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Pairwise intersections and forbidden configurations

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Abstract

Let $f_m(a, b, c, d)$ denote the maximum size of a family $\mathcal{F}$ of subsets of an m-element set for which there is no pair of subsets $A, B \in \mathcal{F}$ with

$$|A \cap B| \geq a, \quad |\bar{A} \cap B| \geq b, \quad |A \cap \bar{B}| \geq c, \quad \text{and} \quad |\bar{A} \cap \bar{B}| \geq d.$$

By symmetry we can assume $a \geq d$ and $b \geq c$. We show that $f_m(a, b, c, d)$ is $\Theta(m^{a+b-1})$ if either $b > c$ or $a, b \geq 1$. We also show that $f_m(0, b, b, 0)$ is $\Theta(m^b)$ and $f_m(a, 0, 0, d)$ is $\Theta(m^a)$. The asymptotic results are as $m \to \infty$ for fixed non-negative integers a, b, c, d. This can be viewed as a result concerning forbidden configurations and is further evidence for a conjecture of Anstee and Sali. Our key tool is a strong stability version of the Complete Intersection Theorem of Ahlswede and Khachatrian, which is of independent interest.

1. Introduction

Questions concerning the maximum size of set systems with certain intersection properties have a rich history in combinatorics. Say that a family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$ is t-intersecting if $|A \cap B| \geq t$ for every pair $A, B \in \mathcal{A}$. (In the case $t = 1$ we just say 'intersecting'.) One of the first observations made on this subject is that if $\mathcal{A}$ is intersecting then $|\mathcal{A}| \leq 2^{m-1}$. The general case was solved by Katona [13], who showed that the maximum t-intersecting family on $[m]$ is $\mathcal{K}(m, t)$, defined to be $\{A \subset [m] : |A| \geq (m + t)/2\}$ when $m + t$ is even, and $\{A \subset [m] : |A\backslash\{1\}| \geq (m + t - 1)/2\}$ when $m + t$ is odd. It is natural to ask the same question under the condition that $\mathcal{A}$ is k-uniform, i.e. $|A| = k$ for every $A \in \mathcal{A}$. Erdős

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et al. [7] determined the maximum size of a k-uniform intersecting family on $[m]$: it is $\binom{m-1}{k-1}$ for $m \geq 2k$ and $\binom{m}{k}$ for $m < 2k$. For general t, the maximum size of uniform t-intersecting families was a long-standing open problem of Frankl, resolved relatively recently by Ahlswede and Khachatrian. We will return to this question shortly, but first we want to introduce the subject of this paper.

We will consider families in which for every pair of subsets we constrain all four parts of the Venn diagram that the two subsets determine. To this end, we let $f_m(a, b, c, d)$ denote the maximum size of a family $\mathcal{F}$ of subsets of an m-element set for which there is no pair of subsets $A, B \in \mathcal{F}$ with

$$|A \cap B| \geq a, \quad |\bar{A} \cap B| \geq b, \quad |A \cap \bar{B}| \geq c, \quad \text{and} \quad |\bar{A} \cap \bar{B}| \geq d.$$

Some motivation for studying this function comes from the forbidden configuration problem for matrices popularised by the first author. We can identify a family $\mathcal{A} = \{A_1, \ldots, A_n\}$ of subsets of $[m]$ with an $m \times n$ (0, 1)-matrix $A = (a_{ij})$ determined by incidence, i.e. a_{ij} is 1 if $i \in A_j$ and otherwise 0. We say a matrix is *simple* if it has no repeated columns. Our incidence matrix A is simple. Let F be a fixed (0, 1)-matrix (not necessarily simple). We say A has F as a *configuration* (or A has the configuration F) if there is a submatrix of A which is a row and column permutation of F. We define forb(m, F) to be the largest n for which there is a simple $m \times n$ (0, 1)-matrix A that does not contain F as a configuration. Alternatively, forb(m, F) is the smallest n for which every simple $m \times (n + 1)$ (0, 1)-matrix A does contain F as a configuration. If we interpret A, F as incidence matrices of systems $\mathcal{A}, \mathcal{F}$ then A has an F configuration exactly when $\mathcal{A}$ has $\mathcal{F}$ as a *trace*, i.e. $\mathcal{F} \subset \{A \cap X : A \in \mathcal{A}\}$ for some $X \subset [m]$. Handling repeated columns/sets in F requires allowing the trace $\mathcal{F}$ to be a multiset.

