the brief proof.

Proof. Suppose F contains the interval $I_{\sigma,r}(x)$. Let $j \geq 0$ be maximal so that $y = x + j \pmod n$ is in $I_{\sigma,r}(x)$ and $I_{\sigma,r}(y) \in F$. We claim that any interval $I_{\sigma,r}(z)$ in F contains y . To see this, note that since $I_{\sigma,r}(z)$ intersects $I_{\sigma,r}(x)$ we either have $z \in I_{\sigma,r}(x)$ or $z + r - 1 \in I_{\sigma,r}(x)$. In the former case we have $z = x + j' \pmod n$ with $0 \leq j' \leq j$ by definition of y , so $y \in I_{\sigma,r}(z)$. In the latter case we must have $z + r - 1 = x + j' \pmod n$ with $j \leq j' \leq r - 1$, or otherwise $I_{\sigma,r}(z)$ would be disjoint from $I_{\sigma,r}(y)$, so again $y \in I_{\sigma,r}(z)$. \square

Now we prove Theorem 4, which is as follows: Suppose $1 \leq r < n/2$, $\delta < \frac{1}{2rn^4}$ and G is an intersecting r -graph on n vertices with $|G| \geq (1 - \delta) \binom{n-1}{r-1}$. Then there is some vertex v so that all but at most $\delta r \binom{n-1}{r-1}$ edges of G contain v .

Proof of Theorem 4. For each cyclic order $\sigma \in C_n$ let $G(\sigma)$ consist of those sets of G that are intervals in σ . We say σ is complete if $|G(\sigma)| = r$, otherwise incomplete. The lemma tells us that if σ is complete then there is some point v belonging to all intervals of $G(\sigma)$. To specify this point we say that σ is v -complete. Let X be the set of incomplete σ . Then

$$r!(n - r)!|G| = \sum_{\sigma \in C_n} |G(\sigma)| \leq \sum_{\sigma \in C_n \setminus X} r + \sum_{\sigma \in X} (r - 1) = r(n - 1)! - |X|,$$

so $|X| \leq r(n-1)! - (1-\delta)r!(n-r)! \binom{n-1}{r-1} = \delta r(n-1)!$. It follows that the number of complete σ is at least $(1-\delta r)(n-1)!$.

Now we make the following claim: if σ is v -complete, τ is complete, and $\tau = \sigma \circ (i \ i + 1)$ for some $i \in [n] \setminus \{v, v-1\}$, then τ is v -complete. To prove this, we start by relabelling (if necessary) so that $v = n$, and so $1 \leq i \leq n-2$. Since σ is n -complete we have $I_{\sigma,r}(x) \in G(\sigma)$ for $n-r+1 \leq x \leq n$. Also, if $n-r+1 \leq x \leq n$, $x \neq i+1$ and $x+r-1 \neq i$ then $I_{\tau,r}(x) = I_{\sigma,r}(x) \in G$ (the order is different but the sets are the same). We have three cases according to the value of i . Firstly, if $i \neq n-r$ and $i \neq r-1$ then $I_{\tau,r}(n-r+1) = I_{\sigma,r}(n-r+1)$ and $I_{\tau,r}(n) = I_{\sigma,r}(n)$ are both in $G(\tau)$, and their only common position is n , so τ must be n -complete. Secondly, if $i = r-1$ then $i+1 \neq n-r+1$ (since $r < n/2$) so $I_{\sigma,r}(x) = I_{\tau,r}(x) \in G(\tau)$ for $n-r+1 \leq x \leq n-1$. These intervals have just two common positions: $n-1$ and n . Since τ is complete $G(\tau)$ must either contain $I_{\tau,r}(n)$ or $I_{\tau,r}(n-r)$. The latter case is impossible, as $I_{\tau,r}(n-r) = I_{\sigma,r}(n-r)$ (since $i+1 = r < n-r$), but this is disjoint to $I_{\sigma,r}(n) \in G(\sigma)$ and G is intersecting. Therefore $I_{\tau,r}(n) \in G(\tau)$, i.e. τ is n -complete. The argument for the third case, when $i = n-r$, is the same as that for the second case (by symmetry), so we will omit it. This proves the claim.

