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On the minimal degree implying equality of the largest triangle-free and bipartite subgraphs

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Abstract

Erdős posed the problem of finding conditions on a graph G that imply $t(G) = b(G)$, where $t(G)$ is the largest number of edges in a triangle-free subgraph and $b(G)$ is the largest number of edges in a bipartite subgraph. Let δ_c be the least number so that any graph G on n vertices with minimum degree $\delta_c n$ has $t(G) = b(G)$. Extending results of Bondy, Shen, Thomassé and Thomassen we show that $0.75 \leqslant \delta_c < 0.791$.

Keywords: Extremal Graphs Theory; Triangle-free graphs; Turán's theorem

1. Introduction

For which graphs do the largest bipartite subgraph and largest triangle-free subgraph have the same number of edges? This question was raised by Erdős [4], who noted that there is equality for the complete graph K_n (by Turán's theorem). Babai, Simonovits and Spencer [2] showed that equality holds almost surely for the random graph where edges are chosen with probability $1/2$.

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A general condition implying equality was given by Bondy, Shen, Thomassé and Thomassen [3], who showed that a minimum degree condition is sufficient.

For a graph G we write $b(G)$ for the number of edges in its largest bipartite subgraph, and $t(G)$ for the number of edges in its largest triangle-free subgraph. Clearly $t(G) \geqslant b(G)$. Write δ_c for the least number so that, for n sufficiently large, any graph G on n vertices with minimum degree $\delta(G) \geqslant (\delta_c + o(1))n$ has $t(G) = b(G)$. Bondy et al. [3] showed that $0.675 \leqslant \delta_c \leqslant 0.85$. We will strengthen this as follows.

Theorem 1.1. $0.75 \leqslant \delta_c < 0.791$.

Moreover, we believe that the lower bound is tight and propose the following conjecture.

Conjecture 1.2. *In any graph on n vertices with minimum degree at least $(3/4 + o(1))n$ the largest triangle-free and largest bipartite subgraphs have equal size.*

This paper is organised as follows. In the next section we will describe some properties of triangle-free graphs under certain minimum degree conditions. Section 3 contains a proof of a slightly weaker form of Theorem 1.1, in which we relax the upper bound to $\delta_c \leqslant 0.8$. This contains the main ideas of the proof, but the bound of 0.791 is more involved, so we defer it to Section 4. In Section 5 we prove a technical lemma needed in Section 4. The final section contains some concluding remarks.

Notation. We usually write $G = (V, E)$ for a graph G with vertex set $V = V(G)$ and edge set $E = E(G)$, setting $n = |V|$ and $e = e(G) = |E(G)|$. If $X \subset V$ is a subset of the vertex set then $G[X]$ denotes the restriction of G to X, i.e. the graph on X whose edges are those edges of G with both endpoints in X. We will also write $e_G(X) = e(G[X])$. Similarly, we write $e_G(X, Y)$ for the number of edges with one endpoint in X and the other in Y. We will usually omit the subscript G unless there is possibility for confusion. The neighbourhood of a vertex v is $N(v)$, and adjacency of u and v is denoted by $u \sim v$.

We will assume throughout the paper that n is sufficiently large. To improve readability we will omit 'floor' and 'ceiling' signs, and all inequalities will be understood to hold up to an additive error of $o(1)$, i.e. a quantity that tends to zero as n tends to infinity.

2. Preliminaries

We start by describing the structure of triangle-free graphs with high minimal degrees. For $d \geqslant 1$ we define a graph F_d as follows. The vertex set $V(F_d)$ consists of the integers modulo $3d - 1$, which we denote by $\mathbb{Z}_{3d-1}$. The vertex $v \in \mathbb{Z}_{3d-1}$ is adjacent to the vertices $v + 1$, $v + 4, v + 7, \dots, v - 1$. Thus F_d is a d-regular graph on $3d - 1$ vertices. For example, $F_1 = K_2$ consists of a single edge, and $F_2 = C_5$ is a 5-cycle. Figure 1 shows F_3 and F_4.

Given a graph H we say that a graph G has H-type if there is a homomorphism from G to H, i.e. a function $f : V(G) \to V(H)$ so that if uv is an edge of G then $f(u)f(v)$ is an edge of H. Equivalently, G is a subgraph of a blow-up of H, with parts $\{f^{-1}(x)\colon x \in V(H)\}$. For example, G has F_1-type if and only if it is bipartite. The following result was proved by Jin [5].

Theorem 2.1. *Let $1 \leqslant d \leqslant 9$ and suppose G is a triangle-free graph with minimum degree $\delta(G) > \frac{d+1}{3d+2}n$. Then G has F_d-type.*

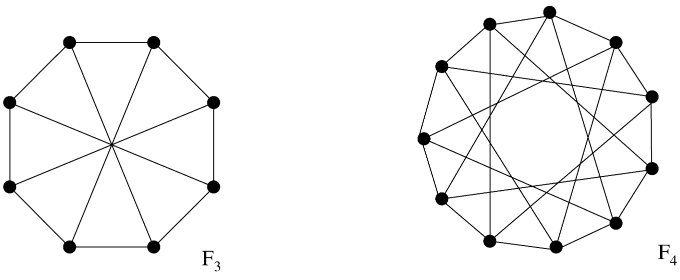

Fig. 1. Triangle-free graphs with high minimal degree.

Next we will need a lemma which describes the behaviour of these graphs under certain minimum degree assumptions.

Lemma 2.2. *Suppose G is a graph on m vertices with minimum degree $\delta(G) \geqslant \gamma m$. Suppose also that G has F_d-type, with parts V_i, $i \in \mathbb{Z}_{3d-1}$, but not F_i-type for any $i < d$.*

(i) *If $d = 2$ then $\gamma \leqslant 2/5$, $m^{-2}e(G) \leqslant 5\gamma^2 - 4\gamma + 1$ and $m^{-2}\sum_i |V_i|^2 \leqslant 30\gamma^2 - 24\gamma + 5$.*
(ii) *If $d = 3$ then $\gamma \leqslant 3/8$ and $m^{-2}e(G) \leqslant 36\gamma^2 - 27\gamma + 21/4$.*
(iii) *If $d = 4$ then $\gamma \leqslant 4/11$ and $m^{-2}e(G) \leqslant \frac{297}{4}\gamma^2 - 54\gamma + 10$.*

This is immediate from the following lemma, except in the case $d = 2$, when a little extra work is needed to get the bound for $e(G)$.

