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On the number of edges not covered by monochromatic copies of a fixed graph ☆

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Abstract

For a fixed graph H, let $f(n,H)$ denote the maximum possible number of edges not belonging to a monochromatic copy of H in a 2-edge-coloring of the complete graph of order n. Let $ex(n,H)$ be the *Turán number* of H, i.e., the maximum number of edges that a graph on n vertices can have without containing a copy of H. An easy lower bound of $f(n,H) \geqslant ex(n,H)$ follows from the 2-edge-coloring in which the edges of one color form the largest H-free graph. In this paper we consider the cases when H is an edge-color-critical graph (e.g., a complete graph) or a 4-cycle. We will show then, that for sufficiently large n, the value of $f(n,H)$ is actually equal to $ex(n,H)$.

1. Introduction

The classical result of Ramsey tells us that for every integer $k \geqslant 2$ there exists a number $R(k,k)$ such that every 2-edge-coloring of the complete graph of order $n > R(k,k)$ contains a monochromatic clique of size k. It is natural then to ask how many such k-cliques there are. For the case of triangles ($k = 3$) this was answered by Goodman [5] (see also [8]), who obtained that in any 2-edge-coloring of K_n, the complete graph of order n, there will be at least $n^3/24 + o(n^3)$ monochromatic triangles. This can be seen to be tight by two different examples: a random coloring,

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or by taking one color to be the complete bipartite graph with both parts of size $n/2$. Erdős conjectured that a random coloring should have the smallest number of monochromatic K_k's for general k, but this was disproved even for $k = 4$ by Thomason [11]. Another question of Erdős motivated by Goodman's result asks for the minimum number of *edge disjoint* monochromatic triangles in any 2-edge-coloring of the complete graph K_n. In [2] he conjectured that here also it is best to take one of the colors to be a complete bipartite graph, which will give $n^2/12 + o(n^2)$ edge disjoint monochromatic triangles. This conjecture remains open, although some progress has been made in [3,7].

In this paper we look at the problem from a different perspective, that of finding the maximum number of the edges that do not belong to a monochromatic copy of some fixed graph. For triangles one might expect, by analogy with the results above, that the maximum occurs when one color class forms a maximum size triangle free graph, thus giving the answer $\lfloor n^2/4 \rfloor$. Let $f(n, \triangle)$ be the maximum possible number of edges not contained in a monochromatic triangle in a 2-edge-coloring of K_n. Erdős mentioned in [2], Problem 10 that, together with Rousseau and Schelp (unpublished), they showed that for sufficiently large n, indeed $f(n, \triangle) = \lfloor n^2/4 \rfloor$. Also, as was pointed out to us by N. Alon, this can be deduced from a result of Pyber [9], which states that for $n \geqslant 2^{1500}$ one needs at most $\lfloor n^2/4 \rfloor + 2$ monochromatic cliques to cover all the edges of a 2-edge-colored K_n. Nevertheless, we feel that it is worth setting out a simple argument which will determine $f(n, \triangle)$ for all n. This is done in the following theorem.

Theorem 1.1. *$f(n, \triangle) = \binom{n}{2}$ for $n \leqslant 5$, $f(6, \triangle) = 10$ and for all $n \geqslant 7$*

$$f(n, \triangle) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

In [2] Erdős also suggests that generalizations of this result should be possible. For a fixed graph H, let $f(n, H)$ denote the maximum possible number of *NIM-H edges*, i.e., edges not belonging to a monochromatic copy of H in a 2-edge-coloring of K_n. Let $ex(n, H)$ be the *Turán number* of H, that is, the maximum number of edges that a graph on n vertices can have without containing a copy of H. Note that an easy lower bound of $f(n, H) \geqslant ex(n, H)$ follows from the 2-edge-coloring in which the edges of one color form the largest H-free graph.

Our first result concerns the case when H is a complete graph, so we recall Turán's theorem, which gives $ex(n, H)$ in this case. Let $T_r(n)$ be the complete r-partite graph on n vertices with class sizes as equal as possible. Let $t_r(n)$ be the number of edges of $T_r(n)$. Then $t_r(n) = \frac{r-1}{2r} n^2 + O(n)$ and a celebrated theorem of Turán states that any K_{r+1}-free graph G on n vertices satisfies $e(G) \leqslant t_r(n)$, with equality iff $G = T_r(n)$. The following result shows that for sufficiently large, but still 'reasonable' n, the value of $f(n, K_{r+1})$ is actually equal to $ex(n, K_{r+1}) = t_r(n)$.

Theorem 1.2. *Let $r \geqslant 2$. Then for $n > e^{20^{r^2}}$, $f(n, K_{r+1}) = t_r(n)$.*

We say that a graph H with chromatic number $\chi(H) = r + 1$ is *edge-color-critical* if there is some edge e of H for which $\chi(H - e) = r$. For such H, it is known (see, e.g., [10]) that $ex(n, H) = t_r(n)$ for sufficiently large n. Noting that K_{r+1} is edge-color-critical we see that the following theorem generalizes Theorem 1.2.