Now consider the Cayley graph C on S_{n-1} defined above. Suppose W is a set of complete cyclic orders, which we may consider as a subset of $V(C)$. Since C is a $1/n^3$ -expander, if $n^3\delta r \leq |W|/(n-1)! \leq 1/2$ we have $|N(W)| > \delta r(n-1)!$, and so there is a complete σ in $N(W)$. It follows that the restriction of C to the set of complete cyclic orders has a connected component C' of size at least $(1-n^3\delta r)(n-1)!$.⁶ By the claim, there is some v so that every σ in C' is v -complete. Write G_v for the sets in G that contain v . Then $r!(n-r)!|G_v| \geq \sum_{\sigma \in C'} |G(\sigma)| \geq (1-\delta r)(n-1)! \cdot r$, so $|G_v| \geq (1-\delta r) \binom{n-1}{r-1}$. Now by the Erdős–Ko–Rado theorem there are at most $\delta r \binom{n-1}{r-1}$ sets of G that do not contain v , as required. \square

Remark. The generators we use in this argument are poor from an expansion point of view, and in fact Kassabov [18] has shown that a constant eigenvalue gap can be obtained with just a constant number of generators (universal constants independent of n). However, this does not imply an improvement to our theorem, as we rely heavily on structural properties of the generating set in our argument.

7. An algebraic generalisation of Lovász’s theorem

In this section we prove an algebraic generalisation of the Lovász version of the Kruskal–Katona theorem. Let G be an r -graph and $s \leq r$. The (higher) inclusion matrix $M_s^r(G)$ is a $\{0, 1\}$ matrix with rows indexed by edges of G and columns indexed by subsets of $V(G)$ of size s : the entry corresponding to an edge e and a set S is 1 if $S \subset e$ and 0 otherwise. Frankl and Tokushige [12] posed the problem of finding the minimum rank of $M_s^r(G)$ in terms of $|G|$.

When $|G| = \binom{n}{r}$ for an integer n then one natural construction is the complete r -graph K_n^r . Here the rank is given by a theorem of Gottlieb ([15], see also [3]):

⁶ Consider the components of C restricted to the complete cyclic orders. Each component must either have size at most $n^3\delta r(n-1)!$ (‘small’) or more than $(n-1)!/2$ (‘large’). Since components are disjoint sets there is at most one large component. Also, the total size of all small components is at most $n^3\delta r(n-1)!$, or we could take W to be a union of small components with $n^3\delta r(n-1)! \leq |W| \leq 2n^3\delta r(n-1)!$ and find a complete σ in $N(W)$, contradicting the definition of components. Therefore there is a large component, and its size is at least $(1-n^3\delta r)(n-1)!$.

Theorem 11. (See Gottlieb [15].) $\text{rk } M_s^r(K_n^r) = \min\{\binom{n}{r}, \binom{n}{s}\}$.

Before describing what might be expected in general we describe some recursive properties of inclusion matrices. We define two operations associated with a vertex x of G giving hypergraphs on $V(G) \setminus \{x\}$. Deletion gives the r -graph $G \setminus x = \{A : A \in G, x \notin A\}$. Contraction gives the $(r - 1)$ -graph $G/x = \{A \setminus \{x\} : x \in A \in G\}$.

Lemma 12. Suppose G is an r -graph, x is a vertex of G and $1 \leq s \leq r - 1$. Then

$$\text{rk } M_s^r(G) \geq \max\{\text{rk } M_s^r(G \setminus x) + \text{rk } M_{s-1}^{r-1}(G/x), \text{rk } M_{s-1}^r(G \setminus x) + \text{rk } M_s^{r-1}(G/x)\}.$$

Proof. First we note an identity for inclusion matrices. Suppose H is a t -graph, $u \leq t$. Let K be the complete u -graph on $V(H)$. Then $M_u^t(H)M_{u-1}^u(K) = (t - u + 1)M_{u-1}^t(H)$. To see this, note that if $A \in H$ and $|S| = u - 1$ then the (A, S) entry on the left-hand side is either 0 if $S \not\subset A$, or otherwise the number of u -sets U with $S \subset U \subset A$, i.e. $t - u + 1$, which agrees with the definition of the right-hand side.

