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Rainbow factors in hypergraphs

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ABSTRACT

For any r -graph H , we consider the problem of finding a rainbow H -factor in an r -graph G with large minimum ℓ -degree and an edge-colouring that is suitably bounded. We show that the asymptotic degree threshold is the same as that for finding an H -factor.

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1. Introduction

A fundamental question in Extremal Combinatorics is to determine conditions on a hypergraph G that guarantee an embedded copy of some other hypergraph H . The Turán problem for an r -graph H asks for the maximum number of edges in an H -free r -graph G on n vertices; we usually think of H as fixed and n as large. For $r = 2$ (ordinary graphs) this problem is fairly well understood (except when H is bipartite), but for general r and

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general H we do not even have an asymptotic understanding of the Turán problem (see the survey [11]). For example, a classical theorem of Mantel determines the maximum number of edges in a triangle-free graph on n vertices (it is $\lfloor n^2/4 \rfloor$), but we do not know even asymptotically the maximum number of edges in a tetrahedron-free 3-graph on n vertices. On the other hand, if we seek to embed a spanning hypergraph then it is most natural to consider minimum degree conditions. Such questions are known as Dirac-type problems, after the classical theorem of Dirac that any graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. There is a large literature on such problems for graphs and hypergraphs, surveyed in [15,17,22,29].

One of these problems, finding hypergraph factors, will be our topic for the remainder of this paper. To describe it we introduce some notation and terminology. Let G be an r -graph on $[n] = \{1, \dots, n\}$. For any $L \subseteq V(G)$ the degree of L in G is the number of edges of G containing L . The minimum ℓ -degree $\delta_\ell(G)$ is the minimum degree in G over all $L \subseteq V(G)$ of size ℓ . Let H be an r -graph with $|V(H)| = h \mid n$. A partial H -factor F in G of size m is a set of m vertex-disjoint copies of H in G . If $m = n/h$ we call F an H -factor. We let $\delta_\ell(H, n)$ be the minimum δ such that $\delta_\ell(G) > \delta n^{r-\ell}$ guarantees an H -factor in G . Then the asymptotic ℓ -degree threshold for H -factors is

$$\delta_\ell^*(H) := \lim_{m \rightarrow \infty} \delta_\ell(H, mh). \quad (1)$$

We refer to Section 2.1 in [29] for a summary of the known bounds on $\delta_\ell^*(H)$. As for the Turán problem, $\delta_\ell^*(H)$ is well-understood for graphs [14,16], but there are few results for hypergraphs. Even for perfect matchings (the case when H is a single edge) there are many cases for which the problem remains open (this is closely connected to the Erdős Matching Conjecture [6]).

Note that the limit in the definition of $\delta_\ell^*(H)$ does indeed exist. The proof of this is a relatively straightforward adaptation of the proof of our main theorem which we sketch in Appendix A.

Let us now introduce colours on the edges of G and ask for conditions under which we can embed a copy of H that is *rainbow*, meaning that its edges have distinct colours. Besides being a natural problem in its own right, this general framework also encodes many other combinatorial problems. Perhaps the most well-known of these is the Ryser-Brualdi-Stein Conjecture [3,23,25] on transversals in Latin squares, which is equivalent to saying that any proper edge-colouring of the complete bipartite graph $K_{n,n}$ has a rainbow matching of size $n - 1$. There are several other well-known open problems that can be encoded as finding certain rainbow subgraphs in graphs with an edge-colouring that is locally k -bounded for some k , meaning that each vertex is in at most k edges of any given colour (so $k = 1$ is proper colouring). For example, a recent result of Montgomery, Pokrovskiy and Sudakov [20] shows that any locally k -bounded edge-colouring of K_n contains a rainbow copy of any tree of size at most $n/k - o(n)$, and this implies asymptotic solutions to the conjectures of Ringel [21] on decompositions by trees and of Graham and Sloane [9] on harmonious labellings of trees.

We now consider rainbow versions of the extremal problems discussed above. The rainbow Turán problem for an r -graph H is to determine the maximum number of edges in a properly edge-coloured r -graph G on n vertices with no rainbow H . This problem was introduced by Keevash, Mubayi, Sudakov and Verstraëte [12], who were mainly concerned with degenerate Turán problems (the case of even cycles encodes a problem from Number Theory), but also observed that a simple supersaturation argument shows that the threshold for non-degenerate rainbow Turán problems is asymptotically the same as that for the usual Turán problem, even if we consider locally $o(n)$ -bounded edge-colourings.

For Dirac-type problems, it seems reasonable to make stronger assumptions on our colourings, as we have already noted that even locally bounded colourings of complete graphs encode many problems that are still open. For example, Erdős and Spencer [7] showed the existence of a rainbow perfect matching in any edge-colouring of $K_{n,n}$ that is $(n-1)/16$ -bounded, meaning that are at most $(n-1)/16$ edges of any given colour. Coulson and Perarnau [4] recently obtained a Dirac-type version of this result, showing that any $o(n)$ -bounded edge-colouring of a subgraph of $K_{n,n}$ with minimum degree at least $n/2$ has a rainbow perfect matching. One could consider this a ‘local resilience’ version (as in [27]) of the Erdős-Spencer theorem. This is suggestive of a more general phenomenon, namely that for any Dirac-type problem, the rainbow problem for bounded colourings should have asymptotically the same degree threshold as the problem with no colours. A result of Yuster [28] on H -factors in graphs adds further evidence (but only for the weaker property of finding an H -factor in which each copy of H is rainbow). For graph problems, the general phenomenon was recently confirmed in considerable generality by Glock and Joos [8], who proved a rainbow version of the blow-up lemma of Komlós, Sárközy and Szemerédi [13] and the Bandwidth Theorem of Böttcher, Schacht and Taraz [2].