Theorem 1.3. *Let H be an edge-color-critical graph of chromatic number $r + 1 > 2$. Then for n sufficiently large, $f(n, H) = ex(n, H) = t_r(n)$.*

For example, an odd cycle C_{2t+1} is edge-color-critical with chromatic number 3, so we have the following corollary.

Corollary 1.4. *For n sufficiently large, $f(n, C_{2t+1}) = \lfloor \frac{n^2}{4} \rfloor$.*

The final case we consider is when $H = C_4$ is a 4-cycle, and we will compute $f(n, C_4)$ for all values of n.

Theorem 1.5. $f(n, C_4) = \binom{n}{2}$ *for* $n \leqslant 5$, $f(6, C_4) = 9$ *and for all* $n \geqslant 7$

$$f(n, C_4) = ex(n, C_4).$$

The rest of this paper is organized as follows. In the next section we determine the value of $f(n, \triangle)$ for all n. In Section 3 we study the number of edges not covered by monochromatic cliques of size $r + 1, r \geqslant 2$ and prove Theorem 1.2. The same method with some modifications extends to prove the more general Theorem 1.3, for which we give a proof without obtaining an explicit bound on n for the sake of clarity. In Section 4 we consider the case when H is a 4-cycle and compute for every n the maximum number of edges not covered by monochromatic copies of C_4. The last section of the paper is devoted to some concluding remarks and related open problems.

2. Edges not in monochromatic triangles

In this section we prove Theorem 1.1. For convenience we use the following abbreviation. Given a 2-edge-coloring of the complete graph K_n, we call an edge a *NIM-$\triangle$ edge* if it is not contained in any monochromatic triangle. First we consider small values of n. For $n \leqslant 5$ obviously there are 2-edge-colorings of K_n without monochromatic triangles, so they will have as many as $\binom{n}{2}$ *NIM-$\triangle$* edges. Next, consider the following edge-coloring of K_6. The red color consists of a 5-cycle together with the three edges which join a sixth vertex to three consecutive vertices on a cycle, and the remaining edges are colored blue. Then the blue color is triangle free, so it is easy to check that this coloring has $10 > 6^2/4 = 9$ *NIM-$\triangle$* edges. A computer search shows that this is the maximum possible value for $n = 6$. Also using a computer search, one can prove that for $n = 7, 8, 9$, the colorings where one color forms a complete bipartite graph give the maximum number of *NIM-$\triangle$* edges, which is $\lfloor n^2/4 \rfloor$. So to finish the proof of Theorem 1.1 it suffices to show the following statement.

Proposition 2.1. *For $n \geqslant 10$, every 2-edge-coloring of K_n has at most $\lfloor n^2/4 \rfloor$ NIM-$\triangle$ edges.*

Proof. Suppose that there is a 2-edge-coloring with more than $\lfloor n^2/4 \rfloor$ *NIM*-$\triangle$ edges. By Turán's theorem there exists a triangle consisting of *NIM*-$\triangle$ edges, whose vertices we denote by x, y, z. Without loss of generality both edges adjacent to x are red, and the edge yz is blue. Let $S = V(K_n) - \{x, y, z\}$ be the set of the remaining vertices.

Note that all edges connecting x to S are blue. Indeed, if xt is red for some vertex $t \in S$, then ty and tz should be blue to avoid a red triangle containing one of two edges xy and xz. Thus yz is not a *NIM*-$\triangle$ edge since tyz forms a blue triangle, contradiction.

Next we show that no edge connecting x to S can be *NIM*-$\triangle$. For suppose there exists a vertex $t \in S$, such that xt is a *NIM*-$\triangle$ edge. Let $S' = S - \{t\}$. Note that for any $s \in S'$, the edge xs is blue, so to avoid creating a monochromatic triangle on xt the edge ts must be red. Thus every edge spanned by S' forms a monochromatic triangle either with x or with t, so is not a *NIM*-$\triangle$ edge. We can see that there are at most $n - 4$ *NIM*-$\triangle$ edges connecting $\{x, t\}$ to S'. For if there is any $s \in S'$ such that xs is a *NIM*-$\triangle$ edge we see that all edges from s to $S' - s$ are red, so all edges from t to S' belong to red triangles. Similarly, if there is any $s \in S'$ such that ts is a *NIM*-$\triangle$ edge we see that all edges from s to $S' - s$ are blue, so all edges from x to S' belong to blue triangles. Hence at most $|S'| = n - 4$ *NIM*-$\triangle$ edges connect $\{x, t\}$ to S'. Together with the edges connecting $\{y, z\}$ to S' and edges spanned by the vertices $\{t, x, y, z\}$, we obtain that the total number of *NIM*-$\triangle$ edges in this coloring is at most $3(n-4) + 6 < \lfloor n^2/4 \rfloor$ (since $n \geqslant 10$), which is a contradiction.