Lemma 2.3. *Suppose $d \geqslant 2$ and the vertices of F_d are weighted by reals, so that vertex i has weight x_i, where $0 \leqslant x_i \leqslant 1$ and $\sum_i x_i = 1$. Write $c = \sum_i x_i^2$, $g_i = \sum_{j:j\sim i} x_j$ and $e = \frac{1}{2}\sum_i x_i g_i = \sum_{i\sim j} x_i x_j$. Suppose $g_i \geqslant \gamma$ for each $i \in \mathbb{Z}_{3d-1}$. Then*

$$\gamma \leqslant \frac{d}{3d-1},$$
$$c \leqslant (3d-1)(3\gamma-1)^2 + \big(d - (3d-1)\gamma\big)\big((21-9d)\gamma + 3d - 6\big),$$
$$e \leqslant \frac{1}{2}d(3d-1)(3\gamma-1)^2 + \frac{3}{4}\big(d-(3d-1)\gamma\big)(3d\gamma + 3\gamma - d).$$

Proof. Note that every $i \in \mathbb{Z}_{3d-1}$ is adjacent to exactly one element of $\{0, 1, 2\}$, apart from 1, which is adjacent to both 0 and 2. Therefore,

$$3\gamma \leqslant g_0 + g_1 + g_2 = x_1 + \sum_i x_i = x_1 + 1,$$

so $x_1 \geqslant 3\gamma - 1$. Also, we have

$$\begin{aligned}(3d-4)\gamma \leqslant g_3 + \cdots + g_{3d-2} &= \sum_i g_i - (g_0 + g_1 + g_2) = d\sum_i x_i - \left(x_1 + \sum_i x_i\right) \\ &= (d-1) - x_1,\end{aligned}$$

so $x_1 \leqslant d - 1 - (3d-4)\gamma$. Combining these inequalities gives $(3d-1)\gamma \leqslant d$. Set

$$y_i = \frac{x_i - (3\gamma - 1)}{d - (3d-1)\gamma}.$$

Then $0 \leqslant y_i \leqslant 1$ and

$$\sum_i y_i = \big(d-(3d-1)\gamma\big)^{-1}\big(1-(3d-1)(3\gamma-1)\big) = 3.$$

We can write

$$\begin{aligned} c &= \sum_i x_i^2 = \sum_i \big(3\gamma - 1 + \big(d-(3d-1)\gamma\big)y_i\big)^2 \\ &= \sum_i (3\gamma-1)^2 + 2(3\gamma-1)\big(d-(3d-1)\gamma\big)\sum_i y_i + \big(d-(3d-1)\gamma\big)^2 \sum_i y_i^2 \\ &= (3d-1)(3\gamma-1)^2 + 6(3\gamma-1)\big(d-(3d-1)\gamma\big) + \big(d-(3d-1)\gamma\big)^2 \sum_i y_i^2, \end{aligned}$$

so c is maximised when $\sum_i y_i^2$ is maximised. Since $0 \leqslant y_i \leqslant 1$ we have $\sum_i y_i^2 \leqslant \sum y_i = 3$. Substituting gives

$$c \leqslant (3d-1)(3\gamma-1)^2 + \big(d-(3d-1)\gamma\big)\big((21-9d)\gamma + 3d - 6\big).$$

Also, we can write

$$\begin{aligned} e &= \sum_{i\sim j} \big(3\gamma - 1 + \big(d-(3d-1)\gamma\big)y_i\big)\big(3\gamma - 1 + \big(d-(3d-1)\gamma\big)y_j\big) \\ &= \sum_{i\sim j} (3\gamma-1)^2 + \sum_i y_i \sum_{j: j\sim i} (3\gamma-1)\big(d-(3d-1)\gamma\big) + \big(d-(3d-1)\gamma\big)^2 \sum_{i\sim j} y_i y_j \\ &= \frac{1}{2}d(3d-1)(3\gamma-1)^2 + 3d(3\gamma-1)\big(d-(3d-1)\gamma\big) + \big(d-(3d-1)\gamma\big)^2 \sum_{i\sim j} y_i y_j. \end{aligned}$$

It is well known, and easy to see by a variational argument, that the maximum of $\sum_{i\sim j} y_i y_j$ subject only to the conditions $\sum_i y_i = 3$, $y_i \geqslant 0$ is achieved when the vertices with $y_i > 0$ form a clique in the graph. Since F_d is triangle-free this clique is just an edge, so $\sum_{i\sim j} y_i y_j \leqslant (3/2)^2$. This bound is not best possible, as we have not used the other conditions that the y_i must satisfy, but it suffices for our purposes. Therefore,

$$\begin{aligned} e &\leqslant \frac{1}{2}d(3d-1)(3\gamma-1)^2 + 3d(3\gamma-1)\big(d-(3d-1)\gamma\big) + \frac{9}{4}\big(d-(3d-1)\gamma\big)^2 \\ &= \frac{1}{2}d(3d-1)(3\gamma-1)^2 + \frac{3}{4}\big(d-(3d-1)\gamma\big)(3d\gamma + 3\gamma - d). \end{aligned}$$

Finally, we observe a little trick that improves the bound to that asserted by Lemma 2.2 when $d = 2$, i.e. $F_2 = C_5$. Set $z_i = 1 - y_i$, so that $0 \leqslant z_i \leqslant 1$ and $\sum_i z_i = 5 - \sum_i y_i = 2$. Then

$$\sum_{i\sim j} y_i y_j = \sum_{i\sim j} (1-z_i)(1-z_j) = 5 - 2\sum_i z_i + \sum_{i\sim j} z_i z_j = 1 + \sum_{i\sim j} z_i z_j.$$

By the argument above $\sum_{i\sim j} z_i z_j \leqslant \frac{1}{4}(\sum_i z_i)^2 = 1$, so $\sum_{i\sim j} y_i y_j \leqslant 2$. Substituting this improved bound above gives $e \leqslant 5\gamma^2 - 4\gamma + 1$. This completes the proof of the lemma. □

Remark. With more careful analysis we can obtain the best possible bound for e in the above lemma, by showing that $\sum_{i\sim j} y_i y_j \leqslant 2$ for any d. The argument is rather more involved, so we will just state the result:

$$e \leqslant \frac{1}{2}d(3d-1)(3\gamma-1)^2 + \bigl(d-(3d-1)\gamma\bigr)\bigl((3d+2)\gamma-d\bigr).$$

3. A slightly weaker bound

Recall that for a graph G, we write $b(G)$ for the number of edges in its largest bipartite subgraph and $t(G)$ for the number of edges in its largest triangle-free subgraph. We write δ_c for the least number so that any graph G on n vertices with minimum degree $\delta(G) \geqslant (\delta_c + o(1))n$ has $t(G) = b(G)$. In this section we will show that $0.75 \leqslant \delta_c \leqslant 0.8$. This will serve to illustrate the ideas involved in the proof, and we will postpone the more involved proof of $\delta_c < 0.791$ to the next section.