The first forbidden configuration result was obtained independently by Sauer [17], Perles, Shelah [18], Vapnik and Chervonenkis [19]. Given a fixed integer k, then for F being the $k \times 2^k$ (0, 1)-matrix with all possible distinct columns they showed that forb$(m, F) = \sum_{i=0}^{k-1} \binom{m}{i}$. For a general k-row matrix F Füredi [10] obtained an $O(m^k)$ upper bound on forb(m, F). By this we mean that for fixed k and a given $k \times \ell$ matrix F that forb$(m, F) \leq cm^k$ for some constant c that depends only on k and ℓ. It seems hard to determine more precise asymptotic estimates for forb(m, F) as $m \to \infty$. This was achieved when F has two rows by Anstee et al. [4] and for three rows by Anstee and Sali [5], but is open in general. Also, it is not hard to see that if F consists of a single column with s 0's and t 1's then forb(m, F) is $\Theta(m^{\max\{s-1,t-1\}})$.

In this paper we solve the problem when F has two columns. Let F_{abcd} be the $(a + b + c + d) \times 2$ (0, 1)-matrix which has a rows of [11], b rows of [10], c rows of [01], d rows of [00]. Then forb$(m, F_{abcd}) = f_m(a, b, c, d)$ as defined above. By interchanging the roles of A and B we see that $f_m(a, b, c, d) = f_m(a, c, b, d)$, and by considering families of complements we see that $f_m(a, b, c, d) = f_m(d, c, b, a)$. Therefore we may and will assume throughout that $a \geq d$ and $b \geq c$. Our result for the function $f_m(a, b, c, d)$ is the following.

Theorem 1.1. *Suppose $a \geq d$ and $b \geq c$ are fixed non-negative integers. Then $f_m(a, b, c, d)$ is $\Theta(m^{a+b-1})$ if either $b > c$ or $a, b \geq 1$. Also $f_m(a, 0, 0, d)$ is $\Theta(m^a)$ and $f_m(0, b, b, 0)$ is $\Theta(m^b)$.*

Our main tool in proving this is a structural result that we will now discuss. Let numbers k, r_1, r_2 be given and suppose G and H are disjoint subsets of a ground set with $|G| = k - r_1 + r_2$. We define $\mathcal{I}^k_{r_1,r_2}$ on the pair (H, G) to be the family consisting of all sets of size k in $G \cup H$ that intersect G in at least $k - r_1 = |G| - r_2$ points. Note that any two sets in $\mathcal{I}^k_{r_1,r_2}$ have at least

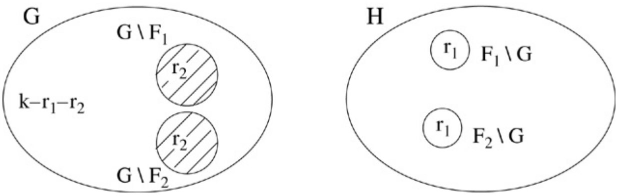

Fig. 1. Illustration of $F_1, F_2 \in \mathcal{I}^k_{r_1,r_2}$ with $|F_1 \cap F_2| = k - r_1 - r_2$.

$|G| - 2r_2 = k - r_1 - r_2$ points in common, i.e. $\mathcal{I}^k_{r_1,r_2}$ is t-intersecting, where $t = k - r$ and $r = r_1 + r_2$. For our purposes the key parameter is r, rather than $t = k - r$, so we will refer to such a family as $(k - r)$-intersecting, rather than t-intersecting (which is generally preferred in the literature). An illustration of two sets of $\mathcal{I}^k_{r_1,r_2}$ having minimum possible intersection size $k - r$ is shown in Fig. 1.