We claim that if v is any vertex for which all edges of the triangle vyz are *NIM*-$\triangle$ then vy and vz are red and thus by the above argument all edges from v to $V - \{v, y, z\}$ are blue and not *NIM*-$\triangle$. Indeed, let $v \neq x$ be a vertex for which all edges of the triangle vyz are *NIM*-$\triangle$. Note that vy and vz cannot both be blue (yz is blue and *NIM*-$\triangle$) so if they are not both red then we can assume that vy is blue and vz is red. Let $W = V - \{v, x, y, z\}$. As above we see that all edges connecting y to W are red. In addition, we already have that all the edges connecting x to W are blue, so W spans no *NIM*-$\triangle$ edges. Also as above we can see that there are at most $n - 4$ *NIM*-$\triangle$ edges connecting $\{x, y\}$ to W. Therefore the same calculation as before leads to a contradiction. This shows that vy and vz are both red, as required.

Recall that all edges between x and $S = V - \{x, y, z\}$ are blue, so only the red edges inside S can be *NIM*-$\triangle$. Also the *NIM*-$\triangle$ red edges clearly contain no triangle, so by Turán's theorem there are at most $\lfloor (n-3)^2/4 \rfloor$ of these. Therefore the number of *NIM*-$\triangle$ edges between xyz and S is at least $\lfloor n^2/4 \rfloor - \lfloor (n-3)^2/4 \rfloor - 3 \geqslant n - 1$, since $n \geqslant 10$. None of these *NIM*-$\triangle$ edges meets x, so there must be two vertices v, w in S joined to both y and z by *NIM*-$\triangle$ edges. By the previous paragraph no other edges at v or w are NIM-$\triangle$. As before, only red edges not incident to $\{v, w, x, y, z\}$ can be NIM-$\triangle$ and they cannot contain a triangle, so there are at most $\lfloor (n-5)^2/4 \rfloor$ such edges. Since $n \geqslant 10$, we see that the total number of NIM-$\triangle$ edges in this coloring is at most

$$\lfloor (n-5)^2/4 \rfloor + 2(n-3) + 3 \leqslant \lfloor n^2/4 \rfloor,$$

a contradiction that completes the proof. □

3. Edge-color-critical graphs

Recall that a graph H with chromatic number $\chi(H) = r+1$ is *edge-color-critical* if there is some edge e of H for which $\chi(H-e) = r$. In this section we determine the value of $f(n, H)$ for such H when n is sufficiently large. A natural example of such a graph is the complete graph K_{r+1}. We will begin by presenting the argument in this simpler case, so that we may illustrate the ideas involved and also to show that the lower bound on n which we require is quite reasonable.

We start by recalling some notation. Let $T_r(n)$ be the complete r-partite graph on n vertices with class sizes as equal as possible. Let $t_r(n)$ be the number of edges of $T_r(n)$. Then it is easy to see that $t_r(n) = \frac{r-1}{2r} n^2 + O(n)$. By Turán's theorem any K_{r+1}-free graph G on n vertices satisfies $e(G) \leqslant t_r(n)$, with equality iff $G = T_r(n)$. Erdős and Stone [4] showed that with slightly more than $t_r(n)$ edges, we not only can guarantee a copy of K_{r+1}, but even a copy of $K_{r+1}(t)$, i.e., a complete $r+1$-partite graph with t vertices in each class. The following quantitative version of this theorem is due to Bollobás and Erdős (see, e.g., [1].)

Theorem 3.1. *Let $r \geqslant 1$ be an integer, and let $\varepsilon > 0$. For $n > \max(3/\varepsilon, 100)$, if a graph G on n vertices contains more than $(\frac{r-1}{2r} + \varepsilon)n^2$ edges, then it contains a $K_{r+1}(t)$, for some $t \geqslant \frac{\varepsilon \log n}{2^{r+1}(r-1)!}$.*

We will use this result to prove the following lemma which we need in the proof of Theorem 1.2.

Lemma 3.2. *Let $r \geqslant 2$ and $n > e^{12^{r^2}}$. Then every 2-edge-coloring of a complete graph K_n in which there are more than $t_r(n)$ NIM-K_{r+1} edges contains a monochromatic copy of $K_r(2r)$ consisting entirely of NIM-K_{r+1} edges.*

Proof. Consider a red–blue edge-coloring of K_n in which there are more than

$$t_r(n) = \frac{r-1}{2r} n^2 - O(n) \geqslant \left(\frac{r-2}{2(r-1)} + \frac{1}{2r^2}\right) n^2$$

NIM-K_{r+1} edges and let $t = 4r(r!4^{r(r-1)})$. An easy but tedious computation, which we omit here, shows that we can apply Theorem 3.1 with $\varepsilon = 1/2r^2$ to conclude that there exists a complete r-partite subgraph $K_r(t)$ consisting entirely of NIM-K_{r+1} edges. Denote its parts by $X_1, \cdots, X_r$.