First we give the lower bound. We remind the reader of the Chernoff large deviations bound (see, e.g., [1, Appendix A]). Suppose $X_1, \ldots, X_m$ are independent identically distributed random variables with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1 - p$, where p is a constant not depending on m. Then $\mathbb{P}(|\sum X_i - mp| > a) < e^{-ca^2/m}$, where c is a constant depending only on p.

Theorem 3.1. *For any $\delta < 3/4$ there is n and a graph G on n vertices with minimum degree at least δn in which the largest triangle-free subgraph has more edges than the largest bipartite subgraph. Therefore $\delta_c \geqslant 3/4$.*

Proof. The vertex set $V = V(G)$ of our graph will be divided into parts V_i, $i \in \mathbb{Z}_5$, each of size $n/5$. All pairs uv with $u, v \in V_i$ or $u \in V_i$, $v \in V_{i+1}$ for some i are edges of G. Also, for every i each pair uv with $u \in V_i$, $v \in V_{i+2}$ is chosen to be an edge randomly and independently with probability θ, for some $\theta < 3/8$ which we specify later.

Consider a vertex $v \in V_i$. It is joined to all $3n/5 - 1$ vertices of $(V_i - v) \cup V_{i+1} \cup V_{i-1}$. In addition, $|N(v) \cap (V_{i+2} \cup V_{i-2})|$ is a sum of $\frac{2}{5}n$ independent indicator random variables each taking the value 1 with probability θ. By the Chernoff bound, the probability that this sum deviates from $\frac{2}{5}\theta n$ by more than $n^{3/4}$ is less than $e^{-\Omega(\sqrt{n})}$. Therefore $|d(v) - \frac{1}{5}(3+2\theta)n| < n^{3/4}$ for every vertex v, with probability at least $1 - ne^{-\Omega(\sqrt{n})} = 1 - o(1)$. Similarly, the probability that the number of edges $e(A_i, A_{i+2})$ between some subsets $A_i \subset V_i$, $A_{i+2} \subset V_{i+2}$ deviates from $\theta|A_i||A_{i+2}|$ by more than $n^{5/3}$ is at most $e^{-\Omega(n^{4/3})}$. Indeed, if $|A_i||A_{i+2}| < n^{5/3}$ this probability is zero, otherwise we can use the Chernoff bound again. Therefore, for every i and every such choice of A_i, A_{i+2} we have $|e(A_i, A_{i+2}) - \theta|A_i||A_{i+2}|| < n^{5/3}$, with probability at least $1 - 2^{2n} \cdot e^{-\Omega(n^{4/3})} = 1 - o(1)$.

By the above discussion there exists a choice of G such that all vertices satisfy $d(v) = \frac{1}{5}(3+2\theta)n + o(n)$ and $e(A_i, A_{i+2}) = \theta|A_i||A_{i+2}| + o(n^2)$ for any two subsets $A_i \subset V_i$, $A_{i+2} \subset V_{i+2}$. We choose θ so that $\delta < \frac{1}{5}(3+2\theta) < 3/4$. Then G has minimum degree at least δn. Now suppose that A_i, $i \in \mathbb{Z}_5$, define the largest cut in G, i.e. $e(\bigcup A_i, V \setminus \bigcup A_i) = b(G)$. Write $|A_i| = x_i n$ and $x = \sum x_i$. Then $0 \leqslant x_i \leqslant 1/5$, and by replacing every A_i by its complement if necessary we can assume $x \leqslant 1/2$. Now

$$\begin{aligned} n^{-2}b(G) < {} & \sum_i x_i(1/5 - x_i) + \sum_i x_i(1/5 - x_{i+1}) + \sum_i x_i(1/5 - x_{i-1}) \\ & + \theta\sum_i x_i(1/5 - x_{i+2}) + \theta\sum_i x_i(1/5 - x_{i-2}) + o(1) \end{aligned}$$

$$= \frac{1}{5}(3+2\theta)x - \sum_i x_i^2 - 2\sum_i x_i x_{i+1} - 2\theta \sum_i x_i x_{i+2} + o(1)$$
$$= \frac{1}{5}(3+2\theta)x - x^2 + 2(1-\theta)\sum_i x_i x_{i+2} + o(1),$$

using the identity $x^2 = \sum_i x_i^2 + 2\sum_i x_i x_{i+1} + 2\sum_i x_i x_{i+2}$. It is not hard to show (see [3]) that

$$\sum_i x_i x_{i+2} \leqslant \begin{cases} x^2/4, & 0 \leqslant x \leqslant 2/5, \\ \frac{1}{5}(x-1/5), & 2/5 \leqslant x \leqslant 1/2. \end{cases}$$

If $0 \leqslant x \leqslant 2/5$ we have $n^{-2}b(G) < \frac{1}{5}(3+2\theta)x - \frac{1}{2}(1+\theta)x^2 + o(1)$. The maximum of the quadratic is at $x = \frac{1}{5}(2+(1+\theta)^{-1}) + o(1) > 6/11 + o(1)$, as $\theta < 3/8$. This is not in the range $[0, 2/5]$ for large n, so the maximum occurs at $x = 2/5$, giving $n^{-2}b(G) < \frac{2}{25}(2+\theta) + o(1) < 0.19 + o(1) < 1/5$. If $2/5 \leqslant x \leqslant 1/2$ then

$$n^{-2}b(G) < \frac{1}{5}(3+2\theta)x - x^2 + \frac{2}{5}(1-\theta)(x-1/5) + o(1)$$
$$= x - x^2 - \frac{2}{25}(1-\theta) + o(1).$$

The maximum occurs at $x = 1/2$, so we see that $n^{-2}b(G) \leqslant \frac{17}{100} + \frac{2}{25}\theta + o(1) < 1/5$, as $\theta < 3/8$ is a constant. However, the graph spanned by the edges joining V_i to V_{i+1} for each i is triangle free and has $n^2/5$ edges, so $t(G) \geqslant n^2/5 > b(G)$. This completes the proof. □

Before proving the upper bound we need two lemmas.