The complete intersection theorem, conjectured by Frankl, and proved by Ahlswede and Khachatrian [1], is that any k-uniform, $(k - r)$-intersecting family of maximum size on a given ground set is isomorphic to $\mathcal{I}^k_{r-p,p}$, for some $0 \le p \le r$ and some choice of G, H, where the choices depends on the size of the ground set. In this context, we have $G \cup H$ being the entire ground set, but this will not be the case elsewhere in the paper. We prove the following result.

Theorem 1.2. *Suppose $\mathcal{A}$ is a k-uniform $(k - r)$-intersecting set system on $[m]$ of size at least $(6r)^{5r+7}m^{r-1}$. Then $\mathcal{A} \subset \mathcal{I}^k_{r-p,p}$ for some $0 \le p \le r$.*

Consider, for example, the case when k is fixed, $r = k - 1$ and $\mathcal{A}$ is an intersecting family of size at least $(6k)^{5k+7}m^{k-2}$. Then $\mathcal{I}^k_{k-1-p,p}$ is only large enough to contain $\mathcal{A}$ if $p = 0$, when it is the system of all sets containing a fixed point, so we deduce that all sets in $\mathcal{A}$ contain some fixed point. This is a special case of the Hilton–Milner Theorem [12]. There are related results of Frankl [8] for t-intersecting families for fixed t ($r = k - t$), but the main power of our result is in the case when r is a constant, and k and m are arbitrary. A stronger version of Theorem 1.2 in the case $r = 1$ is in [3].

Our theorem may be compared to a number of recent stability results in extremal hypergraph theory, in which it is shown that hypergraphs close in size to the extremal configuration are in fact close in structure to the extremal configuration. One of the first such results was proved by Keevash and Mubayi [14] regarding the Turán problem for the hypergraph $\mathcal{F}_5 = \{abc, abd, def\}$. As well as being interesting in their own right, they are often useful tools in proving an exact result for the extremal problem (see for example [2,11,15]). Our result may be regarded as a stability version of the Complete Intersection Theorem, which is stronger than the usual paradigm in two ways. One is that we prove a result for all systems of order $\Omega(m^{r-1})$, where the extremal configuration has order $\Theta(m^r)$; the second is that we deduce that our system is actually contained in the extremal configuration, not just approximately. This immediately gives an independent proof of the Complete Intersection Theorem for those values of the parameters where the maximum system has size at least $(6r)^{5r+7}m^{r-1}$, although not in full generality.

Given k, r_1, r_2 and disjoint sets G, H on a ground set, we define a related family $\mathcal{F}^k_{r_1,r_2}$ on the pair (H, G) to be the family consisting of all sets of size k in $G \cup H$ that intersect G in exactly $k - r_1 = |G| - r_2$ points. We do not require that $G \cup H$ be all of the ground set and in fact in the proof of Lemma 5.4 we consider cases where $G \cup H$ is not the ground set. Now, for a choice of G, H, the system $\mathcal{F}^k_{r_1,r_2}$ is a subsystem of $\mathcal{I}^k_{r_1,r_2}$ and $|\mathcal{I}^k_{r_1,r_2} \backslash \mathcal{F}^k_{r_1,r_2}|$ is generally of a lower order of magnitude than $|\mathcal{I}^k_{r_1,r_2}|$ and $|\mathcal{F}^k_{r_1,r_2}|$. In particular, with $\mathcal{I}^k_{r_1,r_2}$, $\mathcal{F}^k_{r_1,r_2}$ both defined on the ground set $[m]$, then

$$|\mathcal{I}^k_{r_1,r_2}\backslash\mathcal{F}^k_{r_1,r_2}| \le \sum_{i=1}^{\max\{r_1,r_2\}} \binom{k-r_1+r_2}{r_2-i}\binom{m-k+r_1-r_2}{r_1-i} < m^{r-2}.$$

For fixed r_1, r_2 and with $k = \Theta(m)$ and $m-k = \Theta(m)$, then $|\mathcal{I}^k_{r_1,r_2}|$ and $|\mathcal{F}^k_{r_1,r_2}|$ are $\Theta(m^r)$, whereas $|\mathcal{I}^k_{r_1,r_2}\backslash\mathcal{F}^k_{r_1,r_2}| < m^{r-2}$.