To write $M_s^r(G)$ in a convenient form we organise the rows as $R = R_1 \cup R_2$ and columns as $C = C_1 \cup C_2$, where R_1 corresponds to those sets of G that contain x and C_1 corresponds to all s -sets of $V(G)$ that contain x . This gives the block form

$$M_s^r(G) = \begin{pmatrix} M_{s-1}^{r-1}(G/x) & M_s^{r-1}(G/x) \\ 0 & M_s^r(G \setminus x) \end{pmatrix},$$

from which we obtain the first lower bound on the rank. Let M_1, M_2 be the submatrices corresponding to the columns in C_1, C_2 respectively and K be the complete s -graph on $V(G) \setminus \{x\}$. Now we apply the row and column operations

$$M_1' = (s - r)(r - s + 1)^{-1}(M_1 - (r - s)^{-1}M_2M_{s-1}^s(K)).$$

Since $M_{s-1}^{r-1}(G/x)M_{s-1}^s(K) = (r - s)M_{s-1}^{r-1}(G/x)$ and $M_s^r(G \setminus x)M_{s-1}^s(K) = (r - s + 1)M_{s-1}^r(G \setminus x)$ we obtain a matrix with block form

$$\begin{pmatrix} 0 & M_s^{r-1}(G/x) \\ M_{s-1}^r(G \setminus x) & M_s^r(G \setminus x) \end{pmatrix},$$

which gives the second lower bound on the rank. \square

Given this recursion, it is natural to think that for a general size $|G|$ of the r -graph G it may be optimal to take an initial segment of the colex order. To explain this point further we will briefly describe some properties of the order, and we refer the reader to the survey [12] for more information. Write $|G|$ in cascade form: the unique expression $|G| = \binom{n_r}{r} + \binom{n_{r-1}}{r-1} + \dots + \binom{n_j}{j}$ where $n_r > n_{r-1} > \dots > n_j \geq j \geq 1$. Using the natural numbers as our underlying ordered set, the initial segment of size G consists of all r -subsets of $[n_r]$, all r -sets obtained by adding $n_r + 1$ to an $(r - 1)$ -subset of $[n_{r-1}]$, \dots , and all r -sets obtained by adding $n_r + 1, n_{r-1} + 1, \dots, n_{j+1} + 1$ to a j -subset of $[n_j]$. The shadow of this system is the initial segment of the colex order on $(r - 1)$ -sets of length $|\partial G| = \binom{n_r}{r-1} + \binom{n_{r-1}}{r-2} + \dots + \binom{n_j}{j-1}$, and the Kruskal–Katona theorem states that this is the best possible lower bound. Iterating, we obtain that for $s \leq r$ the s -shadow

is the initial segment of the colex order on s -sets of length $|\partial_s^r G| = \binom{n_r}{s} + \binom{n_{r-1}}{s-1} + \dots + \binom{n_j}{j-r+s}$, where $\binom{m}{i}$ is defined to be zero for $i < 0$. Considering the decomposition used in Lemma 12, with $x = n_r + 1$, it is not hard to see that $\text{rk } M_s^r(G) = \text{rk } M_s^r(G \setminus x) + \text{rk } M_{s-1}^{r-1}(G/x) = \binom{n_r}{s} + \text{rk } M_{s-1}^{r-1}(G/x)$ (using Gottlieb’s Theorem), so iterating we obtain $\text{rk } M_s^r(G) = |\partial_s^r G|$.

However, the rank of $M_s^r(G)$ may not be as large as the s -shadow. For example consider the 2-graph C_4 (a 4-cycle). The size of its shadow is 4, which is as small as possible for a graph with 4 edges, but its inclusion matrix $M_1^2(C_4)$ has rank 3. This is not merely an effect for ‘small numbers’ as we can use it as a building block in larger examples: pick a number $n > 5$ and consider the 3-graph $K_{n-1}^3 \cup \{12n, 23n, 34n, 14n\}$. Thus there is no direct algebraic analogue of the Kruskal–Katona theorem. There is an algebraic analogue of Lovász’s theorem, at least for large r -graphs, and that is the content of Theorem 5, which we will soon prove.

First we need the following lemma, which expresses a rigidity property of $M_s^r(K_n^r)$ that seems independently interesting.