Lemma 3.2. *Suppose Γ is a bipartite subgraph of a graph G and there are m edges incident to the vertices in $V(G) \setminus V(\Gamma)$. Then G has a bipartite subgraph of size at least $e(\Gamma) + m/2$.*

Proof. Let (A_0, B_0) be the bipartition of Γ. Consider a bipartite subgraph G' of G with parts (A, B), where $A_0 \subset A$, $B_0 \subset B$ and we place each vertex $v \in V(G) \setminus V(\Gamma)$ in A or B randomly and independently with probability $1/2$. All edges of Γ are edges of G', and each edge incident to a vertex in $V(G) \setminus V(\Gamma)$ appears in G' with probability $1/2$. By linearity of expectation $\mathbb{E}[e(G')] = e(\Gamma) + m/2$, so some bipartite subgraph of G has at least this many edges. □

Lemma 3.3. *Let G be a graph with vertices partitioned as $V(G) = \bigcup_{i \in \mathbb{Z}_5} V_i$. Say that an edge uv of G has* type t *if $t \in \{0, 1, 2\}$ and $u \in V_i$, $v \in V_{i+t}$ for some i. Write e_t for the number of edges of type t, and $e = e(G) = e_0 + e_1 + e_2$. Then*

$$b(G) \geqslant \frac{2}{5}(e + e_1) - \frac{3}{10}e_0.$$

Proof. Choose $i \in \mathbb{Z}_5$ uniformly at random, and then randomly partition V_i as $A_i \cup B_i$, by placing $v \in V_i$ in A_i or B_i randomly and independently with probability $1/2$. Consider the bipartite subgraph G' with parts (A, B) where $A = A_i \cup V_{i+1} \cup V_{i-2}$ and $B = B_i \cup V_{i-1} \cup V_{i+2}$. For each edge of G we compute the probability that it appears in G'.

Consider an edge uv of type 0, with $u, v \in V_j$. This will appear in G' if $i = j$ and then u, v are placed with one in A_i and the other in B_i, an event with probability $1/10$. Next consider an edge uv of type 1 with $u \in V_j$ and $v \in V_{j+1}$. This appears in G' if one of the following three mutually exclusive events occurs: (i) $i \in \{j-1, j-2, j+2\}$, (ii) $i = j$, $u \in B_i$, (iii) $i = j+1$, $v \in A_i$. The total probability of these events is $4/5$. Finally, consider an edge uv of type 2 with $u \in V_j$ and $v \in V_{j+2}$. This appears in G' if one of the following three mutually exclusive events occurs: (i) $i = j+1$, (ii) $i = j$, $u \in A_i$, (iii) $i = j+2$, $v \in B_i$. The total probability of these events is $2/5$.

Since $e = e_0 + e_1 + e_2$, by linearity of expectation

$$\mathbb{E}\big[e(G')\big] = \frac{1}{10}e_0 + \frac{4}{5}e_1 + \frac{2}{5}e_2 = \frac{2}{5}(e + e_1) - \frac{3}{10}e_0.$$

Therefore there is a bipartite subgraph of G with at least this many edges. □

Theorem 3.4. *Suppose G is a graph on n vertices with minimum degree $\frac{4}{5}n + 1$, where n is large. Then the largest triangle-free and largest bipartite subgraphs of G have equal size. Therefore $\delta_c \leqslant 4/5$.*

Proof. Let G be a graph on n vertices with minimum degree $\frac{4}{5}n + 1$. Then $e(G) \geqslant \frac{2}{5}n^2 + \frac{1}{2}n$. We will suppose that $b(G) < t(G)$ and derive a contradiction. Let H be a triangle-free subgraph of G with $e(H) = t(G)$ maximal, and write $e(H) = tn^2$. Since $t(G) > b(G) \geqslant e(G)/2$ we have $t > 1/5 + 1/(4n)$.

Construct a sequence of graphs $H = H_n, H_{n-1}, \dots,$ where if H_k has a vertex of degree less than $\frac{11}{30}k$ we delete that vertex to obtain H_{k-1}. Let Γ be the final (possibly empty) graph of this sequence and write $|V(\Gamma)| = \alpha n$. Then Γ is a triangle-free graph with minimal degree $\delta(\Gamma) \geqslant \frac{11}{30}|V(\Gamma)|$, and $e(\Gamma) \geqslant e(H) - \frac{11}{30}\big(\binom{n+1}{2} - \binom{\alpha n+1}{2}\big) = e(H) - \frac{11}{60}(1-\alpha^2)n^2 - \frac{11}{60}(1-\alpha)n$, i.e.

$$n^{-2}e(\Gamma) \geqslant t - \frac{11}{60}\big(1-\alpha^2\big) - \frac{11}{60}(1-\alpha)/n. \tag{1}$$

As $11/30 > 4/11$, by Theorem 2.1, Γ has F_d-type for some $d \leqslant 3$. Choose d so that Γ does not have F_i-type for any $i < d$. If $d = 3$, then by Lemma 2.2(ii) (with $\gamma = 11/30$) we have

$$n^{-2}e(\Gamma) \leqslant \big(36(11/30)^2 - 27(11/30) + 21/4\big)\alpha^2 = \frac{19}{100}\alpha^2.$$

Since $\alpha \leqslant 1$, this implies $t - O(1/n) \leqslant \frac{19}{100}\alpha^2 + \frac{11}{60}(1-\alpha^2) = \frac{11}{60} + \frac{1}{150}\alpha^2 \leqslant 1/5 - 1/100$, which is a contradiction.

Next consider the case when $d = 1$, i.e. Γ is bipartite. The number of edges of G incident to vertices in $V(G) \setminus V(\Gamma)$ is

$$\begin{aligned} m &= \sum_{v \in V(G)\setminus V(\Gamma)} d(v) - e\big(V(G) \setminus V(\Gamma)\big) > (1-\alpha)n \cdot \frac{4}{5}n - \binom{(1-\alpha)n}{2} \\ &= \left(\frac{4}{5}(1-\alpha) - \frac{1}{2}(1-\alpha)^2\right)n^2 + \frac{1}{2}(1-\alpha)n. \end{aligned}$$

Applying Lemma 3.2 we have

$$\begin{aligned} t &= n^{-2}t(G) > n^{-2}b(G) \geqslant n^{-2}\big(e(\Gamma) + m/2\big) \\ &> t - \frac{11}{60}\big(1-\alpha^2\big) - \frac{11}{60}(1-\alpha)/n + \frac{2}{5}(1-\alpha) - \frac{1}{4}(1-\alpha)^2 + \frac{1}{4}(1-\alpha)/n. \end{aligned}$$

This gives $0 > -\frac{11}{60}(1-\alpha^2) + \frac{2}{5}(1-\alpha) - \frac{1}{4}(1-\alpha)^2$, which simplifies to $\frac{1}{30}(1-\alpha)(1-2\alpha) > 0$, i.e. $\alpha < 1/2$. However $e(\Gamma) \leqslant |V(\Gamma)|^2/4$ by Turán's theorem. Hence by inequality (1)

$$t \leqslant \frac{1}{4}\alpha^2 + \frac{11}{60}\left(1-\alpha^2\right) + \frac{11}{60}(1-\alpha)/n \leqslant \frac{11}{60} + \frac{1}{15}\alpha^2 + \frac{11}{60n} < 1/5 + 1/(4n),$$

which is a contradiction.