The rest of this paper is organised as follows. In the next section we give the constructions that establish the lower bounds asserted by Theorem 1.1, and notice that they support a conjecture of Anstee and Sali. Two general inductive lemmas concerning forb(m, F) are presented in Section 3. Section 4 contains the proof of the structural Theorem 1.2. In Section 5 we use the structure theorem to establish two important cases of Theorem 1.1, which form the basis of an inductive proof of the full theorem, given in the final section.

2. Constructive lower bounds

We start by describing the constructions that give the lower bounds in Theorem 1.1, and how they relate to a conjecture of Anstee and Sali. Suppose we are given set systems $\mathcal{F}_i$ on a disjoint collection of ground sets G_i for $1 \le i \le t$. We define the product system $\mathcal{F}_1 \times \cdots \times \mathcal{F}_t$ to consist of all sets of the form $\bigcup_{i=1}^t A_i$ with $A_i \in \mathcal{F}_i$ for $1 \le i \le t$. Observe that if $\mathcal{F}_i$ has incidence matrix F_i for $1 \le i \le t$, then the incidence matrix of $\mathcal{F}_1 \times \cdots \times \mathcal{F}_t$ is $F_1 \times \cdots \times F_t$, which is defined to be the matrix consisting of all columns $[x_1 \cdots x_t]^T$, where x_i^T is any column of F_i.

There are three natural matrices that arise in the study of forbidden configurations. The identity matrix I_k is the $k \times k$ matrix with 1's on the main diagonal and 0's elsewhere. The identity-complement matrix C_k is the $k \times k$ matrix with 0's on the main diagonal and 1's elsewhere. The triangular matrix T_k is the $k \times k$ matrix with 1's above the main diagonal and 0's on and below the main diagonal. It will be convenient to work only with the matrix formulation of the problem in this section, but for comparison with other work (such as [6]) we briefly describe the set systems corresponding to these matrices: I_k corresponds to the k-singleton $\{\{i\} : 1 \le i \le k\}$, C_k to the k-co-singleton $\{[k]\backslash\{i\} : 1 \le i \le k\}$ and T_k to the k-chain $\{\emptyset, [1], [2], \ldots, [k-1]\}$.

We need one more piece of notation before we can state the Anstee–Sali conjecture. Given a (0, 1)-matrix F define $t(F)$ to be the largest number t for which there exist $A^1, \ldots, A^t \in \{I, C, T\}$ so that $A^1_k \times \cdots \times A^t_k$ does not contain an F configuration for any k (however large). For example, suppose that $F = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. It is easy to see that I_2 is not a configuration in T_k for any k, but is a configuration in each of I_k, C_k and $T_k \times T_k$ for all $k \ge 2$, and so I_2 is a configuration in any product $A^1_k \times \cdots \times A^t_k$ with $t \ge 2$ and $A^i \in \{I, C, T\}$ for $1 \le i \le t$. This shows that $t(I_2) = 1$. To see that $t(F)$ is well defined in general, we claim that if F is an $a \times b$ matrix then $t(F) \le a$. For consider a product matrix $A^1_k \times \cdots \times A^{a+1}_k$ for some $A^1, \ldots, A^{a+1} \in \{I, C, T\}$ and some $k \ge 2$. In each factor A^i_k with $1 \le i \le a$ we can choose a row R_i that contains at least one 0 and at least one 1. Considering the matrix $A^1_k \times \cdots \times A^a_k$ restricted to the rows $R_1, \ldots, R_a$ we can find an $a \times 2^a$ submatrix P_a with all possible distinct columns. If $k \ge b$ then considering $A^1_k \times \cdots \times A^{a+1}_k$ restricted to the rows $R_1, \ldots, R_a$ we can find b disjoint copies of P_a, and this in turn contains F as a configuration.