Our main result establishes the same phenomenon for hypergraph factors. We will use the following boundedness assumption for our colourings, in which we include the natural r -graph generalisations of both the local boundedness and boundedness assumptions from above (for $r = 2$ boundedness implies local boundedness, but in general they are incomparable assumptions).

Definition 1.1. An edge-colouring of an r -graph on n vertices is μ -bounded if for every colour c :

- i. there are at most μn^{r-1} edges of colour c ,
- ii. for any set I of $r-1$ vertices, there are at most μn edges of colour c containing I .

Note that we cannot expect any result without some “global” condition as in Definition 1.1.i, since any H -factor contains linearly many edges. Similarly, some “local” condition along the lines of Definition 1.1.ii is also needed. Indeed, consider the edge-colouring of the complete r -graph K_n^r by $\binom{n}{r-1}$ colours identified with $(r-1)$ -subsets

of $[n]$, where each edge is coloured by its $r - 1$ smallest elements. Suppose H has the property that every $(r - 1)$ -subset of $V(H)$ is contained in at least 2 edges of H (e.g. suppose H is also complete). Then there are fewer than n edges of any given colour, but there is no rainbow copy of H (let alone an H -factor), as in any embedding of H all edges containing the $r - 1$ smallest elements have the same colour.

Our main theorem is as follows (we use the notation $a \ll b$ to mean that for any $b > 0$ there is some $a_0 > 0$ such that the statement holds for $0 < a < a_0$).

Theorem 1.2. *Let $1/n \ll \mu \ll \varepsilon \ll 1/h \leq 1/r < 1/\ell \leq 1$ with $h|n$. Let H be an r -graph on h vertices and G be an r -graph on n vertices with $\delta_\ell(G) \geq (\delta_\ell^*(H) + \varepsilon)n^{r-\ell}$. Then any μ -bounded edge-colouring of G admits a rainbow H -factor.*

Throughout the remainder of the paper we fix $\ell, r, h, \varepsilon, \mu, n, H$ and G as in the statement of Theorem 1.2. We also fix an integer m with $\mu \ll 1/m \ll \varepsilon$ and define $\gamma = (mh)^{-m}$.

2. Proof modulo lemmas

The outline of the proof of Theorem 1.2 is the same as that given by Erdős and Spencer [7] for the existence of Latin transversals: we consider a uniformly random H -factor \mathcal{H} in G (there is at least one by definition of $\delta_\ell^*(H)$) and apply the Lopsided Lovász Local Lemma (Lemma 3.2) to show that \mathcal{H} is rainbow with positive probability. We will show that the local lemma hypotheses hold if there are many feasible switchings, defined as follows.

Definition 2.1. Let F_0 be an H -factor in G and $H_0 \in F_0$. An (H_0, F_0) -switching is a partial H -factor Y in G with $V(H_0) \subseteq V(Y)$ such that

- i. for each $H' \in F_0$ we have $V(H') \subseteq V(Y)$ or $V(H') \cap V(Y) = \emptyset$, and
- ii. each $H' \in Y$ shares at most one vertex with H_0 .

Let Y' be obtained from Y by deleting all vertices in $V(H_0)$ and their incident edges. We call Y feasible if Y' is rainbow and does not share any colour with any $H' \in F_0 \setminus V(Y)$.

The idea of the switching defined above is that we may replace a small number of copies of H in the H -factor F_0 with different copies in order to remove the “bad copy” H_0 which prevents F_0 from being rainbow. See Fig. 1 for an example of a switching.

The following lemma, proved in Section 4, reduces the proof of Theorem 1.2 to showing the existence of many feasible switchings.

Lemma 2.2. *Suppose that for every H -factor F_0 of G and $H_0 \in F_0$ there are at least γn^{m-1} feasible (H_0, F_0) -switchings of size m . Then G has a rainbow H -factor.*

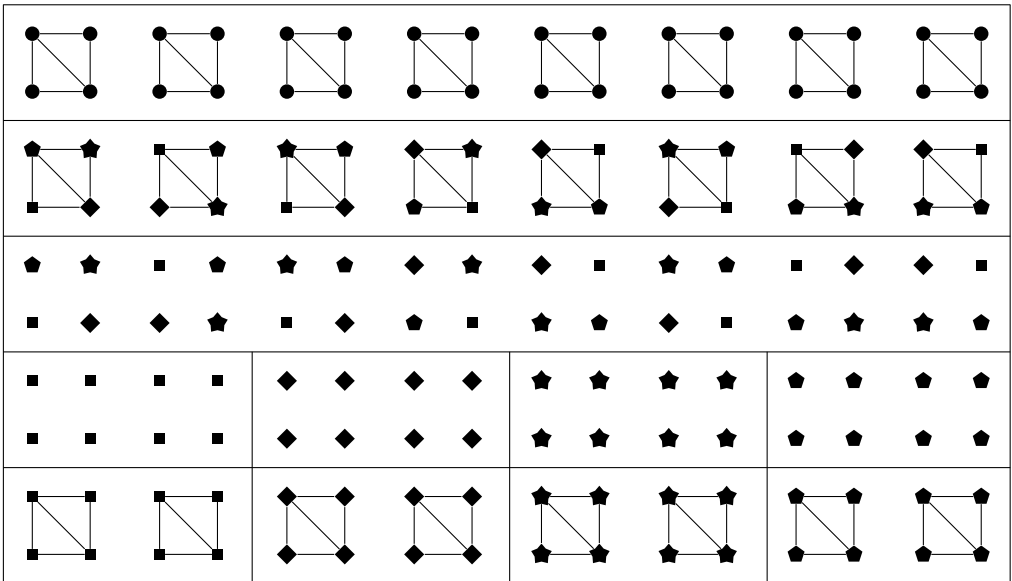


Fig. 1. The process of an (H_0, F_0) -switching of size 8. We start with a partial H -factor of size 8 in the first line, produce a transverse partition as seen in the fourth line, and pick a new partial H -factor within each part of the transverse partition in the fifth line.