Call a vertex u a *red neighbor* (*blue neighbor*) of vertex v if vu is colored red (blue). Let x be an arbitrary vertex of X_1. Without loss of generality we can assume that at least half of its neighbors in X_r are red, and denote the set of these vertices by W. Note that every vertex $y \in X_1 \cup \cdots \cup X_{r-1}$ can have at most $4^r - 1$ blue neighbors in W. Indeed, suppose that some y has at least 4^r blue neighbors in W and the set of the neighbors is W'. Then by the well known bounds on diagonal Ramsey numbers (see, e.g., [6]) W' contains a monochromatic copy of K_r. This copy together with either x (if it is red) or with y (if it is blue) forms a monochromatic copy of K_{r+1}. This contradicts the fact that all the edges between x, y and W are NIM-K_{r+1}.

For all $1 \leqslant i \leqslant r-1$ choose arbitrary subsets $Y_i \subset X_i$ of size $2r(r-1)!4^{r(r-2)}$ and remove from W all blue neighbors of every vertex in $\bigcup_{i=1}^{r-1} Y_i$. Denote the remaining set by Y_r. Then it is easy to see that also

$$\begin{aligned} |Y_r| &\geqslant |W| - \left|\bigcup_{i=1}^{r-1} Y_i\right| 4^r \geqslant t/2 - (r-1)\left(2r(r-1)!4^{r(r-2)}\right)4^r \\ &= 2r(r!4^{r(r-1)}) - 2r(r-1)(r-1)!4^{r(r-1)} \geqslant 2r(r-1)!4^{r(r-2)}. \end{aligned}$$

Also, for all $1 \leqslant i \leqslant r-1$, all the edges between Y_i and Y_r are red. Similarly as before this implies that every vertex in Y_i has at most 4^r blue neighbors in Y_j, for any $1 \leqslant i \neq j \leqslant r-1$. Now we can repeat the above construction for $Y_1, \cdots, Y_{r-1}$ and obtain the sets $Z_i \subset Y_i$ of size $2r(r-2)!4^{r(r-3)}$ such that, for all $1 \leqslant i \leqslant r-2$, all the edges between Z_i and Z_{r-1} are red. Clearly we can continue this process for $r-3$ additional iterations. In the end we obtain sets $X_i' \subset X_i$ such that $|X_i'| \geqslant 2r$, and for every $1 \leqslant i \neq j \leqslant r$ all the edges between X_i' and X_j' are red and NIM-K_{r+1}. □

Having finished all the necessary preparations, we are now ready to complete the proof of our second theorem.

Proof of Theorem 1.2. Suppose the result is not true and consider a red–blue edge-coloring of K_n with more than $t_r(n)$ NIM-K_{r+1} edges. By Lemma 3.2 there are sets of vertices $X_1, \cdots, X_r$ of size $2r$, such that all edges of the r-partite graph $H = (X_1, \cdots, X_r)$ are NIM-K_{r+1}, and red, say. Then it is easy to see that all the edges spanned by the sets X_i individually are blue, since every such red edge will form a red copy of K_{r+1} which contains edges of H. Thus each X_i forms a blue clique of size $2r$ and therefore contains no NIM-K_{r+1} edges. Let Y be the set of vertices not in H. Note that no vertex $y \in Y$ can have a red neighbor in every X_i as this would create a monochromatic copy of K_{r+1} which uses edges of H. So for every vertex $y \in Y$ there exists an index i such that all the edges from y to X_i are blue. Note that these edges are not NIM-K_{r+1} since they are obviously contained in a blue clique of size $r+1$. Let Y_i be the set of vertices $y \in Y$ such that all the edges from y to X_i are blue and i is the smallest index with this property. This gives a partition of Y into disjoint subsets Y_i.

First we consider the case when all the edges in $\bigcup_i E(Y_i)$ are blue. In that case we obtain that the set of vertices of the entire graph is a disjoint union of r blue cliques spanned by the sets $X_i \cup Y_i$. Clearly none of these sets contains NIM-K_{r+1} edges. Hence the subgraph of K_n consisting of NIM-K_{r+1} edges is r-partite and therefore has size at most $t_r(n)$, contradiction.

Next, without loss of generality suppose that Y_1 contains a red edge ab. Note that a and b cannot have common red neighbors simultaneously in each set $X_2, \cdots, X_r$, since this will create a red copy of K_{r+1} which uses NIM-K_{r+1} edges. Therefore, for at least one index $i \geqslant 2$, the red neighborhoods of a and b inside X_i are disjoint and thus one of them has size at most r. This implies that either a or b is connected by at least r blue edges with X_i and these edges are obviously not NIM-K_{r+1}.