Lemma 6. *Suppose $0 \leq s \leq r < n/2$ and $G = K_n^r \setminus F$ is an r -graph on $[n]$ with $|F| < \binom{r}{s}^{-1} \binom{n}{r-s}$. Then $\text{rk } M_s^r(G) = \binom{n}{s}$.*

Proof. We argue by induction on s and $r - s$. The two base cases are straightforward: if $r = s$ then $|F| < 1$, so $G = K_n^r$ and $\text{rk } M_s^r(K_n^r) = \binom{n}{s}$ by Gottlieb’s Theorem (since $n > 2r$); if $s = 0$ then $|G| > 0$ and $\text{rk } M_0^r(G) = 1$. For the induction step we choose a vertex x of minimum degree in F , so that

$$d_F(x) \leq n^{-1} \sum_{v \in V(G)} d_F(v) = r|F|/n < \frac{r}{n} \binom{r}{s}^{-1} \binom{n}{r-s} = \binom{r-1}{s}^{-1} \binom{n-1}{r-s-1}.$$

By relabelling we can assume that $x = n$. Now $G/x = K_{n-1}^{r-1} \setminus (F/x)$ and $|F/x| = d_F(x)$, so by induction hypothesis $\text{rk } M_{s-1}^{r-1}(G/x) = \binom{n-1}{s-1}$. Also $G \setminus x = K_{n-1}^r \setminus (F \setminus x)$ and $|F \setminus x| \leq |F| < \binom{r}{s}^{-1} \binom{n}{r-s} < \binom{r}{s-1}^{-1} \binom{n-1}{r-s+1}$ (since $n > 2r$), so by induction hypothesis $\text{rk } M_{s-1}^r(G \setminus x) = \binom{n-1}{s-1}$. By Lemma 12 we have $\text{rk } M_s^r(G) \geq \text{rk } M_{s-1}^r(G \setminus x) + \text{rk } M_{s-1}^{r-1}(G/x) = \binom{n-1}{s-1} + \binom{n-1}{s-1} = \binom{n}{s}$. \square

Now we prove Theorem 5, which is as follows: For every $r \geq s \geq 0$ there is a number $n_{r,s}$ so that if G is an r -graph with $|G| = \binom{x}{r} \geq n_{r,s}$ then $\text{rk } M_s^r(G) \geq \binom{x}{s}$. Also, if $r > s > 0$ then equality holds only if x is an integer and $G = K_x^r$.

Proof of Theorem 5. We argue by induction on r and s . The cases $s = 0$ and $r = s$ are trivial, so suppose $r > s > 0$. Suppose that $G \neq K_x^r$ is an r -graph with $|G| = \binom{x}{r}$ and $\text{rk } M_s^r(G) = \binom{x}{s} - h$ with $h \geq 0$. We will show that if $n_{r-1,s-1} \geq \binom{u_0(r,s)}{r}$ (where $u_0(r,s)$ is given by Lemma 8) and $|G| \geq n_{r-1,s-1}$ then there is some vertex v so that $G \setminus v$ is an r -graph with $\binom{z}{r}$ edges and $\text{rk } M_s^r(G \setminus v) = \binom{z}{s} - h - h'$, where $z > x - 1$ and $h' > (3r)^{-r} x^{-1}$. Then we can iterate this fact to obtain an r -graph G_0 , such that $|G_0| = \binom{z_0}{r}$ with $n_{r-1,s-1} < |G_0| < 2n_{r-1,s-1}$ and

$$\text{rk } M_s^r(G_0) < \binom{z_0}{s} - (3r)^{-r} \sum_{i=z_0+1}^x 1/i < 2n_{r-1,s-1} - (3r)^{-r} \log \frac{x+1}{z_0+1},$$

using the estimate $\sum_{i=z_0+1}^x 1/i > \int_{z_0+1}^x dt/t = \log \frac{x+1}{z_0+1}$. This is less than 0 if we suppose that $|G| = \binom{x}{r} \geq n_{r,s}$ with $n_{r,s}$ sufficiently large, so we will have a contradiction to the existence of such G , which is the required result.

Now we show how to find the vertex v . We claim that there is a vertex v with $1 \leq d(v) \leq \binom{x-1}{r-1} - \frac{1}{2r!}x^{r-s-1}$. Write $n = |V(G)|$. Then we can bound the minimum degree as $\delta(G) \leq r|G|/n = \frac{x}{n} \binom{x-1}{r-1}$. If $n \geq x + 1$ then we get $\delta(G) \leq \binom{x-1}{r-1} - \frac{1}{x+1} \binom{x-1}{r-1} < \binom{x-1}{r-1} - \frac{1}{2r!}x^{r-s-1}$ for large x , so we can suppose that $n < x + 1$. This rules out the case when x is an integer, as we are supposing $G \neq K_x^r$. Therefore $n = \lceil x \rceil = x + \theta$ for some $0 < \theta < 1$. Also, since $\text{rk } M_s^r(G) \leq \binom{x}{s} < \binom{n}{s}$ we have $|K_n^r \setminus G| \geq \binom{r}{s}^{-1} \binom{n}{r-s}$ by Lemma 6. This gives $\binom{r}{s}^{-1} \binom{n}{r-s} \leq \binom{n}{r} - \binom{n-\theta}{r} < \theta \binom{n}{r-1}$, so $\theta > \frac{s}{r} \binom{n-r+s}{s-1}^{-1}$. Now