Note the exponent $m - 1$ in Lemma 2.2 comes from the fact that we use $m - 1$ additional copies of H in addition to H_0 in order to perform an (H_0, F_0) -switching of size m . Thus Lemma 2.2 states that a constant fraction of candidate switchings being feasible suffices to find a rainbow H -factor.

We will construct switchings by randomly choosing some copies of H from F_0 and considering a random transverse partition in the sense of the following definition.

Definition 2.3. Let F_0 be an H -factor in G and $H_0 \in F_0$. Let $X \subseteq F_0$ be a partial H -factor in G with $H_0 \in X$. We call $S \subseteq V(X)$ transverse if $|H' \cap S| \leq 1$ for all $H' \in X$. We call a partition of $V(X)$ transverse if each part is transverse. For any edges e and f let $X(e, f) = \{H' \in X : |V(H') \cap (e \cup f)| \geq 2\}$. We call X suitable if

- i. for any transverse $I \subseteq V(X) \setminus V(H_0)$ with $|I| = r - 1$ there are at most $\varepsilon|X|/4$ vertices $v \in V(X)$ such that $I \cup \{v\} \in E(G)$ shares a colour with some $H' \in F_0$, and
- ii. for any transverse edges e and f disjoint from $V(H_0)$ of the same colour we have $X(e, f) \neq \emptyset$, and furthermore if $e \cap f = \emptyset$ then $|X(e, f)| \geq 2$.

The following lemma, proved in Section 5, shows that a suitable partial H -factor has an associated feasible switching if it has a transverse partition whose parts each satisfy the minimum degree condition for an H -factor.

Lemma 2.4. *Let F_0, H_0 and X be as in Definition 2.3, suppose X is suitable and $|X| = m$. Let $\mathcal{P} = (V_1, \dots, V_h)$ be a transverse partition of $V(X)$ and suppose $\delta_\ell(G[V_i]) \geq (\delta_\ell^*(H) + \varepsilon/2)m^{r-\ell}$ for all $i \in [h]$. Then there is a partial H -factor Y in G with $V(Y) = V(X)$ such that Y is a feasible (H_0, F_0) -switching.*

The following lemma, proved in Section 6, gives a lower bound on the number of partial H -factors X with some transverse partition \mathcal{P} satisfying the conditions of the previous lemma.

Lemma 2.5. *Let F_0 be an H -factor in G and $H_0 \in F_0$. Let $X \subseteq F_0$ be a random partial H -factor where $H_0 \in X$ and each $H' \in F_0 \setminus \{H_0\}$ is included independently with probability $p = \frac{m}{n/h-1}$. Let $\mathcal{P} = (V_1, \dots, V_h)$ be a transverse partition of $V(X)$ chosen uniformly at random. Then with probability at least $1/m$ we have X suitable, $|X| = m$ and all $\delta_\ell(G[V_i]) \geq (\delta_\ell^*(H) + \varepsilon/2)m^{r-\ell}$.*

We conclude this section by showing how Theorem 1.2 follows from the above lemmas.

Proof of Theorem 1.2. By Lemma 2.2, it suffices to show that for every H -factor F_0 of G and $H_0 \in F_0$ there are at least γn^{m-1} feasible (H_0, F_0) -switchings of size m . There are $\binom{n/h-1}{m-1} \geq (n/mh - 1)^{m-1}$ partial H -factors X of size m with $H_0 \in X \subseteq F_0$. By Lemma 2.5, at least $m^{-1}(n/mh - 1)^{m-1} > \gamma n^{m-1}$ of these are suitable and have a transverse partition $\mathcal{P} = (V_1, \dots, V_h)$ with all $\delta_\ell(G[V_i]) \geq (\delta_\ell^*(H) + \varepsilon/2)m^{r-\ell}$. By Lemma 2.4, each such X has an associated feasible (H_0, F_0) -switching. \square

3. Probabilistic methods

In this section we collect various probabilistic tools that will be used in the proofs of the lemmas stated in the previous section. We start with a general version of the local lemma which follows easily from that given by Spencer [24].

Definition 3.1. Let \mathcal{E} be a set of events in a finite probability space. Suppose Γ is a graph with $V(\Gamma) = \mathcal{E}$ and $p \in [0, 1]^\mathcal{E}$. We call Γ a p -dependency graph for \mathcal{E} if for every $E \in \mathcal{E}$ and $\mathcal{E}' \subseteq \mathcal{E}$ such that $EE' \notin E(\Gamma)$ for all $E' \in \mathcal{E}'$ and $\mathbb{P}[\bigcap_{E' \in \mathcal{E}'} \overline{E'}] > 0$, we have $\mathbb{P}[E | \bigcap_{E' \in \mathcal{E}'} \overline{E'}] \leq p_E$.

Lemma 3.2. *Under the setting of Definition 3.1, if $\sum\{p_{E'} : EE' \in E(\Gamma)\} \leq 1/4$ for all $E \in \mathcal{E}$ then with positive probability none of the events in \mathcal{E} occur.*

We also require Talagrand’s Inequality, see e.g. [19, page 81].

Lemma 3.3. *Let $X \geq 0$ be a random variable determined by n independent trials which is:*

c-Lipschitz. Changing the outcome of any one trial can affect X by at most c .

r-certifiable. If $X \geq s$ then there is a set of at most rs trials whose outcomes certify $X \geq s$.

Then for any $0 \leq t \leq \mathbb{E}[X]$,

$$\mathbb{P}[|X - \mathbb{E}[X]| > t + 60c\sqrt{r\mathbb{E}[X]}] \leq 4e^{-t^2/(8c^2r\mathbb{E}[X])}.$$

Next we state an inequality of Janson [10].