Therefore we conclude that $d = 2$, i.e. Γ has C_5-type. By Lemma 2.2(i) we have $n^{-2}e(\Gamma) \leqslant (5(11/30)^2 - 4(11/30) + 1)\alpha^2 = \frac{37}{180}\alpha^2$, so by inequality (1)

$$t \leqslant \frac{11}{60}\left(1-\alpha^2\right) + \frac{37}{180}\alpha^2 + O(1/n) = \frac{11}{60} + \frac{1}{45}\alpha^2 + O(1/n). \tag{2}$$

Write

$$p = 2n^{-2}e_G\big(V(G)\setminus V(\Gamma)\big), \qquad q = 2n^{-2}e_G\big(V(\Gamma)\big),$$
$$r = n^{-2}e_G\big(V(\Gamma), V(G)\setminus V(\Gamma)\big).$$

By the minimum degree condition on G we have

$$\frac{4}{5}\alpha \leqslant n^{-2}\sum_{v\in V(\Gamma)} d(v) = q + r, \qquad \frac{4}{5}(1-\alpha) \leqslant n^{-2}\sum_{v\notin V(\Gamma)} d(v) = p + r. \tag{3}$$

Label the parts of Γ as V_i, $i \in \mathbb{Z}_5$, so that edges of Γ have type 1 in the terminology of Lemma 3.3. By Lemma 3.3 we have

$$n^{-2}b\big(G\big[V(\Gamma)\big]\big) \geqslant \frac{2}{5}(q/2 + e_1) - \frac{3}{10}e_0,$$

where we have denoted the number of edges of $G[V(\Gamma)]$ of type i by $e_i n^2$ (slightly modifying the notation used in the lemma).

Now by Lemma 3.2

$$t > n^{-2}b(G) \geqslant n^{-2}b\big(G\big[V(\Gamma)\big]\big) + \frac{1}{2}(p/2 + r) \geqslant \frac{1}{4}p + \frac{1}{5}q + \frac{1}{2}r + \frac{2}{5}e_1 - \frac{3}{10}e_0.$$

Also, we have $e_1 \geqslant n^{-2}e(\Gamma) \geqslant t - \frac{11}{60}(1-\alpha^2) - O(1/n)$, so for large n we get

$$t \geqslant \frac{1}{4}p + \frac{1}{5}q + \frac{1}{2}r + \frac{2}{5}\left(t - \frac{11}{60}\left(1-\alpha^2\right) - O(1/n)\right) - \frac{3}{10}e_0,$$
$$t \geqslant \frac{5}{12}p + \frac{1}{3}q + \frac{5}{6}r - \frac{11}{90}\left(1-\alpha^2\right) - \frac{1}{2}e_0 - O(1/n).$$

Next we substitute $q \geqslant \frac{4}{5}\alpha - r$ and $p \geqslant \frac{4}{5}(1-\alpha) - r$ from inequalities (3) to get

$$t \geqslant \frac{1}{3} - \frac{1}{15}\alpha + \frac{1}{12}r - \frac{11}{90}\left(1-\alpha^2\right) - \frac{1}{2}e_0 - O(1/n).$$

Also, Lemma 2.2(i) (with $\gamma = 11/30$) gives $\frac{1}{2}e_0 \leqslant \frac{1}{4}(30(11/30)^2 - 24(11/30) + 5)\alpha^2 = \frac{7}{120}\alpha^2$. Since $r \geqslant 0$ we have

$$t \geqslant \frac{1}{3} - \frac{1}{15}\alpha - \frac{11}{90}\left(1-\alpha^2\right) - \frac{7}{120}\alpha^2 - O(1/n) = \frac{23}{360}\alpha^2 - \frac{1}{15}\alpha + \frac{19}{90} - O(1/n). \tag{4}$$

Combining this with inequality (2) gives

$$\frac{11}{60} + \frac{1}{45}\alpha^2 + O(1/n) \geqslant \frac{23}{360}\alpha^2 - \frac{1}{15}\alpha + \frac{19}{90} - O(1/n),$$

i.e. $\frac{1}{24}\alpha^2 - \frac{1}{15}\alpha + \frac{1}{36} \leqslant O(1/n)$. However, this quadratic is always at least $1/900$. This contradiction completes the proof. □

4. Proof of the full result

In this section we will show how to extend the above argument to deal with the case $\delta > 0.791$, which will complete the proof of Theorem 1.1. A possible method is that instead of merely combining inequalities (2) and (4) in the preceding argument, we could use the lower bound on t given by inequality (4) to 'bootstrap' the argument—with each iteration we will improve the lower bound on t and be able to delete vertices of slightly higher degree from H in forming Γ, until we arrive at a contradiction. Equivalently (and this is the approach we will take) we can delete vertices from H according to some degree condition depending on the unknown parameter t, and then conclude the argument by showing that inequalities (2) and (4) have no common solution. The necessary computations are rather involved, so we will state them altogether in the following lemma so as not to clutter the proof of the theorem.

Lemma 4.1. *Let $\delta = 0.791$ and suppose $1 \geqslant t \geqslant \delta/4$. Then there exists $\epsilon > 0$ so that the following hold with $\gamma = \frac{4t-(2\delta-1)^2}{6-8\delta+8t} - \epsilon$:*

(i) *$5/14 < \gamma < 2t$, and*
(ii) *$(1-2\gamma)(4t-2\gamma) > (1+2\gamma-2\delta)^2$.*

Suppose also that $t < ((9-10\delta)^2+4)/20$. Then:

(iii) *if $\gamma \leqslant 3/8$ then $t > 36\gamma^2 - 27\gamma + 21/4$;*
(iv) *if $\gamma \leqslant 4/11$ then $t > \frac{297}{4}\gamma^2 - 54\gamma + 10$;*
(v) *$\frac{2t-\gamma}{10\gamma^2-9\gamma+2} > 1/4$; and*
(vi) *the inequalities*

$$\left(5\gamma^2 - \frac{9}{2}\gamma + 1\right)\alpha^2 + \frac{1}{2}\gamma \geqslant t \geqslant \frac{5}{12}(1-\alpha)^2 + \frac{1}{3}\left(\alpha^2 - (1-\delta)(2\alpha-1)\right) + \frac{5}{6}\left(\delta(1-\alpha) - (1-\alpha)^2\right) - \frac{1}{3}\gamma\left(1-\alpha^2\right) - \frac{1}{4}\left(30\gamma^2 - 24\gamma + 5\right)\alpha^2$$

have no solution with $0 \leqslant \alpha \leqslant 1$.