$$\begin{aligned} \binom{x-1}{r-1} - \delta(G) &\geq \binom{x-1}{r-1} - \frac{x}{n} \binom{x-1}{r-1} = \frac{\theta}{n} \binom{x-1}{r-1} > \frac{s}{rn} \binom{n-r+s}{s-1}^{-1} \binom{x-1}{r-1} \\ &= (1 + o(1)) \frac{s!}{r!} x^{r-s-1} > \frac{1}{2r!} x^{r-s-1}, \end{aligned}$$

for large x , as required.

Write $\binom{z}{r} = |G \setminus v| = |G| - d(v)$. Since $d(v) < \binom{x-1}{r-1}$ we have $z > x - 1$. We consider the cases $d(v) \leq n_{r-1,s-1}$ and $d(v) > n_{r-1,s-1}$ separately. First suppose that $d(v) \leq n_{r-1,s-1}$. Then $n_{r-1,s-1} \geq \binom{x}{r} - \binom{z}{r} > (x-z) \binom{z-1}{r-1} > (x-z) \binom{x-2}{r-1}$ and $\binom{x}{s} - \binom{z}{s} < (x-z) \binom{x}{s-1} < n_{r-1,s-1} \binom{x}{s-1} \binom{x-2}{r-1}^{-1} < 1/2$ for large x . Since $d(v) \geq 1$ we have $\text{rk } M_{s-1}^{r-1}(G/v) \geq 1$, so by Lemma 12 we have $\text{rk } M_s^r(G \setminus v) \leq \text{rk } M_s^r(G) - \text{rk } M_{s-1}^{r-1}(G/v) \leq \binom{x}{s} - h - 1 < \binom{z}{s} - h - 1/2$.

Now suppose that $d(v) > n_{r-1,s-1}$. Write $d(v) = \binom{w}{r-1}$. Then by the induction hypothesis we have $\text{rk } M_{s-1}^{r-1}(G/v) \geq \binom{w}{s-1}$. Now $\binom{x}{r} = \binom{z}{r} + \binom{w}{r-1}$, where $\binom{w}{r-1} = d(v) < \frac{x-1}{r-1} - \frac{1}{2r!}x^{r-s-1}$, so by Lemma 8 we have $\binom{x}{s} = \binom{z}{s} + \binom{w}{s-1} - h'$, with $h' > (3r)^{-r} x^{-1}$. Now by Lemma 12 we have $\text{rk } M_s^r(G \setminus v) \leq \text{rk } M_s^r(G) - \text{rk } M_{s-1}^{r-1}(G/v) \leq \binom{x}{s} - h - \binom{w}{s-1} = \binom{z}{s} - h - h'$, as required.

Either way we obtain the vertex v required in the first paragraph of the proof, so we are done. \square

8. Concluding remarks

The argument for proving Theorem 4 via expansion in the Cayley graph applies generally to any extremal problem on k -graphs with the property that when one restricts to an interval there can be at most k sets, with equality exactly when they all contain some fixed point. More generally, it gives a strategy to prove a stability theorem for any extremal problem that can be uniformly covered by ‘simpler instances’ via the action of a group G , provided that we have a characterisation of the maximum constructions for the simpler instances and a set of generators for G that is ‘well behaved’ with respect to the constructions and give a Cayley graph with good expansion. Such a stability result could in turn be used as part of the stability method for solving the original problem (see [20] for an example of the stability method and references to many other examples). We hope to return to this idea in future work.

We have proved an algebraic analogue of Lovász’s theorem, but it is natural to ask if there can be any algebraic analogue of the Kruskal–Katona theorem, even though we have observed that

there are other factors to take into account, and so the full description of the optimal constructions may be very complicated.