Definition 3.4. Let $\{I_i\}_{i \in \mathcal{I}}$ be a finite family of indicator random variables. We call a graph Γ on \mathcal{I} a strong dependency graph if the families $\{I_i\}_{i \in A}$ and $\{I_i\}_{i \in B}$ are independent whenever A and B are disjoint subsets of \mathcal{I} with no edge of Γ between A and B .

Theorem 3.5. In the setting of Definition 3.4, let $S = \sum_{i \in \mathcal{I}} I_i$, $\mu = \mathbb{E}[S]$, $\delta = \max_{i \in \mathcal{I}} \sum \{p_j : ij \in E(\Gamma)\}$ and $\Delta = \sum \{\mathbb{E}[I_i I_j] : ij \in E(\Gamma)\}$. Then for any $0 < \eta < 1$,

$$\mathbb{P}[S < (1 - \eta)\mu] \leq \exp(-\min\{(\eta\mu)^2/(8\Delta + 2\mu), \eta\mu/(6\delta)\}).$$

We conclude with a standard bound on the probability that a binomial is equal to its mean.

Lemma 3.6. Let X be a binomial random variable with parameters n and p such that $np = m \in \mathbb{N}$ and $m^2 = o(n)$. Then $\mathbb{P}[X = m] \geq 1/(4\sqrt{m})$.

Proof. The stated bound follows from $\mathbb{P}[X = m] = \binom{n}{m} p^m (1-p)^{n-m} \geq m!^{-1} (n-m)^m p^m (1-p)^{n-m} = m!^{-1} m^m (1-p)^n, (1-p)^n = e^{-np+O(np^2)}$ and $m! \leq e^{1-m} m^{m+1/2}$. \square

4. Applying the local lemma

In this section we prove Lemma 2.2, which applies the local lemma to reduce the proof of Theorem 1.2 to finding many feasible switchings.

Proof of Lemma 2.2. Suppose that for every H -factor F_0 of G and $H_0 \in F_0$ there are at least γn^{m-1} feasible (H_0, F_0) -switchings of size m . We need to show that G has a rainbow H -factor.

We will apply Lemma 3.2 to a uniformly random H -factor \mathcal{H} in G , where $\mathcal{E} = \mathcal{A} \cup \mathcal{B}$ consists of all events of the following two types. For every copy H_0 of H in G and any two edges e and f in H_0 of the same colour we let $A(e, f, H_0)$ be the event that $H_0 \in \mathcal{H}$; we let $\mathcal{A} = \{A(e, f, H_0) : \mathbb{P}[A(e, f, H_0)] > 0\}$. For every pair H_1, H_2 of vertex-disjoint copies of H in G and edges e_1 of H_1 and e_2 of H_2 of the same colour we let $B(e_1, e_2, H_1, H_2)$ be the event that $H_1 \in \mathcal{H}$ and $H_2 \in \mathcal{H}$; we let

$\mathcal{B} = \{B(e_1, e_2, H_1, H_2) : \mathbb{P}[B(e_1, e_2, H_1, H_2)] > 0\}$. Then \mathcal{H} is rainbow iff none of the events in \mathcal{E} occur.

We define the supports of $A = A(e, f, H_0)$ as $\text{supp}(A) = V(H_0)$ and of $B = B(e_1, e_2, H_1, H_2)$ as $\text{supp}(B) = V(H_1) \cup V(H_2)$. Let Γ be the graph on $\mathcal{A} \cup \mathcal{B}$ where $E_1, E_2 \in V(\Gamma)$ are adjacent if and only if $\text{supp}(E_1) \cap \text{supp}(E_2) \neq \emptyset$. Our goal is to show that there exist suitably small p_A, p_B such that Γ is a p -dependency graph for $\mathcal{A} \cup \mathcal{B}$, where $p_A = p_A$ for all $A \in \mathcal{A}$ and $p_B = p_B$ for all $B \in \mathcal{B}$. For $\mathcal{X} \in \{\mathcal{A}, \mathcal{B}\}$, let $d_{\mathcal{X}}$ be the maximum over $E \in V(\Gamma)$ of the number of neighbours of E in \mathcal{X} . To apply Lemma 3.2, it suffices to show $p_A d_A + p_B d_B \leq 1/4$.

To bound the degrees, we will first estimate the number of events in \mathcal{A} and \mathcal{B} whose support contains any fixed vertex $v \in V(G)$. We claim that there are at most $2^{r+1} h! \mu n^{h-1}$ events $A(e, f, H_0) \in \mathcal{A}$ with $v \in V(H_0)$. To see this, first consider those events with $v \notin e \cup f$. For any $s < r$, as the colouring is μ -bounded, the number of choices of e and f of the same colour with $|e \cap f| = s$ is at most $n^r \cdot \binom{r}{s} \mu n^{r-s}$. For any such e and f with $v \notin e \cup f$, there are at most $h! n^{h-(2r-s+1)}$ copies of H containing $e \cup f \cup \{v\}$, so summing over s we obtain at most $2^r h! \mu n^{h-1}$ such events. Now we consider events $A(e, f, H_0)$ with $v \in e \cup f$. The number of choices of e and f of the same colour with $|e \cap f| = s$ and $v \in e \cup f$ is at most $n^{r-1} \cdot \binom{r}{s} \mu n^{r-s}$. For any such e and f there are at most $h! n^{h-(2r-s)}$ copies of H containing $e \cup f \cup \{v\}$, so summing over s we obtain at most $2^{r+1} h! \mu n^{h-1}$ such events. The claim follows.