We defer the proof of this lemma to Section 5, and first show how the theorem follows.

Theorem 4.2. *Suppose G is a graph on n vertices with minimum degree $0.791n$, where n is large. Then the largest triangle-free and largest bipartite subgraphs of G have equal size. Therefore $\delta_c < 0.791$.*

Proof. Let G be a graph on n vertices with minimum degree δn. Then $e(G) \geqslant \frac{1}{2}\delta n^2$. We will suppose that $b(G) < t(G)$ and show that we can derive a contradiction when $\delta = 0.791$. This will show that $\delta_c < 0.791$.

Let H be a triangle-free subgraph of G with $e(H) = t(G)$ maximal, and write $e(H) = tn^2$. Since $t(G) \geqslant b(G) \geqslant e(G)/2$ we have $t \geqslant \delta/4$. Set $\gamma = \frac{4t-(2\delta-1)^2}{6-8\delta+8t} - \epsilon$, where ϵ is chosen as in the statement of Lemma 4.1. Construct a sequence of graphs $H = H_n, H_{n-1}, \dots,$ where if H_k has a vertex of degree less than γk we delete that vertex to obtain H_{k-1}. Let Γ be the final (possibly empty) graph of this sequence. Write $|V(\Gamma)| = \alpha n$ and $e(\Gamma) = \beta |V(\Gamma)|^2$. Then Γ is a triangle-free graph with minimal degree $\delta(\Gamma) \geqslant \gamma |V(\Gamma)|$ and $e(\Gamma) > e(H) - \gamma(\binom{n+1}{2} - \binom{\alpha n+1}{2})$, i.e.

$$\beta\alpha^2 = n^{-2} e(\Gamma) > t - \frac{1}{2}\gamma\bigl(1-\alpha^2\bigr) - O(1/n),$$

or equivalently $(2\beta - \gamma)\alpha^2 > 2t - \gamma - O(1/n)$. By Lemma 4.1(i) $2t - \gamma > 0$, so then $2\beta - \gamma > 0$ for large n, and.

$$\alpha^2 > \frac{2t-\gamma}{2\beta-\gamma}. \tag{5}$$

By Lemma 4.1(i) $\gamma > 5/14$, so by Theorem 2.1 Γ has F_d-type for some $d \leqslant 4$. Let d be such that Γ has F_d-type but does not have F_i-type for any $i < d$.

Suppose first that $d = 1$, i.e. Γ is bipartite. In this case $e(\Gamma) \leqslant |V(\Gamma)|^2/4$ by Turán's theorem, i.e. $\beta \leqslant 1/4$. The number of edges of G incident to vertices in $V(G) \setminus V(\Gamma)$ is.

$$m = \sum_{v \in V(G)\setminus V(\Gamma)} d(v) - e\bigl(V(G) \setminus V(\Gamma)\bigr) \geqslant (1-\alpha)n \cdot \delta n - \binom{(1-\alpha)n}{2}.$$

Applying Lemma 3.2 we have.

$$\begin{aligned} t &= n^{-2} t(G) \geqslant n^{-2} b(G) \geqslant n^{-2}\bigl(e(\Gamma) + m/2\bigr) \\ &> t - \frac{1}{2}\gamma\bigl(1-\alpha^2\bigr) + \frac{1}{2}\left(\delta(1-\alpha) - \frac{1}{2}(1-\alpha)^2\right). \end{aligned}$$

Cancelling a factor $1-\alpha$ gives $\gamma(1+\alpha) + \frac{1}{2}(1-\alpha) > \delta$, and since $\gamma < 1/2$ this can be rewritten as.

$$\alpha < \frac{1+2\gamma-2\delta}{1-2\gamma}.$$

Combining this with inequality (5) gives $(2t-\gamma)(1-2\gamma)^2 < (1+2\gamma-2\delta)^2(2\beta-\gamma)$, and since $\beta \leqslant 1/4$ we have $(1-2\gamma)(4t-2\gamma) < (1+2\gamma-2\delta)^2$. This contradicts Lemma 4.1(ii), so this case leads to a contradiction. Note that if $t \geqslant \frac{1}{20}((9-10\delta)^2+4)$ then we may choose $\gamma = 2/5$ to satisfy inequalities (i) and (ii) of Lemma 4.1, which immediately gives a contradiction, as by Theorem 2.1 we know that Γ can only be bipartite. Therefore we can assume that.

$$t < \frac{(9-10\delta)^2+4}{20}. \tag{6}$$

For the case $d = 4$ by Lemmas 2.2 and 4.1(iv) we have $\beta \leqslant \frac{297}{4}\gamma^2 - 54\gamma + 1 < t$, and then by inequality (5) we get the contradiction $\alpha > 1$. Likewise, in the case $d = 3$ we get $\beta \leqslant 36\gamma^2 - 27\gamma + 21/4 < t$, which again gives the contradiction $\alpha > 1$. Therefore we conclude that $d = 2$, i.e. Γ has C_5-type.

By Lemma 2.2 we have $\beta \leqslant 5\gamma^2 - 4\gamma + 1$, so by inequality (5) $\alpha^2 > \frac{2t-\gamma}{2\beta-\gamma} \geqslant \frac{2t-\gamma}{10\gamma^2-9\gamma+2}$. Now by Lemma 4.1(v) we have $\alpha^2 > 1/4$, i.e. $\alpha > 1/2$. It will also be useful later to rewrite inequality (5) as

$$\left(5\gamma^2 - \frac{9}{2}\gamma + 1\right)\alpha^2 + \frac{1}{2}\gamma > t. \tag{7}$$

Write

$$p = 2n^{-2}e_G\big(V(G)\setminus V(\Gamma)\big), \qquad q = 2n^{-2}e_G\big(V(\Gamma)\big),$$
$$r = n^{-2}e_G\big(V(\Gamma), V(G)\setminus V(\Gamma)\big).$$

Recalling that $|V(\Gamma)| = \alpha n$ we have inequalities

$$0 \leqslant p \leqslant (1-\alpha)^2, \qquad 0 \leqslant q \leqslant \alpha^2, \qquad 0 \leqslant r \leqslant \alpha(1-\alpha). \tag{8}$$

Also, by the minimum degree condition on G we have

$$\delta\alpha \leqslant n^{-2}\sum_{v\in V(\Gamma)} d(v) = q + r, \qquad \delta(1-\alpha) \leqslant n^{-2}\sum_{v\notin V(\Gamma)} d(v) = p + r. \tag{9}$$

Label the parts of Γ as V_i, $i \in \mathbb{Z}_5$, so that edges of Γ have type 1 in the terminology of Lemma 3.3. By Lemma 3.3 we have

$$n^{-2}b\big(G\big[V(\Gamma)\big]\big) \geqslant \frac{2}{5}(q/2 + e_1) - \frac{3}{10}e_0,$$

where we have denoted the number of edges of $G[V(\Gamma)]$ of type i by $e_i n^2$ (slightly modifying the notation used in the lemma).