Constructions of explicit rigid matrices can be used to obtain lower bounds in various notions of complexity used in Theoretical Computer Science (see [17,23,25]). As far as we can see, our rigidity result (Lemma 6) does not give any non-trivial result in this arena, but perhaps some new ideas could turn it into a useful construction. The lemma is not exactly tight, but it is tight up to a constant, as may be seen by fixing some set S of size s and letting F consist of all r -sets that contain S . Then $|F| = \binom{n-s}{r-s} = \Theta(n^{r-s})$ and $\partial_s^r G$ does not contain S , so $M_s^r(G)$ does not have full rank.

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Appendix A. Proofs of binomial coefficient estimates

This appendix contains the proofs of Lemmas 7 and 8. First we recall an identity for binomial coefficients (see Ex. 1.42(i) in [24]).

$$\binom{x+y}{r} = \sum_{j=0}^r \binom{x+j-1}{j} \binom{y-j}{r-j}. \tag{9}$$

Another exercise in [24], 1.43(e), states that

$$\frac{d}{dx} \binom{x}{r} = \sum_{i=1}^r \frac{1}{i} \binom{x-i}{r-i}. \tag{10}$$

Note that this is a strictly increasing function of x for $x \geq r - 1$. We also need the Mean Value Theorem from Calculus, that if $f(x)$ is a real differentiable function and $a > b$ then $\frac{f(a)-f(b)}{a-b} = f'(c)$ for some $a \geq c \geq b$. Furthermore, if $f'(t)$ is a strictly increasing function we can take $a < c < b$.

Proof of Lemma 7. Suppose $x > y \geq r - 1$. Write $f(z) = \binom{z}{r}$. By the Mean Value Theorem we can write $\frac{f(x)-f(y)}{x-y} = f'(c)$, for some $y < c < x$. Then

$$\binom{x}{r} - \binom{y}{r} = (x - y)f'(c) > (x - y)f'(y) > (x - y)\binom{y-1}{r-1},$$

by Eq. (10). Also

$$\binom{x}{r} - \binom{y}{r} = (x - y)f'(c) < (x - y)f'(x) \leq (x - y) \sum_{i=1}^r \binom{x-i}{r-i} = (x - y)\binom{x}{r-1},$$

applying Eq. (9) with x replaced by 1, y replaced by $x - 1$ and r replaced by $r - 1$. \square

Next we will prove Lemma 8, which can be regarded as a defect form of Ex. 13.31(a) in [24], and indeed our proof involves a more careful analysis of what is going on inside Lovász’s proof. First we need to give a separate argument for the case $r = 2$, which is easy.

Lemma 13. *Suppose $C > 0$ and $\binom{u}{2} = \binom{v}{2} + w$ with $1 \leq w < u - 1 - C$. Then $u < v + 1 - C/u$.*

Proof. Since $w < u - 1$ we have $v > u - 1$. Also $0 = v(v - 1) + 2w - u(u - 1) = 2w - (u - v)(u + v - 1)$, so $1 + v - u = 1 - \frac{2w}{u+v-1} > 1 - \frac{2(u-1-C)}{2u-2} = C/(u - 1) > C/u$. \square

Now we can assume $r \geq 3$. We prove the following lemma, in which statement (3) is the result we want, Lemma 8.

Lemma 14. *Suppose $r \geq 3$, $\binom{u}{r} = \binom{v}{r} + \binom{w}{r-1}$ with $C' \leq \binom{w}{r-1} < \binom{u-1}{r-1} - C$, for some $C, C' > 0$ and u is sufficiently large. For $1 \leq s \leq r$ write $X_s = \binom{v}{s} + \binom{w}{s-1} - \binom{u}{s}$. Then*

- (1) if $w > u - 3r$ then $X_{r-1} > C/3u$,
- (2) if $w < u - 2r$ then $X_{r-1} > \min\{1/4r!, C'\}$, and
- (3) if $s \leq r - 1$, $C' = 1$ and $C = \frac{1}{2r!}u^{r-s-1}$ then $X_s > (3r)^{-r}u^{-1}$.

Proof. First consider the (possibly non-existent) case $C' \leq \binom{w}{r-1} \leq 1$, when we have $r - 2 < w \leq r - 1$. Now $(\binom{u}{r} - \binom{v}{r}) - (\binom{u}{r-1} - \binom{v}{r-1}) = \binom{u-1}{r-1} - \binom{v-1}{r-1} > 0$ since $u > v > r - 1$ so $X_{r-1} > \binom{w}{r-2} + \binom{v}{r} - \binom{u}{r} = \binom{w}{r-2} - \binom{w}{r-1} = (\frac{r-1}{w-r+2} - 1)\binom{w}{r-1} \geq (r - 2)\binom{w}{r-1} \geq C'$.