Now we can bound the number of the potential NIM-K_{r+1} edges in this coloring. H contains $2r^2(r^2 - r)$ such edges. The $n - 2r^2$ vertices of Y can each send $2r(r-1)$ NIM-K_{r+1} edges to vertices of H and at least one vertex of Y (a or b) can send at most $2r(r-1) - r$ such edges. Recall that $f(n, K_{r+1})$ denotes the maximum possible number of NIM-K_{r+1} edges in a 2-edge-coloring of K_n. So there are at most $f(n - 2r^2, K_{r+1})$ NIM-K_{r+1} edges inside Y. Altogether we obtain the following bound:

$$\begin{aligned} f(n, K_{r+1}) &\leqslant f(n - 2r^2, K_{r+1}) + 2r(r-1)(n - 2r^2) - r + 2r^2(r^2 - r) \\ &= f(n - 2r, K_{r+1}) + 2r(r-1)n - 2r^2(r^2 - r) - r. \end{aligned}$$

Let $g(n) = f(n, K_{r+1}) - t_r(n)$. Using the identity

$$t_r(n) - t_r(n - 2r^2) = 2r(r-1)n - 2r^2(r^2 - r),$$

and the previous inequality we obtain that

$$g(n) \leqslant g(n - 2r^2) - r. \tag{1}$$

On the other hand, for every m we have the trivial bound $g(m) \leqslant \binom{m}{2} - t_r(m) < m^2/2r$. Since we are assuming that $g(n) > 0$, by repeatedly applying (1) we conclude

$$0 < g(n) \leqslant g(m) - r\frac{n-m}{2r^2} = g(m) - \frac{n-m}{2r}.$$

This inequality holds provided $n - m$ is divisible by $2r^2$ and $m > e^{12r^2}$ so that we can apply Lemma 3.2. Therefore picking m a little larger than e^{12r^2} such that $n - m$ is divisible by $2r^2$ we obtain $m^2/2r \geqslant g(m) \geqslant \frac{n-m}{2r}$. This contradicts the fact that $n > e^{20r^2}$ and completes the proof of the theorem. $\square$

Now we give the proof of Theorem 1.3. For simplicity we do not calculate the smallest value of n for which the argument works. First we note the following lemma, the proof of which is essentially the same as that of Lemma 3.2.

Lemma 3.3. *Let H have chromatic number $\chi(H) = r + 1 > 2$ and $t > 0$ be an integer. Then there exists $n(H,t)$ so that for all $n > n(H,t)$, every 2-edge-coloring of a complete graph K_n in which there are more than $t_r(n)$ NIM-H edges contains a monochromatic copy of $K_r(t)$ consisting entirely of NIM-H edges.*

Proof of Theorem 1.3. Suppose the result is not true and consider a red–blue edge-coloring of K_n with more than $t_r(n)$ *NIM-H* edges. Let s be the number of vertices of H. By Lemma 3.3 there are sets of vertices $X_1, \dots, X_r$ of size $3s$, such that all edges of the r-partite graph $J = (X_1, \dots, X_r)$ are *NIM-H*, and red, say. Then it is easy to see that all the edges spanned by the sets X_i individually are blue, since every such red edge will form a red copy of H which contains edges of J. Thus each X_i forms a blue clique of size $3s$ and therefore contains no *NIM-H* edges.

Let Y be the set of vertices not in J. Note that no vertex $y \in Y$ can have a red neighbor in every X_i and in addition at least s red neighbors in all but one of the X_i's. Indeed, let vertex y have s red neighbors in X_i for every $1 \leqslant i \leqslant r-1$ and some red neighbor in X_r. Then we can take these neighbors of y together with y itself and $s-2$ other arbitrary vertices in X_r to find a red copy of H which contains edges of J. So we can partition the vertices of Y into $r+1$ classes as follows. For $1 \leqslant i \leqslant r$ let Y_i be the set of vertices $y \in Y$ such that all the edges from y to X_i are blue and i is the smallest index with this property. Note that these edges are not *NIM-H* since they are obviously contained in a blue copy of H. Now let $Y_{r+1} = Y - \bigcup_{i=1}^{r} Y_i$ be the remaining vertices. For any y in Y_{r+1} there exists at least two indices $1 \leqslant i \neq j \leqslant r$ for which y has at most $s-1$ red neighbors in each of X_i and X_j. This gives at least $2s+1$ blue edges from y to each of X_i and X_j, which are not *NIM-H*. For y in Y we let $h(y)$ denote the number of edges from y to J are not *NIM-H*. Then by the above we always have $h(y) \geqslant 3s$, and if $y \in Y_{r+1}$ we have $h(y) > 4s$. We note one further restriction on the numbers $h(y)$. Suppose that Y_1 contains a red edge ab. Then a and b cannot have s common red neighbors simultaneously in each set $X_2, \dots, X_r$, since we can take these neighbors of a, b together with a, b and $s-2$ other arbitrary vertices in X_1 to find a red copy of H which uses *NIM-H* edges. Therefore, for at least one index $i \geqslant 2$, the red neighborhoods of a and b inside X_i have total size at most $4s$. Then the blue neighborhoods inside X_i have total size at least $2s$, and so at least s of the edges from $\{a, b\}$ to X_i must be *NIM-H*. Together with those to X_1 we get $h(a) + h(b) \geqslant 7s$. The same is true for a red edge ab in any Y_i, for $1 \leqslant i \leqslant r$.