Now by Lemma 3.2

$$t \geqslant n^{-2}b(G) \geqslant n^{-2}b\big(G\big[V(\Gamma)\big]\big) + \frac{1}{2}(p/2 + r) \geqslant \frac{1}{4}p + \frac{1}{5}q + \frac{1}{2}r + \frac{2}{5}e_1 - \frac{3}{10}e_0.$$

Also, we have $e_1 \geqslant n^{-2}e(\Gamma) > t - \frac{1}{2}\gamma(1-\alpha^2) - O(1/n)$, so taking n large we have

$$t \geqslant \frac{1}{4}p + \frac{1}{5}q + \frac{1}{2}r + \frac{2}{5}\left(t - \frac{1}{2}\gamma\big(1-\alpha^2\big)\right) - \frac{3}{10}e_0,$$
$$t \geqslant \frac{5}{12}p + \frac{1}{3}q + \frac{5}{6}r - \frac{1}{3}\gamma\big(1-\alpha^2\big) - \frac{1}{2}e_0. \tag{10}$$

To make further progress we want to see how small the right-hand side of this inequality can be, subject to inequalities (8) and (9) that we know for p, q, r. This is a simple linear program, which can be solved as follows.

We need to purchase units of p, q, r at prices $5/12$, $1/3$, $5/6$ to satisfy inequalities (9) as cheaply as possible. Since a unit of r is the most expensive, and contributes the same as p or q to either inequality, we want to make r as small as possible, subject to being able to satisfy inequalities (8). Therefore,

$$r = \max\big\{\delta\alpha - \alpha^2, \delta(1-\alpha) - (1-\alpha)^2, 0\big\}, \qquad p = \delta(1-\alpha) - r, \qquad q = \delta\alpha - r.$$

Now $\delta(1-\alpha) - (1-\alpha)^2 > 0$, since $\delta > 1/2 > 1-\alpha$, and $\delta(1-\alpha) - (1-\alpha)^2 - (\delta\alpha - \alpha^2) = (1-\delta)(2\alpha - 1) > 0$, since $\alpha > 1/2$, so we have

$$r = \delta(1-\alpha) - (1-\alpha)^2, \qquad p = (1-\alpha)^2, \qquad q = \alpha^2 - (1-\delta)(2\alpha - 1).$$

Also, Lemma 2.2(i) gives $\frac{1}{2}e_0 \leqslant \frac{1}{4}(30\gamma^2 - 24\gamma + 5)\alpha^2$, so substituting into inequality (10) we have

$$t \geqslant \frac{5}{12}(1-\alpha)^2 + \frac{1}{3}\big(\alpha^2 - (1-\delta)(2\alpha - 1)\big) + \frac{5}{6}\big(\delta(1-\alpha) - (1-\alpha)^2\big)$$
$$- \frac{1}{3}\gamma\big(1-\alpha^2\big) - \frac{1}{4}\big(30\gamma^2 - 24\gamma + 5\big)\alpha^2. \tag{11}$$

Combining this with inequality (7) and applying Lemma 4.1(vi) we find that there is no solution with $0 \leqslant \alpha \leqslant 1$. This contradiction completes the proof. □

5. Proof of Lemma 4.1

Before starting, we remark that the reader may find a computer algebra package helpful in verifying some of the following computations. Let $\delta = 0.791$, suppose $1 \geqslant t \geqslant \delta/4$, $\epsilon > 0$ and set $\gamma = \frac{4t-(2\delta-1)^2}{6-8\delta+8t} - \epsilon$. We will use the notation $x = y \pm z$ to mean $y - z < x < y + z$. We also write $s = 4t - 4\delta + 3$ and note that $s \geqslant 3(1-\delta) > 0$.

(i) We can compute $\frac{d\gamma}{dt} = 8(1-\delta)^2 s^{-2}$ so $0 < \frac{d\gamma}{dt} \leqslant 8/9$. Therefore to show that $5/14 < \gamma < 2t$ it suffices to check it for $t = \delta/4$. Then we may compute $\gamma = \frac{\delta-(2\delta-1)^2}{6(1-\delta)} - \epsilon = 0.36 \pm 0.001$ for small ϵ. Since $5/14 = 0.357 \pm 0.001$ and $2t = 0.791/2 = 0.3955$ we have $5/14 < \gamma < 2t$.

(ii) We have

$$(1-2\gamma)(4t-2\gamma) - (1+2\gamma-2\delta)^2 = 2s\epsilon > 0.$$

Now suppose also that $t < t^* = ((9-10\delta)^2 + 4)/20$. Then we have $s = 4t - 4\delta + 3 < 20(1-\delta)^2$.

(iii) Suppose that $\gamma \leqslant 3/8$. Let $g_1(\gamma) = 36\gamma^2 - 27\gamma + 21/4$. Then we can compute

$$\frac{d^2 g_1(\gamma(t))}{dt^2} = 576(1-\delta)^2\big(24(1-\delta)^2 - s\big)s^{-4} + O(\epsilon).$$

Since $s < 20(1-\delta)^2$ we have $\frac{d^2 g_1}{dt^2} > 0$. Therefore $t - g_1(\gamma)$ is a concave function of t so to show that it is positive it suffices to check the extreme values $t = \delta/4$ and $t = t'$, where $t' = 0.215 \pm 0.001$ is the value of t at which $\gamma = 3/8$. We have

$$\delta/4 - g_1\big(\gamma(\delta/4)\big) = \frac{1}{4}\big(105\delta - 64\delta^2 - 43\big) + O(\epsilon) = 0.002 \pm 0.001 + O(\epsilon) > 0,$$

and

$$t' - g_1(3/8) = 0.02 \pm 0.01 + O(\epsilon) > 0,$$

for small ϵ, as required.