We claim that forb$(m, F) = \Omega(m^{t(F)})$. To see this, write $t = t(F)$ and use the definition to obtain $A^1, \ldots, A^t \in \{I, C, T\}$ such that $A^1_k \times \cdots \times A^t_k$ does not contain an F configuration for any k. Suppose m is given, and write $m = pt + q$ for some $0 \le q \le t - 1$. Consider the matrix $M = A^1_{k_1} \times \cdots \times A^t_{k_t}$ where $k_1 = \cdots = k_q = p + 1$ and $k_{q+1} = \cdots = k_t = p$. Then M is a simple matrix with m rows that does not contain an F configuration, as this in turn would be contained in $A^1_{p+1} \times \cdots \times A^t_{p+1}$, which is contrary to the definition of t. Since M has $\prod_{i=1}^t k_i \ge p^t$ columns we have $\text{forb}(m, F) \ge p^t = \Omega(m^{t(F)})$.

Anstee and Sali [5] conjecture that in fact $\text{forb}(m, F) = \Theta(m^{t(F)})$ for any F, i.e., the best construction, up to a constant, for any forbidden configuration, can be obtained by products of identity, identity-complement and triangular matrices. The constructions that we present in this section will be of this form, and when we later prove the upper bounds we will have established the Anstee–Sali conjecture for all two column configurations F.

Lemma 2.1. *There is a matrix with m rows and $\Omega(m^{a+b-1})$ columns containing no F_{abcd} configuration, i.e. $f_m(a, b, c, d)$ is $\Omega(m^{a+b-1})$. Also $f_m(a, 0, 0, d)$ is $\Omega(m^a)$.*

Proof. Suppose $m = pt + q$ for some t and p, q with $0 \le q \le t - 1$. Consider the matrix $M = I_{k_1} \times \cdots \times I_{k_t}$ where $k_1 = \cdots = k_q = p + 1$ and $k_{q+1} = \cdots = k_t = p$. This has m rows and at least $p^t = \Omega(m^t)$ columns. When $t = a + b - 1$, every column contains $a + b - 1$ 1's, yet the first column of F_{abcd} contains $a + b$ 1's, so it does not appear as a configuration. Furthermore, when $t = a$ each column with a 1's appears only once, so F_{a00d} is not a configuration. □

Lemma 2.2. *There is a matrix with m rows and $\Omega(m^b)$ columns containing no F_{0bb0} configuration, i.e. $f_m(0, b, b, 0)$ is $\Omega(m^b)$.*

Proof. Write $m = pb + q$ for some $0 \le q \le b - 1$, and consider the matrix $M = I_{k_1} \times \cdots \times I_{k_{b-1}} \times T_{k_b}$, where $k_1 = \cdots = k_q = p + 1$ and $k_{q+1} = \cdots = k_b = p$. This has m rows and at least $p^b = \Omega(m^b)$ columns. In the matrix F_{0bb0} the rows [10] and [01] each appear b times. In any two columns of M and the rows corresponding to a given factor I_{k_i}, each of [10] and [01] appears at most once (as column sums of I_{k_i} are one). In any two columns of M and the rows corresponding to the factor T_p we cannot find two rows with both [10] and [01] appearing (as the factor T_p does not have a configuration I_2). Therefore F_{0bb0} is not a configuration in M. □

Remark. It is easy to give constructions with better constants. For Lemma 2.1 we can take all columns with at most $a + b - 1$ 1's, and for Lemma 2.2 we can take $M' \times T_p$, where M' is a matrix with $m - p$ rows and all columns with at most $b - 1$ 1's. We gave product constructions to illustrate the Anstee–Sali conjecture.