First we consider the case when Y_{r+1} is empty and all the edges in $\bigcup_i E(Y_i)$ are blue. In that case we obtain that the set of vertices of the entire graph is a disjoint union of r blue cliques spanned by the sets $X_i \cup Y_i$. Clearly none of these sets contains *NIM-H* edges. Hence the subgraph of K_n consisting of *NIM-H* edges is r-partite and therefore has size at most $t_r(n)$, contradiction.

Otherwise we have $\sum_{y \in Y}(h(y) - 3s) \geqslant s$ and we can bound the total number of *NIM-H* edges in this coloring as follows. Recall that $f(n,H)$ denotes the maximum possible number of such edges in a 2-edge-coloring of K_n, so there are at most

$f(n-3sr, H)$ inside Y. There are at most $\binom{r}{2}(3s)^2$ in J, so in total we have

$$f(n,H) \leqslant f(n-3sr,H) + \binom{r}{2}(3s)^2 + \sum_{y \in Y}(3sr - h(y)).$$

Let $g(n) = f(n,H) - t_r(n)$. Using the identity

$$t_r(n) - t_r(n-3sr) = 3s(r-1)(n-3sr) + \binom{r}{2}(3s)^2,$$

and the previous inequality we obtain that

$$g(n) \leqslant g(n-3sr) - \sum_{y \in Y}(h(y) - 3s) \leqslant g(n-3sr) - s. \quad (2)$$

On the other hand, for every m we have the trivial bound $g(m) \leqslant \binom{m}{2} - t_r(m) < m^2/2r$. Since we are assuming that $g(n) > 0$, by repeatedly applying (2) we conclude

$$0 < g(n) \leqslant g(m) - s\frac{n-m}{3sr} = g(m) - \frac{n-m}{3r}.$$

This inequality holds provided $n-m$ is divisible by $3sr$ and m is large enough that we can apply Lemma 3.2. We choose a minimal such m, which we can bound independently of n. But we obtain $m^2/2r \geqslant g(m) \geqslant \frac{n-m}{3r}$, which is a contradiction for n sufficiently large and completes the proof of the theorem. □

4. Edges not in monochromatic C_4

Finally we present the proof of our last theorem. Let $ex(n, C_4)$ be the maximum number of edges in a C_4-free graph on n vertices. It is well known that $ex(n, C_4) = (1 + o(1))n^{3/2}/2$. But for our purposes we need a lower bound which is true also for small values of n. The following simple construction shows that $ex(n, C_4) \geqslant \lfloor 3n/2 \rfloor - 1$ for every $n \geqslant 7$. If n is odd then we can take $(n-1)/2$ triangles which sharing a common vertex, and if n is even we can take a cycle of length n and add to it the path of length $n/2 - 1$ that connects vertices $1, 3, 5, \ldots, n-1$ along the cycle. It is also rather easy to show that $ex(6, C_4) = 7$.

Recall that given a 2-edge-coloring of the complete graph K_n, we call an edge $NIM\text{-}C_4$ if it is not contained in any monochromatic 4-cycle. First we consider small values of n. For $n \leqslant 5$ obviously there are 2-edge-colorings of K_n without monochromatic C_4, so they will have as many as $\binom{n}{2}$ $NIM\text{-}C_4$ edges. Also one can check (e.g., using a computer search) that $f(6, C_4) = 9 > ex(6, C_4)$. So to finish the proof of Theorem 1.5 it suffices to consider only $n \geqslant 7$.

Proof of Theorem 1.5. Suppose the result is not true and consider a red–blue edge-coloring of K_n with more than $ex(n, C_4)$ $NIM\text{-}C_4$ edges. Then, by definition, the graph of $NIM\text{-}C_4$ edges contains a 4-cycle. Denote its vertices by a, b, c, d and suppose that this is an order in which they appear on the cycle. We consider a few cases according to the colors of the edges of this C_4.

Case 1: There are two edges of each color and they alternate (red/blue) along the cycle. Then the diagonals *ac* and *bd* cannot have the same color, as this would create a monochromatic C_4 with one of the pairs of opposite edges of this 4-cycle. So without loss of generality we may assume that *ab*, *bd* and *cd* are red, and *ac*, *bc* and *da* are blue. Let x be some other vertex of K_n. Then it cannot be the case that both edges *xa* and *xd* are red, as this would create a red C_4 *abdx* using the edge *ab*. We can partition the vertices other than a, b, c, d into two disjoint sets X, Y, where all the edges from a to X are blue and all the edges from a to Y are red. Then by the above discussion all the edges from d to Y are blue. Now for $x \in X$, consider the 4-cycle *xacb*. Since *bc* is a *NIM*-C_4 edge we conclude that *xb* is red. Similarly, cycle *xbdc* shows that all the edges from c to X are blue and cycle *xcad* shows that all the edges from d to X are red. Also, considering cycle *ydac*, $y \in Y$, we obtain that all the edges from c to Y are red, and considering cycle *ycdb* we obtain that all the edges from b to Y are blue. Then it follows that either X or Y is empty since for any edge *xy* one of the cycles *xdcy* or *xady* is monochromatic, contradiction. Without loss of generality assume that $X = \emptyset$. Now note that for any two distinct $y_1, y_2 \in Y$ cycle $y_1 a y_2 c$ is red and cycle $y_1 b y_2 d$ is blue. Therefore all the edges between Y and $\{a, b, c, d\}$ are not *NIM*-C_4 edges. Since b is connected to Y by blue edges and c is connected to Y by red edges, we obtain that *NIM*-C_4 edges inside Y cannot contain a monochromatic path of length 2. Therefore they form two matchings and there are at most $|Y|$ of them. So the total number of *NIM*-C_4 edges is at most $|Y| + 6 = n - 4 + 6 = n + 2$, contradiction.