Now we suppose that $\binom{w}{r-1} > 1$ and, following [24], introduce the change of variables $w = t + r - 1$, $u' = u - t$, $v' = v - t$. Note that $t > 0$, $u' = (u - w) + r - 1 > r$ and $v' > r - 1$. By identity (9) we have $\binom{u}{r} = \sum_{j=0}^r \binom{t+j-1}{j} \binom{u'-j}{r-j}$, $\binom{v}{r} = \sum_{j=0}^r \binom{t+j-1}{j} \binom{v'-j}{r-j}$, and $\binom{w}{r-1} = \sum_{j=0}^{r-1} \binom{t+j-1}{j}$, so

$$0 = X_r = \sum_{j=0}^{r-1} \binom{t+j-1}{j} \phi_{r-j}, \tag{11}$$

where $\phi_i = \binom{v'-r+i}{i} + 1 - \binom{u'-r+i}{i}$. Similarly we have

$$X_{r-1} = \sum_{j=1}^{r-1} \binom{t+j-1}{j-1} \phi_{r-j} = \sum_{j=1}^{r-1} \frac{j}{t} \binom{t+j-1}{j} \phi_{r-j}. \tag{12}$$

Now

$$\phi_{i-1} = \left(\frac{i}{u'-r+i}\right)\phi_i + \left(1 - \frac{i}{u'-r+i}\right) + \left(\frac{i}{v'-r+i} - \frac{i}{u'-r+i}\right)\binom{v'-r+i}{i},$$

so if $\phi_i > 0$ we have $\phi_{i-1} > 0$. Therefore there is some $0 \leq k \leq r$ so that $\phi_i > 0$ for $1 \leq i \leq k$ and $\phi_i \leq 0$ for $k < i \leq r$.

Note that $\phi_1 = v' + 1 - u' = v + 1 - u > 0$, so $k \geq 1$. We have $\binom{v}{r} = \binom{u}{r} - \binom{w}{r-1} > \binom{u}{r} - \binom{u-1}{r-1} + C = \binom{u-1}{r} + C$ so $C < \binom{v}{r} - \binom{u-1}{r} < (v - u + 1)\binom{v}{r-1}$ and $\phi_1 > C\binom{v}{r-1}^{-1}$. Let $k' = \max\{k, 3/2\}$. By Eqs. (12) and (11) we have

$$X_{r-1} > \frac{r-k'}{t} \sum_{j=0}^{r-1} \binom{t+j-1}{j} \phi_{r-j} + \frac{1}{2t} \binom{t+r-2}{r-1} \phi_1 = \frac{1}{2t} \binom{t+r-2}{r-1} \phi_1. \tag{13}$$

If $w > u - 3r$ then Eq. (13) gives

$$\begin{aligned} X_{r-1} &> \frac{1}{2u} \binom{u-3r-1}{r-1} \cdot C \binom{v}{r-1}^{-1} \\ &= (1 + o(1)) \frac{1}{2u} \cdot \frac{u^{r-1}}{(r-1)!} \cdot C \cdot \frac{(r-1)!}{u^{r-1}} \sim C/2u > C/3u \end{aligned}$$

for large u , which proves (1). Now suppose $w < u - 2r$, so $u' = u - w + r - 1 > 3r - 1$ and $v' > u' - 1 > 3r - 2$. If $\phi_1 \geq 1/2$ then Eq. (13) gives $X_{r-1} \geq \frac{1}{2t} \binom{t+r-2}{r-1} \cdot 1/2 = \frac{\prod_{i=1}^{r-2} (t+i)}{4(r-1)!} > 1/4r!$, i.e. (2) holds, so we can suppose $\phi_1 < 1/2$. Then $u' - v' = 1 - \phi_1 > 1/2$ and $1 - \phi_2 = \binom{u'-r+2}{2} - \binom{v'-r+2}{2} = \frac{1}{2}(u' - v')(u' + v' - 2r + 3) > r$, so $\phi_2 < 1 - r \leq -2$. Then $k = 1$ and by Eqs. (12) and (11) we have

$$\begin{aligned} X_{r-1} &> \frac{r-1}{t} \sum_{j=0}^{r-1} \binom{t+j-1}{j} \phi_{r-j} - \frac{1}{t} \binom{t+r-3}{r-2} \phi_2 = \frac{1}{t} \binom{t+r-3}{r-2} (-\phi_2) \\ &= \frac{\prod_{i=1}^{r-3} (t+i)}{(r-2)!} (-\phi_2) > 2/(r-2)! > 1/4r!, \end{aligned}$$

so (2) holds in either case.