3. Two forbidden configuration lemmas

The following two forbidden configuration lemmas will be useful in this paper, and are stated in greater generality than necessary for use in other investigations of forbidden configurations. We extend our previous notation to incorporate families of configurations as follows: if $\{F_1, \ldots, F_k\}$ is a set of (0, 1)-matrices then $\text{forb}(m, \{F_1, \ldots, F_k\})$ is defined to be the largest n for which there is a simple $m \times n$ (0, 1)-matrix that does not contain an F_i configuration for any $1 \le i \le k$.

Lemma 3.1. *Let F be a $(0, 1)$-matrix with k rows. Let F_i denote the matrix with $k - 1$ rows obtained from F by deleting row i. If $\text{forb}(m, \{F_1, F_2, \ldots, F_k\})$ is $O(m^t)$ then $\text{forb}(m, F)$ is $O(m^{t+1})$.*

Proof. The proof follows what is referred to as the standard argument in [4]. Let A be a simple $m \times n$ matrix with no configuration F. Then we can decompose A as

$$A = \begin{bmatrix} 00\cdots0 & 00\cdots0 & 11\cdots1 & 11\cdots1 \\ B_1 & B_2 & B_2 & B_3 \end{bmatrix}$$

where B_2 is chosen as those columns which are repeated in the matrix obtained from A by deleting the first row, and then we have reordered the columns of A to obtain the decomposition above. Thus $B_1 B_2 B_3$ is a simple matrix with no configuration F, and so has at most forb$(m-1, F)$ columns. Also B_2 is simple and has no configuration F_i for any $1 \le i \le t$. For if it did, then A would have F as a configuration, since a row and column permutation of F is contained in

$$\begin{bmatrix} 00\cdots0 & 11\cdots1 \\ F_i & F_i \end{bmatrix},$$

and this in turn is contained in A, in the columns containing the two copies of B_2. We deduce that the number of columns in B_2 is at most forb$(m-1, \{F_1, F_2, \ldots, F_k\})$. Now the number of columns in A is the number of columns in $B_1 B_2 B_3$ added to the number of columns in B_2. Thus forb$(m, F) \le$ forb$(m-1, F) +$ forb$(m-1, \{F_1, F_2, \ldots, F_k\})$. With forb$(m, \{F_1, F_2, \ldots, F_k\})$ being $O(m^t)$, we deduce using induction that forb(m, F) is $O(m^{t+1})$. □

Lemma 3.2. *Let F be a $(0,1)$-matrix with k rows for which* forb(m, F) *is $O(m^t)$. Then with*

$$F' = \begin{bmatrix} 11\cdots1 \\ 00\cdots0 \\ F \end{bmatrix}$$

we have forb(m, F') *being $O(m^{t+1})$.*

Proof. Let A be a simple $m \times n$ matrix with no configuration F'. We consider those columns which have i 0's in the first i rows and a 1 in row $i+1$. When restricted to the remaining $m-i-1$ rows, these columns yield a simple matrix with no F configuration, and hence there are at most forb$(m-i-1, F)$ such columns. Similarly, the number of columns with i 1's in the first i rows and a 0 in row $i+1$ is at most forb$(m-i-1, F)$. This accounts for all columns in A, except possibly an all 0 and an all 1 column. Hence the number of columns in A is at most $2 + 2\sum_{i=1}^{m-1}$ forb$(m-i-1, F)$ which, by hypothesis on forb(m, F), yields that forb(m, F') is $O(m^{t+1})$. □

4. Structure of uniform set systems with large pairwise intersections

In this section we prove Theorem 1.2, which is a strong stability version of the Complete Intersection Theorem, as described in the introduction. Let positive integers $k \ge r$ be given, suppose $r = r_1 + r_2$ for some non-negative integers r_1, r_2, and suppose G and H are disjoint sets with $|G| = k - r_1 + r_2$. We define $\mathcal{I}^k_{r_1,r_2}$ and $\mathcal{F}^k_{r_1,r_2}$ on the pair (H, G) as in the introduction. Recall that $\mathcal{I}^k_{r_1,r_2}$ and $\mathcal{F}^k_{r_1,r_2}$ are both $(k-r)$-intersecting.

We will deduce our theorem from the following lemma.