Case 2: There are two edges of each color but not alternating. So without loss of generality we can assume that *ab*, *ad* are red, and *bc*, *cd* are blue. Note that for any other vertex x, one of the edges *xb*, *xd* must be red and the other blue, since otherwise either a or c together with x, b, d will form a monochromatic C_4. Let X be those vertices x for which *xd* is red and *xb* blue, and Y be those y for which *yd* is blue and *yb* red.

Suppose first that *bd* is blue, the case when it is red can be treated similarly. Then considering the 4-cycle *dbcy*, $y \in Y$ and *xbdc*, $x \in X$ we obtain that all edges from c to Y and X must be red. Now considering cycle *cyba* (or cycle *xdac* if Y is empty) we see that *ac* is blue. Finally considering cycles *acbx*, $x \in X$ and *acdy*, $y \in Y$ we see that all edges from a to X and Y are red. Since *ab* and *ad* are *NIM*-C_4 edges then for x_1, x_2 in X the cycle $x_1 a d x_2$ shows that internal edges of X are blue, the cycle $y_1 a b y_2$ shows that internal edges of Y are blue, and cycle *xaby* shows that edges from X to Y are blue. If both X and Y are nonempty then every cycle of the form *xbdy* is blue. This implies that edge *bd*, all the blue edges from b to X, from d to Y and from X to Y are not *NIM*-C_4. Also red cycles $y_1 c y_2 b$ and $y_1 a y_2 c$ guarantee that if $|Y| > 1$ then all the red edges from Y are also not *NIM*-C_4 and the same is true for X. Moreover any path of length two inside Y together with d will form a blue 4-cycle. So Y contains *NIM*-C_4 edges only if $|Y| = 2$.

Therefore, if both X and Y are nonempty and have size at least two then there are at most 7 *NIM*-C_4 edges, i.e., 2 inside $X \cup Y$ and 5 inside $\{a, b, c, d\}$. If one of them, e.g., $X = \{x\}$ is of size one and another of size at least two then there are 5 *NIM*-C_4 edges inside $\{a, b, c, d\}$, at most one inside Y, and edge *xd*, since the edges *xa* and *xc* form a red 4-cycle with any vertex from Y. Altogether we have 7 NIM-C_4 edges. Finally if X or Y is empty, then another has size at least three and thus only the 6 edges inside $\{a,b,c,d\}$ are NIM-C_4 edges. All these three possibilities lead to contradiction.

Case 3: There are three edges of one color and one of the other. Say there is just one blue edge. Then at least one of the diagonals of this 4-cycle is blue. First consider the case when both diagonals are blue. Then by symmetry we can assume that da, ab, bc are red and bd, cd, ac are blue. Let x be some other vertex. The cycle $xdab$ shows that at least one of xd, xb is blue. So we can partition the vertices other than a,b,c,d into two disjoint set X and Y so that all edges from d to X are blue, from d to Y are red and thus all the edges from b to Y are blue.

Considering cycle $cdby$ for some $y\in Y$ we obtain that all edges from c to Y are red. A cycle $abcy$ shows that all edges from a to Y are blue. Next $acdx, x\in X$ shows that all the edges from a to X are red, $abcx$ shows that all the edges from c to X are blue and cycle $bdcx$ shows that all the edges from b to X are red. Also note that for any pair $x_1, x_2\in X$ at least one of 4-cycles x_1x_2cd and x_1x_2ab is monochromatic. Therefore X has size at most one. If indeed $X=\{x\}$, then cycle $xady$ guarantee that all the edges from x to Y are blue and form blue 4-cycles $xcay$ and $xdby$. Therefore these edges together with xc, xd, ac and bd are not NIM-C_4. Since b is connected to Y by blue edges and c is connected to Y by red edges, we obtain that NIM-C_4 edges inside Y cannot contain a monochromatic path of length 2. Therefore they form two matchings and there are at most $|Y|$ of them. Also we have that $|Y|\geqslant 2$ and therefore using cycles y_1dy_2c and y_1ay_2b we conclude that all edges from Y to a,b,c,d are not NIM-C_4. So the total number of NIM-C_4 edges when $|X|=1$ is at most $|Y|+2+4=n-5+6=n+1$. Similarly if X is empty we obtain that the total number of NIM-C_4 edges is bounded by $|Y|+6=n+2$. In both cases it is at most $ex(n, C_4)$, contradiction.