Finally, suppose $C' = 1$ and $C = \frac{1}{2r!} u^{r-s-1}$. To prove (3) we consider the cases $w < u - 2r$ and $w \geq u - 2r$ separately. If $w < u - 2r$ then for $1 \leq i \leq r - s$ we prove $X_{r-i} > 1/4r!$ by induction on i . The base case $i = 1$ holds by (2). For the induction step, suppose $X_{r-i} > 1/4r!$ for some $i \geq 1$. Define u_i, w_i by $\binom{w_i}{r-i-1} = \binom{w}{r-i-1} - 1/4r!$ and $\binom{u_i}{r-i} = \binom{v}{r-i} + \binom{w_i}{r-i-1}$. Then $\binom{u}{r-i} < \binom{u_i}{r-i} = \binom{v}{r-i} + \binom{w_i}{r-i-1}$, $w_i < w < u - 2r < u_i - 2r$ and $\binom{w_i}{r-i-1} > 1 - 1/4r! > 1/4r!$, so applying (2) with r replaced by $r - i$ we have

$$\begin{aligned} X_{r-i-1} &= \binom{v}{r-i-1} + \binom{w}{r-i-2} - \binom{u}{r-i-1} \\ &> \binom{v}{r-i-1} + \binom{w_i}{r-i-2} - \binom{u_i}{r-i-1} > 1/4r!, \end{aligned}$$

as required. Since $1/4r! > (3r)^{-r} u^{-1}$, we have proved (3) in the case $w < u - 2r$.

On the other hand, if $w \geq u - 2r$ then for $1 \leq i \leq r - s$ we prove $X_{r-i} > x_i = (2r!)^{-1} 3^{-i} u^{r-s-1-i}$ by induction on i . The base case $i = 1$ holds by (1), since $w \geq u - 2r > u - 3r$. For the induction step suppose $X_{r-i} > x_i$ for some $i \geq 1$. Define u_i, w_i by $\binom{w_i}{r-i-1} = \binom{w}{r-i-1} - x_i$ and $\binom{u_i}{r-i} = \binom{v}{r-i} + \binom{w_i}{r-i-1}$. Then $u < u_i$ and $\binom{u_i}{r-i} = \binom{v}{r-i} + \binom{w_i}{r-i-1} <$

$\binom{u}{r-i} + \binom{u}{r-i-1} = \binom{u+1}{r-i}$, so $u_i < u + 1$. Also $w - w_i = o(u)$, since $x_i = O(u^{r-i-2})$. In fact $x_i = \binom{w}{r-i-1} - \binom{w_i}{r-i-1} > (w - w_i) \binom{w_i}{r-i-2}$, so

$$\begin{aligned} w - w_i &< x_i \binom{w_i}{r-i-2}^{-1} < (2r!)^{-1} u^{r-s-1-i} \cdot \frac{(r-i-2)!}{(w_i-r)^{r-i-2}} \\ &< \left(\frac{u}{w_i-r} \right)^{r-i-2} u^{1-s} < 1 + o(1), \end{aligned}$$

which implies $w_i > w - 2$. Therefore $w_i > w - 2 > u - 2r - 2 > u_i - 2r - 3 \geq u_i - 3r$. Since $\binom{w_i}{r-i-1} = \binom{w}{r-i-1} - x_i < \binom{u_i-1}{r-i-1} - x_i$ we can apply (1), or Lemma 13 in the case $s = 1$ and $i = r - 2$, with r replaced by $r - i$ to get

$$\begin{aligned} X_{r-i-1} &= \binom{v}{r-i-1} + \binom{w}{r-i-2} - \binom{u}{r-i-1} \\ &> \binom{v}{r-i-1} + \binom{w_i}{r-i-2} - \binom{u_i}{r-i-1} > x_i/3u = x_{r-i-1}. \end{aligned}$$

Then $X_s > 3^{s-r} (2r!)^{-1} u^{-1} > (3r)^{-r} u^{-1}$, so we have (3) in both cases, and the lemma is proved. \square

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