Similarly, we claim that there are at most $2(h!)^2 \mu n^{2h-2}$ events $B(e_1, e_2, H_1, H_2) \in \mathcal{B}$ with $v \in V(H_1) \cup V(H_2)$. To see this, first consider those events with $v \in e_1 \cup e_2$. By definition of \mathcal{B} , we may consider only disjoint edges e_1, e_2 . There are at most $h! n^{h-r}$ choices for each of H_1 and H_2 given e_1 and e_2 . Also, the number of choices for e_1 and e_2 is at most $n^{r-1} \cdot \mu n^{r-1} = \mu n^{2r-2}$. Thus, we obtain at most $(h!)^2 \mu n^{2h-2}$ such events. A similar argument applies to events $B(e_1, e_2, H_1, H_2)$ with $v \notin e_1 \cup e_2$, and the claim follows.

In particular, there is some constant $C = C(r, h)$ so that

$$d_A < C \mu n^{h-1} \quad \text{and} \quad d_B < C \mu n^{2h-2}. \tag{2}$$

Now we will bound p_A and p_B using switchings. For p_A we need to bound $\mathbb{P}[A \mid \cap_{E \in \mathcal{E}'} \overline{E}]$ for any $A = A(e, f, H_0) \in \mathcal{A}$ and $\mathcal{E}' \subseteq \mathcal{E}$ such that $AE \notin E(\Gamma)$ for all $E \in \mathcal{E}'$ and $\mathbb{P}[\cap_{E \in \mathcal{E}'} \overline{E}] > 0$. Let \mathcal{F} be the set of H -factors of G that satisfy $\cap_{E \in \mathcal{E}'} \overline{E}$; then $\mathcal{F} \neq \emptyset$ as $\mathbb{P}[\cap_{E \in \mathcal{E}'} \overline{E}] > 0$. Let $\mathcal{F}_0 = \{F_0 \in \mathcal{F} : H_0 \in F_0\}$. We consider the auxiliary bipartite multigraph \mathcal{G}_A with parts $(\mathcal{F}_0, \mathcal{F} \setminus \mathcal{F}_0)$, where for each $F_0 \in \mathcal{F}_0$ and feasible (H_0, F_0) -switching Y of size m we add an edge from F_0 to F obtained by replacing $F_0[V(Y)]$ with Y ; we note that $F \in \mathcal{F} \setminus \mathcal{F}_0$ as Y is rainbow and shares no colours with $H' \in F_0 \setminus V(Y)$ by Definition 2.1, hence F still satisfies $\cap_{E \in \mathcal{E}'} \overline{E}$. Let δ_A be the minimum degree in \mathcal{G}_A of vertices in \mathcal{F}_0 and Δ_A be the maximum degree in \mathcal{G}_A of vertices in $\mathcal{F} \setminus \mathcal{F}_0$. By double-counting the edges of \mathcal{G}_A we obtain $\mathbb{P}[A \mid \cap_{E \in \mathcal{E}'} \overline{E}] = |\mathcal{F}_0|/|\mathcal{F}| \leq \Delta_A/\delta_A$.

We therefore need an upper bound for Δ_A and a lower bound for δ_A . By the hypotheses of the lemma, we have $\delta_A \geq \gamma n^{m-1}$. To bound Δ_A , we fix any $F \in \mathcal{F} \setminus \mathcal{F}_0$ and bound the number of pairs (F_0, Y) where $F_0 \in \mathcal{F}_0$ and Y is a feasible (H_0, F_0) -switching of size m that produces F . Each vertex of $V(H_0)$ must belong to a different copy of H in F , as otherwise there are no (H_0, F_0) -switchings that could produce F . Thus we identify h copies of H in F whose vertex set must be included in $V(Y)$. There are at most n^{m-h} choices for the other copies of H to include in $V(Y)$ and then at most $(hm)!$ choices for Y , so $\Delta_A \leq (hm)!n^{m-h}$. We deduce

$$\mathbb{P}[A \mid \cap_{E \in \mathcal{E}'} \overline{E}] \leq (hm)! \gamma^{-1} n^{1-h} =: p_A. \tag{3}$$

The argument to bound p_B is very similar. We need to bound $\mathbb{P}[B \mid \cap_{E \in \mathcal{E}'} \overline{E}]$ for any $B = B(e_1, e_2, H_1, H_2) \in \mathcal{B}$ and $\mathcal{E}' \subseteq \mathcal{E}$ such that $BE \notin E(\Gamma)$ for all $E \in \mathcal{E}'$ and $\mathbb{P}[\cap_{E \in \mathcal{E}'} \overline{E}] > 0$. Let \mathcal{F} be the set of H -factors of G that satisfy $\cap_{E \in \mathcal{E}'} \overline{E}$; then $\mathcal{F} \neq \emptyset$. Let $\mathcal{F}' = \{F' \in \mathcal{F} : \{H_1, H_2\} \subseteq F'\}$. We consider the auxiliary bipartite multigraph \mathcal{G}_B with parts $(\mathcal{F}', \mathcal{F} \setminus \mathcal{F}')$, where there is an edge from $F' \in \mathcal{F}'$ to $F \in \mathcal{F} \setminus \mathcal{F}'$ for each pair (Y, Z) , where Y is a feasible (H_1, F') -switching of size m producing some H -factor F'' containing H_2 but not H_1 , and Z is a feasible (H_2, F'') -switching of size m with $V(Z) \cap V(H_1) = \emptyset$ producing F ; note that then $F \in \mathcal{F} \setminus \mathcal{F}'$.

We have $\mathbb{P}[B \mid \cap_{E \in \mathcal{E}'} \overline{E}] \leq \Delta_B / \delta_B$, where Δ_B and δ_B are defined analogously to Δ_A and δ_A . The condition $V(Z) \cap V(H_1) = \emptyset$ rules out at most hn^{m-2} choices of Z given H_1 , and similarly the condition that F'' contains H_2 and not H_1 rules out at most hn^{m-2} choices of Y given H_2 . So $\delta_B \geq (\gamma n^{m-1} - hn^{m-2})^2 > \frac{1}{2} \gamma^2 n^{2m-2}$. Similarly to before we have $\Delta_B \leq ((hm)!n^{m-h})^2$, so

$$\mathbb{P}[B \mid \cap_{E \in \mathcal{E}'} \overline{E}] \leq 2(hm)!^2 \gamma^{-2} n^{2-2h} =: p_B. \tag{4}$$

Combining (2), (3) and (4) we have $p_A d_A + p_B d_B \leq 1/4$, so the lemma follows from Lemma 3.2. \square

5. Switchings

In this section we prove Lemma 2.4, which shows how to obtain a feasible switching from a suitable partial H -factor and transverse partition whose parts have high minimum degree.