Finally consider the second possibility when one diagonal of C_4 is red and another is blue. Again by symmetry we can assume that da, ab, bc, ac are red and bd, cd are blue. Then it is easy to see that for any other vertex x the edges xb and xc cannot have the same color. Let X be those vertices for which xb is blue, xc is red and Y be those vertices for which yb is red, yc is blue. Similarly as before, considering cycles $xcad$, $xcba$ and $ybca$, $ybad$ we obtain that all the edges from a,d to X and Y are blue. Note also that one of the cycles $xcby$ and $xdcy$ is always monochromatic. Therefore there are no edges from X to Y and thus one of them is empty. If $X=\emptyset$ then $|Y|\geqslant 3$, and the cycle y_1cdy_2 shows that all the edges inside Y are red. Therefore they form a red 4-cycle together with vertex b. Also cycles y_1cy_2d and y_1ay_2d shows that the blue edges from Y are not NIM-C_4. So the total number of NIM-C_4 edges is at most 6, contradiction.

Now let $Y=\emptyset$ and $|X|\geqslant 3$. If all the edges inside X are red then we can finish the proof as in the previous paragraph, so X has at least one blue edge. Then cycles x_1bx_2a, x_1bx_2d and x_1bdx_2 show that all the edges from a,b,d to X, edge bd and all the blue edges inside X are not NIM-C_4. Since c is connected to X by red edges we obtain that the red NIM-C_4 edges inside X cannot contain a monochromatic path of length 2. Therefore they form a matching and there are at most $\lfloor |X|/2 \rfloor$ of them. In addition, there are 5 NIM-C_4 edges inside $\{a, b, c, d\}$ and $|X|$ such edges from c to X. So the total number of NIM-C_4 edges is at most $5 + |X| + \lfloor |X|/2 \rfloor = \lfloor 3(n-4)/2 \rfloor + 5 = \lfloor 3n/2 \rfloor - 1 \leqslant ex(n, C_4)$, contradiction.

This covers all cases and completes the proof of the theorem. □

5. Concluding remarks

In this paper we study the maximum possible number of edges not covered by a monochromatic copy of a fixed graph H in a 2-edge-coloring of K_n. Clearly this number is always at least the Turán number of H. In the cases when H is a clique or a 4-cycle, or more generally any edge-color-critical graph we managed to prove that for sufficiently large n this number is actually equal to $ex(n, H)$. This suggests the following intriguing open problem.

Problem 5.1. *Let H be a fixed graph. Is it true that for n sufficiently large, $f(n, H) = ex(n, H)$?*

Note that using a slight adaptation of Lemma 3.2 one can prove that $f(n, H)$ equals $ex(n, H)$ at least asymptotically for every non-bipartite graph H. Indeed, let H be a fixed graph with chromatic number $r + 1$. Then by Theorem 3.1, $ex(n, H) = (\frac{r-1}{2r} + o(1))n^2$. Say H has m vertices, so that H is a subgraph of the complete $(r+1)$-partite graph $K_{r+1}(m)$. Let $t = 2(m+1)m!4^{m(m-1)}$ and suppose that there is a 2-edge-coloring of K_n with at least $(\frac{r-1}{2r} + \delta)n^2$ NIM-H edges for some $\delta > 0$. Then by Theorem 3.1, if n is sufficiently large then the set of NIM-H edges will contain a complete $(r+1)$-partite graph $K_{r+1}(t)$. Now applying the argument of Lemma 3.2 we deduce that this $K_{r+1}(t)$ contains a monochromatic $K_{r+1}(m)$, and so a monochromatic H. This is a contradiction, so there cannot be so many NIM-H edges.

For bipartite graphs the situation is less clear, as even the asymptotics of the Turán numbers are known only in a few cases. However, it is interesting that we were able to obtain an exact result for C_4 with only a weak lower bound on $ex(n, C_4)$. This suggests that it may be possible to extend our result to some other bipartite graphs such as the even cycles, or complete bipartite graphs.

We were able to determine $f(n, \triangle)$ and $f(n, C_4)$ for all values of n. Nevertheless, for general complete graphs K_r we only prove the results for n extremely large. The question arises of finding the correct order of magnitude for the minimum value of n for which Theorem 1.2 holds. Well-known lower bounds on Ramsey numbers show that such n should be at least exponential in r, and we think that a similar upper bound ought to hold.

Finally, note that in both cases when $H = K_{r+1}$ and $H = C_4$ our proofs show that for sufficiently large n, $f(n, H) = ex(n, H)$ only if the NIM-H edges form an extremal graph, i.e., the H-free graph with $ex(n, H)$ edges. On the other hand it is worth mentioning that a 2-edge-coloring for which $f(n, H) = ex(n, H)$ is not necessarily unique. For example for $H = K_{r+1}$, let the red edges form a Turán graph $T_r(n)$ minus any subgraph with maximum degree one, and let the rest of the edges be blue. Then it is easy to see that any such coloring will still have $t_r(n)$ NIM-H edges.

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