Lemma 4.1. *Suppose $\mathcal{B}$ is a k-uniform $(k-r)$-intersecting set system and $\mathcal{B} \subset \mathcal{F}^k_{p,p}$ for some $p > r/2$. Then either*

(1) *there is $\mathcal{C} \subset \mathcal{B}$ with $|\mathcal{C}| \ge \binom{p(p+1)}{2p-r}^{-1} |\mathcal{B}|$ and $\mathcal{C} \subset \mathcal{F}^k_{p,r-p}$ or $\mathcal{C} \subset \mathcal{F}^k_{r-p,p}$,*

or
(2) *there is* $\mathcal{C} \subset \mathcal{B}$ *with* $|\mathcal{C}| \geq p^{-4}|\mathcal{B}|$ *and* $\mathcal{C} \subset \mathcal{F}^k_{p-1,p-1}$.

Proof. Suppose $\mathcal{B} \subset \mathcal{F}^k_{p,p}$ defined on the pair (H, G) with $|G| = k$. Say $\mathcal{B} = \{B_i : i \in I\}$ and for each $i \in I$ write $B_i = (G \backslash X_i) \cup Y_i$, where $X_i \subset G$ and $Y_i \subset H$ each have size p. Set $\mathcal{X} = \{X_i : i \in I\}$ and $\mathcal{Y} = \{Y_i : i \in I\}$. For any $i, j \in I$ we have $k - r \leq |B_i \cap B_j| = |G| - |X_i \cup X_j| + |Y_i \cap Y_j| = k - (2p - |X_i \cap X_j|) + |Y_i \cap Y_j|$, so

$$|X_i \cap X_j| + |Y_i \cap Y_j| \geq 2p - r. \tag{1}$$

First we consider the case when $\mathcal{X}$ contains $p + 1$ pairwise disjoint sets, say $X_1, \ldots, X_{p+1}$. Then given any X_i in $\mathcal{X}$, since $|X_i| = p$ there exists an index $t(i)$ with $1 \leq t(i) \leq p + 1$ so that $X_{t(i)}$ is disjoint from X_i. By (1) we have $|Y_i \cap Y_{t(i)}| \geq 2p - r > 0$. For each $i \in I$ choose a set Z_i of size $2p - r$ with $Z_i \subset Y_i \cap \bigcup_{s=1}^{p+1} Y_s$. Now $|\bigcup_{s=1}^{p+1} Y_s| \leq p(p + 1)$, so there are at most $\binom{p(p+1)}{2p-r}$ possibilities for each Z_i. Let Z be the most common of these sets, and let $\mathcal{C} = \{B_i \in \mathcal{B} : Z_i = Z\}$. Then $|\mathcal{C}| \geq \binom{p(p+1)}{2p-r}^{-1} |\mathcal{B}|$ and $\mathcal{C} \subset \mathcal{F}^k_{r-p,p}$ defined on the pair $(H \backslash Z, G \cup Z)$.

A similar argument deals with the case when $\mathcal{Y}$ contains $p + 1$ pairwise disjoint sets, say $Y_1, \ldots, Y_{p+1}$. Then any Y_i in $\mathcal{Y}$ is disjoint from some $Y_{t(i)}$ with $1 \leq t(i) \leq p + 1$, so $|X_i \cap X_{t(i)}| \geq 2p - r > 0$. For each $i \in I$ choose $Z_i \subset X_i \cap \bigcup_{s=1}^{p+1} X_s$ of size $2p - r$, let Z be the most common of these, and set $\mathcal{C} = \{B_i \in \mathcal{B} : Z_i = Z\}$. Then $|\mathcal{C}| \geq \binom{p(p+1)}{2p-r}^{-1} |\mathcal{B}|$ and $\mathcal{C} \subset \mathcal{F}^k_{p,r-p}$ defined on the pair $(H \cup Z, G \backslash Z)$.