Proof of Lemma 2.4. Let F_0 be an H -factor in G and $H_0 \in F_0$. Let $X \subseteq F_0$ be a suitable partial H -factor in G of size m with $H_0 \in X$. Let $\mathcal{P} = (V_1, \dots, V_h)$ be a transverse partition of $V(X)$ such that all $\delta_\ell(G[V_i]) \geq (\delta_\ell^*(H) + \varepsilon/2)m^{r-\ell}$. We need to find a partial H -factor Y in G with $V(Y) = V(X)$ such that Y is a feasible (H_0, F_0) -switching.

We construct Y by successively choosing H -factors Y_i of $G[V_i]$ for $1 \leq i \leq h$. For each i we let $V'_i = V_i \setminus V(H_0)$ and we will show that $G[V'_i]$ is rainbow by Definition 2.3.ii.

This is because every subset of V'_i is transverse by definition. However, if edges e and f are both transverse and have the same colour then by Definition 2.3.ii their union is not transverse. At step i , we let G_i be the r -graph obtained from $G[V_i]$ by deleting all edges disjoint from $V(H_0)$ that share a colour with any H' in F_0 or $\cup_{j<i} Y_j$. It suffices to show that G_i has an H -factor Y_i , as then $Y = \cup_{i=1}^h Y_i$ will be feasible.

By definition of $\delta_\ell^*(H)$, it suffices to show for each $L \subseteq V_i$ with $|L| = \ell$ that we delete at most $\frac{\varepsilon}{2} m^{r-\ell}$ edges containing L . We can assume L is disjoint from $V(H_0)$, as otherwise we do not delete any edges containing L . There are $\binom{m-\ell}{r-1-\ell}$ choices of I of size $r-1$ with $L \subseteq I \subseteq V_i$. For each such I , by Definition 2.3.i, the number of edges containing I deleted due to sharing a colour with any $H' \in F_0$ is at most $\varepsilon m/4$. Thus we delete at most $\frac{\varepsilon}{4} m^{r-\ell}$ such edges containing L .

It remains to consider edges containing L that are deleted due to sharing a colour with any H' in $\cup_{j<i} Y_j$. As $G[V'_i]$ is rainbow, any colour in $\cup_{j<i} Y_j$ accounts for at most one deleted edge. In the case $\ell \leq r-2$ we can crudely bound the number of deleted edges by the total number of edges in $\cup_{j<i} Y_j$, which is at most $ie(H)m < mh^{r+1} < \frac{\varepsilon}{4} m^{r-\ell}$.

Now we may suppose $\ell = r-1$. Consider any edge e containing L that is deleted due to having the same colour as some edge f in some Y_j with $j < i$. By Definition 2.3.ii and $|e \setminus L| = 1$ there is a copy H' of H in X that intersects both L and f . To bound the number of choices for e , note that there are $|L| = r-1$ choices for H' and $i-1$ choices for j . These choices determine a vertex in V_j , and so a copy of H in Y_j , which contains at most h^{r-1} choices for f . Then the colour of f determines at most one deleted edge in e . Thus the number of such deleted edges e containing L is at most $(r-1)(i-1)h^{r-1} < \frac{\varepsilon}{4} m$, as required. \square

6. Transverse partitions

To complete the proof of Theorem 1.2, it remains to prove Lemma 2.5, which bounds the probability that a random partial H -factor and transverse partition satisfy the hypotheses of Lemma 2.4.

Proof of Lemma 2.5. Let F_0 be an H -factor in G and $H_0 \in F_0$. Let $X \subseteq F_0$ be a random partial H -factor where $H_0 \in X$ and each $H' \in F_0 \setminus \{H_0\}$ is included independently with probability $p = \frac{m-1}{n/h-1} \leq \frac{hm}{n}$. Let $\mathcal{P} = (V_1, \dots, V_h)$ be a uniformly random transverse partition of $V(X)$. Note that each copy H' of H in X has one vertex in each V_i , according to a uniformly random bijection between $V(H')$ and $[h]$, and that these bijections are independent for different choices of H' . Consider the events

$$\begin{aligned} \mathcal{E}_1 &= \{|X| = m\}, & \mathcal{E}_2 &= \{X \text{ satisfies Definition 2.3.ii}\}, \\ \mathcal{E}_3 &= \{X \text{ satisfies Definition 2.3.i}\}, & \mathcal{E}_4 &= \cap_{i=1}^h \{\delta_\ell(G[V_i]) \geq (\delta_\ell^*(H) + \varepsilon/2)m^{r-\ell}\}. \end{aligned}$$

We need to show that $\mathbb{P}[\cap_{i=1}^4 \mathcal{E}_i] > 1/m$. To do so, we first recall from Lemma 3.6 that $\mathbb{P}[\mathcal{E}_1] \geq 1/(4\sqrt{m})$. To complete the proof, we will show that $\mathbb{P}[\mathcal{E}_i] \geq 1 - 1/m$ for

$i = 2, 3, 4$. Throughout, for $I \subseteq V(G)$ we let $F_I \subseteq F_0$ be the partial H -factor consisting of all copies of H in F_0 that intersect I .

Bounding $\mathbb{P}[\mathcal{E}_2]$.