(iv) Suppose that $\gamma \leqslant 4/11$. Let $g_2(\gamma) = \frac{297}{4}\gamma^2 - 54\gamma + 10$. Then we can compute

$$\frac{d^2 g_2(\gamma(t))}{dt^2} = 1296(1-\delta)^2\big(22(1-\delta)^2 - s\big)s^{-4} + O(\epsilon).$$

Since $s < 20(1-\delta)^2$ we have $\frac{d^2 g_1}{dt^2} > 0$. Again $t - g_2(\gamma)$ is concave, so we need to show that it is positive at the values $t = \delta/4$ and $t = t''$, where $t'' = 0.201 \pm 0.001$ is the value of t at which $\gamma = 4/11$. We have

$$\delta/4 - g_2\big(\gamma(\delta/4)\big) = \frac{1}{16}\big(844\delta - 528\delta^2 - 337\big) + O(\epsilon) = 0.015 \pm 0.001 + O(\epsilon) > 0,$$

and

$$t'' - g_2(4/11) = 0.02 \pm 0.01 + O(\epsilon) > 0.$$

(v) We need to show $\frac{2t-\gamma}{10\gamma^2-9\gamma+2} > 1/4$, i.e. $8t > g_3(\gamma) = 10\gamma^2 - 5\gamma + 2$. We compute

$$\frac{dg_3(\gamma(t))}{dt} = 40(1-\delta)^2\big(s - 8(1-\delta)^2\big)s^{-3} + O(\epsilon),$$

$$\frac{d^2 g_3(\gamma(t))}{dt^2} = 320(1-\delta)^2\big(12(1-\delta)^2 - s\big)s^{-4} + O(\epsilon).$$

Now $12(1-\delta)^2 - s \leqslant 12(1-\delta)^2 - 3(1-\delta) = 3(1-\delta)(3-4\delta) < 0$, so $\frac{d^2 g_3(\gamma(t))}{dt^2} < 0$. We deduce that $\frac{dg_3(\gamma(t))}{dt}$ is minimised at $t = t^*$, where its value is $\frac{3}{50(1-\delta)^2} + O(\epsilon)$. Therefore, the derivative of $8t - g_3(\gamma)$ is at least $8 - \frac{3}{50(1-\delta)^2} + O(\epsilon) > 0$, so it is enough to verify the inequality $8t > g_3(\gamma)$ at $t = \delta/4$. We have $8(\delta/4) - g_3(\gamma(\delta/4)) = \frac{4}{9}(17\delta - 10\delta^2 - 7) + O(\epsilon) = 0.08 \pm 0.01 + O(\epsilon) > 0$, as required.

(vi) We write the given inequalities as $f_1(\alpha) \geqslant 0$ and $f_2(\alpha) \geqslant 0$, where $f_i(\alpha) = c_{2,i}\alpha^2 + c_{1,i}\alpha + c_{0,i}$ for $i = 1, 2$, with $c_{2,1} = 5\gamma^2 - \frac{9}{2}\gamma + 1$, $c_{1,1} = 0$, $c_{0,1} = \gamma/2 - t$ and $c_{2,2} = \frac{15}{2}\gamma^2 - \frac{19}{3}\gamma + \frac{4}{3}$, $c_{1,2} = -\frac{1}{6}(1-\delta)$, $c_{0,2} = \frac{1}{3}\gamma - \frac{1}{2}\delta + \frac{1}{12} + t$. Write $\Delta_i = c_{1,i}^2 - 4c_{2,i}c_{0,i}$ for the corresponding discriminants. We compute

$$c_{2,1} = (1-\delta)^2\big(20(1-\delta)^2 - s\big)s^{-2} + O(\epsilon) > 0, \quad \text{and}$$
$$c_{2,2} = \frac{1}{24}\big(20(1-\delta)^2 - s\big)\big(36(1-\delta)^2 - s\big)s^{-2} + O(\epsilon) > 0.$$

The roots of f_1 are $\pm r_1$, where

$$r_1 = \frac{1}{2}s^{1/2}(1-\delta)^{-1}\big(20(1-\delta)^2 - s\big)^{-1/2}(4t + 1 - 2\delta) > 0,$$

since $4t + 1 - 2\delta \geqslant 1 - \delta > 0$. Since $c_{2,1} > 0$ and $\alpha \geqslant 0$ we must have $\alpha > r_1$.

We find

$$\frac{dr_1}{ds} = \frac{1}{2}(1-\delta)^{-1}\big(20(1-\delta)^2 + s\big(30(1-\delta)^2 - s\big)\big)s^{-1/2}\big(20(1-\delta)^2 - s\big)^{-3/2} > 0.$$

Numerical computation shows that $r_1 = 1$ for some t in the range 0.2054 ± 0.0001. If $t \geqslant 0.2055$ we would have the contradiction $\alpha > r_1 > 1$, so we must have $t < 0.2055$.

Also

$$\Delta_2 = (-85 \pm 1)(8t - 0.32 \pm 0.01)(t \pm 0.1)(t - 0.2058 \pm 0.0001) \times (t - 0.23 \pm 0.01)(t - 0.4 \pm 0.1).$$

Since $0.197 \pm 0.001 = \delta/4 \leqslant t < 0.2055$ we see that Δ_2 is positive so f_2 has real roots. Denote these roots by $r_{2,1}$ and $r_{2,2}$, with $r_{2,1} < r_{2,2}$. Numerical computations show that $r_{2,2} > 1$ for $t < 0.2056$, so we must have $\alpha < r_{2,1}$. Now we have $r_1 < \alpha < r_{2,1}$. But computations show that $r_1 > r_{2,1}$ for $t < 0.2055$, so we have a contradiction. This shows that the inequalities $f_1(\alpha) \geqslant 0$ and $f_2(\alpha) \geqslant 0$ have no common solution $0 \leqslant \alpha \leqslant 1$, so we are done.

6. Concluding remarks

Our proofs actually show that in a graph on n vertices with minimum degree at least $0.791n$ the largest triangle-free graph must be bipartite. There is some slight room for improvement in our bound using the same methods (say to give $\delta_c < 0.7909$), but a significant improvement seems to require a new idea. We believe that the lower bound in Theorem 1.1 gives the correct value, i.e. $\delta_c = 3/4$. There are two obstacles to pushing our approach towards this. The first is our analysis of the C_5-type case, which is certainly not optimal. The second is the classification of triangle-free graphs with given minimum degree, as Theorem 2.1 does not help for $\delta < 10/29$. One can check that when $\delta = 3/4 + \epsilon$ we need to choose $\gamma = 1/3 + O(\epsilon)$ (with the same notation as in the proof) to deal with the bipartite case. For such γ Thomassen [6] showed that triangle-free graphs with minimum degree γn have chromatic number bounded by a function of ϵ, but probably more precise information on the structure would be needed to make this approach work.

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