For $s \in [r - 1]$ let \mathcal{Z}_s be the set of pairs (e, f) of transverse edges disjoint from $V(H_0)$ of the same colour with $|e \cap f| = s$ and $X(e, f) = \emptyset$. As the colouring is μ -bounded, we have $|\mathcal{Z}_s| \leq n^r \cdot \binom{r}{s} \mu n^{r-s}$. For any $(e, f) \in \mathcal{Z}_s$ we have $|F_{e \cup f}| = 2r - s$, so $\mathbb{P}[e \cup f \subseteq V(X)] = p^{2r-s}$. By a union bound, the probability that any such event holds is at most $\sum_{s=1}^{r-1} \binom{r}{s} \mu n^{2r-s} p^{2r-s} < (hm)^r (hm + 1)^r \mu < 1/2m$.

Similarly, let \mathcal{Z}_0 be the set of pairs (e, f) of transverse edges disjoint from $V(H_0)$ of the same colour with $e \cap f = \emptyset$ and $|X(e, f)| \leq 1$. As the colouring is μ -bounded, we have $|\mathcal{Z}_0| \leq n^r \cdot \mu n^{r-1}$. For any $(e, f) \in \mathcal{Z}_0$, $|F_{e \cup f}| \geq 2r - 1$ and $\mathbb{P}[e \cup f \subseteq V(X)] \leq p^{2r-1}$. Thus the probability that any such event holds is at most $\mu(hm)^{2r-1} < 1/2m$.

Bounding $\mathbb{P}[\mathcal{E}_3]$.

For any transverse $I \subseteq V(X) \setminus V(H_0)$ with $|I| = r - 1$ we let B_I be the set of $v \in V(G) \setminus (V(F_I) \cup V(H_0))$ such that $I \cup \{v\}$ is an edge sharing a colour with some $H' \in F_0$. Write $Y_I = |V(X) \cap B_I|$. It suffices to bound the probability that there is any $I \subseteq V(X)$ with $Y_I > \varepsilon m/5$. Indeed, the number of $v \in V(F_I) \cup V(H_0)$ such that $I \cup \{v\}$ is an edge is at most $rh < \varepsilon m/20$.

First we show that X is unlikely to contain any I in $\mathcal{B} := \{I : |B_I| > \varepsilon n/10h\}$. Indeed, as the colouring is μ -bounded, there are at most $e(F_0)\mu n^{r-1} = \mu e(H)n^r/h$ edges with colours in F_0 , so $|\mathcal{B}| < \mu \varepsilon^{-2} n^{r-1}$. For each transverse I we have $\mathbb{P}[I \subseteq V(X)] = p^{r-1}$, so by a union bound, the probability that X contains any I in \mathcal{B} is at most $\mu \varepsilon^{-2} (hm)^{r-1} < 1/2m$.

Now for each $I \notin \mathcal{B}$ we bound Y_I by Talagrand’s inequality, where the independent trials are the decisions for each $H' \in F_0 \setminus \{H_0\}$ of whether to include H' in X . As $I \notin \mathcal{B}$ we have $\mathbb{E}[Y_I] = p|B_I| \leq \varepsilon m/10$. Also, Y_I is clearly h -Lipschitz as $|H| = h$ and 1-certifiable as we can simply list the successful trials containing the vertices of $V(X) \cap B_I$. We apply Lemma 3.3 to $Y'_I = Y_I + \varepsilon m/30$, with $t = \varepsilon m/30 \leq \mathbb{E}[Y'_I]$, $c = h$ and $r = 1$ to deduce $\mathbb{P}[Y_I > \varepsilon m/5] \leq 4e^{-10^{-4}h^{-2}\varepsilon^2 m} < m^{-2r}$.

As we excluded $V(F_I)$ from B_I , the events $\{I \subseteq V(X)\}$ and $Y_I > \varepsilon m/5$ are independent, so both occur with probability at most $p^{r-1}m^{-2r}$. Taking a union bound over at most n^{r-1} choices of I , we obtain $\mathbb{P}[\mathcal{E}_3] < 1/m$.

Bounding $\mathbb{P}[\mathcal{E}_4]$.

For $L \subseteq V(G)$ with $|L| = \ell$ and $i \in [h]$ we define

$$\mathcal{J}_L = \{J \subseteq V(G) \setminus V(H_0) : F_L \cap F_J = \emptyset \text{ and } L \cup J \in E(G) \text{ is transverse}\}.$$

We say L is i -bad if $L \subseteq V_i$ and $d'_i(L) := |\{J \in \mathcal{J}_L : J \subseteq V_i\}| < (\delta_\ell^*(H) + \varepsilon/2)m^{r-\ell}$. We will give an upper bound on the probability that there is any i -bad L .

First we note that the events $\{L \subseteq V_i\}$ and $\{J \subseteq V_i\}$ are independent for any $J \in \mathcal{J}_L$. There are at most n^ℓ choices of L with $L \cap V(H_0) = \emptyset$, each of which has $\mathbb{P}[L \subseteq$

$V_i] = (p/h)^\ell$, and at most $hn^{\ell-1}$ choices of L with $|L \cap V(H_0)| = 1$, each of which has $\mathbb{P}[L \subseteq V_i] \leq (p/h)^{\ell-1}$. By a union bound, it suffices to show for every transverse L and $i \in [h]$ that $\mathbb{P}[d'_i(L) < (\delta_\ell^*(H) + \varepsilon/2)m^{r-\ell}] < m^{-2r}$.

We also note that $|\mathcal{J}_L| \geq (\delta_\ell^*(H) + 0.9\varepsilon)n^{r-\ell}$, as there are at least $(\delta_\ell^*(H) + \varepsilon)n^{r-\ell}$ choices of J with $L \cup J \in E(G)$, of which the number excluded due to $J \cap V(H_0) \neq \emptyset$, $F_L \cap F_J \neq \emptyset$ or $L \cup J$ not being transverse is at most $hn^{r-\ell-1} + \ell hn^{r-\ell-1} + \frac{n}{h} \binom{h}{2} n^{r-\ell-2} < 0.1\varepsilon n^{r-\ell}$.

We will apply Janson’s inequality to $d'_i(L) = \sum_{J \in \mathcal{J}_L} I_J$, where each I_J is the indicator of $\{J \subseteq V_i\}$. As $\mathbb{P}[J \subseteq V_i] = (p/h)^{r-\ell}$ for each $J \in \mathcal{J}_L$, we have $\mu = \mathbb{E}[d'_i(L)] > (\delta_\ell^*(H) + 0.9\varepsilon)m^{r-\ell}$. We use the dependency graph Γ where JJ' is an edge iff $F_J \cap F_{J'} \neq \emptyset$. Note that for any $J \in \mathcal{J}_L$ and $s \in [r-\ell]$ the number of choices of J' with $|F_J \cap F_{J'}| = s$ is at most $\binom{r-\ell}{s} h^s n^{r-\ell-s}$, and for each we have $\mathbb{P}[J \cup J' \subseteq V_i] = (p/h)^{2(r-\ell)-s}$. Thus we can bound the parameter Δ in Theorem 3.5 as $\Delta \leq |\mathcal{J}_L| \sum_{s=1}^{r-\ell} \binom{r-\ell}{s} h^s n^{r-\ell-s} (p/h)^{2(r-\ell)-s} \leq m^{r-\ell} \sum_{s=1}^{r-\ell} \binom{r-\ell}{s} h^s m^{r-\ell-s} < 2h(r-\ell)m^{2(r-\ell)-1}$. We also have $\delta \leq \sum_{s=1}^{r-\ell} \binom{r-\ell}{s} h^s \times n^{r-\ell-s} (p/h)^{r-\ell-s} \leq \sum_{s=1}^{r-\ell} \binom{r-\ell}{s} h^s m^{r-\ell-s} < 2h(r-\ell)m^{r-\ell-1}$. By Theorem 3.5, there is some constant $c = c(r, \varepsilon, h)$ independent of m so that $\mathbb{P}[d'_i(L) < (\delta_\ell^*(H) + \varepsilon/2)m^{r-\ell}] < e^{-cm} < m^{-2r}$, as required. \square

7. Concluding remarks

Our result and those of [4,8] suggest that for any Dirac-type problem, the rainbow problem for bounded colourings should have asymptotically the same degree threshold as the problem with no colours. In particular, it may be interesting to establish this for Hamilton cycles in hypergraphs (i.e. a Dirac-type generalisation of [5]). The local resilience perspective emphasises analogies with the recent literature on Dirac-type problems in the random setting (see the surveys [1,26]), perhaps suggests looking for common generalisations, e.g. a rainbow version of [18]: in the random graph $G(n, p)$ with $p > C(\log n)/n$, must any $o(pn)$ -bounded edge-colouring of any subgraph H with minimum degree $(1/2 + o(1))pn$ have a rainbow Hamilton cycle?

Appendix A. Proof of existence for $\delta_\ell^*(H)$

The authors were unable to find a proof that $\lim_{m \rightarrow \infty} \delta_\ell(H, mh)$ exists in prior literature, therefore we provide a proof here where we omit the parts that are analogous to the proofs in this paper.

Lemma A.1. *The limit, $\delta_\ell^* = \lim_{m \rightarrow \infty} \delta_\ell(H, mh)$ exists.*

Proof. To prove the existence of this limit, we show convergence to the \liminf . So, let $\delta_\ell^- = \liminf_{m \rightarrow \infty} \delta_\ell(H, mh)$. Let $\varepsilon > 0$, by definition of the \liminf there exists m_ε such that $\delta_\ell(H, m_\varepsilon h) \leq \delta_\ell^- + \varepsilon$. That is every graph with $m_\varepsilon h$ vertices and minimum ℓ -degree at least $(\delta_\ell^- + \varepsilon)(m_\varepsilon h)^{r-\ell}$ has an H -factor. Pick $n \gg m_\varepsilon h^2$ such that $h|n$ and let G be

any r -graph on n vertices with minimum ℓ -degree at least $(\delta_\ell^- + 2\varepsilon)n^{r-\ell}$. We shall show that G has an H -factor.

If G has an H -factor, then we are done. So suppose for a contradiction that G has no H -factor. Let F_0 be a largest partial H -factor of G . Extend F_0 to F_0^* in G by arbitrarily adding vertex disjoint copies of any graph on h vertices such that F_0^* spans $V(G)$. Let H_0 be any $H' \in F_0^*$ which is not a copy of H . The proof of Lemma 2.5 can be adapted to such F_0^* , hence let $\mathcal{P} = (V_1, \dots, V_h)$ be the random transverse partition provided by the lemma with $m = m_\varepsilon h$. With positive probability, for all $i \in [h]$ we have $|V_i| = m$ and $\delta_\ell(G[V_i]) \geq (\delta_\ell^- + \varepsilon)m^{r-\ell}$. So there exists a set X with $|X| = m$ whose corresponding partition satisfies these properties. By definition of m_ε , we may pick an H -factor in each $G[V_i]$. By removing X from F_0 and adding these new H -factors we obtain a partial H -factor that is strictly larger than F_0 , as H_0 was not a copy of H . This is a contradiction with the maximality of F_0 . Hence G has an H -factor.

Thus, for all $\varepsilon > 0$ we have $\delta_\ell(H, mh) \leq \delta_\ell^- + 2\varepsilon$ provided that m is sufficiently large. In particular, $\limsup_{m \rightarrow \infty} \delta_\ell(H, mh) \leq \delta_\ell^- + 2\varepsilon$ for any $\varepsilon > 0$. Taking $\varepsilon \rightarrow 0$ allows us to deduce that the \liminf and \limsup are both the same and hence the limit in (1) exists. \square

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