Finally we have the case when neither $\mathcal{X}$ nor $\mathcal{Y}$ contains $p + 1$ pairwise disjoint sets. Let $X_1, \ldots, X_t$ be a maximal collection of disjoint sets from $\mathcal{X}$. Every other set in $\mathcal{X}$ meets $\bigcup_{s=1}^{t} X_s$, which has $tp \leq p^2$ points. Therefore there is some $x \in \bigcup_{s=1}^{t} X_s$ so that $\mathcal{B}' = \{B_i \in \mathcal{B} : x \in X_i\}$ has $|\mathcal{B}'| \geq p^{-2}|\mathcal{B}|$. Similarly there is some y so that $\mathcal{C} = \{B_i \in \mathcal{B}' : y \in Y_i\}$ has $|\mathcal{C}| \geq p^{-2}|\mathcal{B}'| \geq p^{-4}|\mathcal{B}|$. Also $\mathcal{C} \subset \mathcal{F}^k_{p-1,p-1}$ defined on the pair $(H \backslash \{y\} \cup \{x\}, G \cup \{y\} \backslash \{x\})$, which completes the proof of the lemma. □

Proof of Theorem 1.2. Assume $\mathcal{A}$ is a k-uniform $(k - r)$-intersecting set system on $[m]$ of size at least $(6r)^{5r+7} m^{r-1}$. Our first step is to show that there exists $\mathcal{C} \subset \mathcal{A}$ with $|\mathcal{C}| \geq 6rm^{r-1}$ and $\mathcal{C} \subset \mathcal{F}^k_{r-p,p}$, for some $0 \leq p \leq r$.

Choose a set $A_0 \in \mathcal{A}$. Then $k - r \leq |A_0 \cap B| \leq k - 1$ for every other $B \in \mathcal{A}$, so there are r possible intersection sizes. Let $k - p$ be the most common intersection size and set $\mathcal{B} = \{B \in \mathcal{A} : |A_0 \cap B| = k - p\}$. Then $\mathcal{B} \subset \mathcal{F}^k_{p,p}$ defined on the pair $([m] \backslash A_0, A_0)$, and $|\mathcal{B}| \geq (|\mathcal{A}| - 1)/r$ by choice of p.

Now form a sequence $\mathcal{B}_0, \mathcal{B}_1, \ldots$, where $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_{i+1}$ is the result of applying Lemma 4.1 to $\mathcal{B}_i$ (if possible). The sequence terminates if the outcome of Lemma 4.1 is ever option (1), but while it remains option 2 we have $\mathcal{B}_{i+1} \subset \mathcal{B}_i$ with $|\mathcal{B}_{i+1}| \geq (p - i)^{-4}|\mathcal{B}_i|$ and $\mathcal{B}_{i+1} \subset \mathcal{F}^k_{p-i-1,p-i-1}$, provided that $p - i > r/2$, so that we can apply Lemma 4.1.

The sequence must terminate while $p - i \geq r/2$. For suppose we arrive at $\mathcal{B}_i$ with $p - i \leq (r - 1)/2$. Then we have $|\mathcal{B}_i| > (r + 1)!^{-4}|\mathcal{A}|$, using the inequalities $|\mathcal{B}_{j+1}| \geq (p - j)^{-4}|\mathcal{B}_j|$ for $j = 0, 1, \ldots$ as necessary, and $|\mathcal{B}_0| \geq (|\mathcal{A}| - 1)/r$. Now $\mathcal{B}_i \subset \mathcal{F}^k_{p-i,p-i}$ and $|\mathcal{F}^k_{p-i,p-i}| < m^{2(p-i)} \leq m^{r-1}$ so we obtain $|\mathcal{A}| < (r + 1)!^4|\mathcal{B}_i| < (r + 1)!^4 m^{r-1}$, which contradicts the lower bound on $|\mathcal{A}|$ assumed for the theorem.

There are two possibilities for the termination of the sequence: either we arrive at $\mathcal{B}_i$ with $p - i = r/2$, or there is some stage at which the outcome of Lemma 4.1 is option 1, when we lose a size factor of $\binom{(p-i+1)(p-i+2)}{2(p-i+1)-r} \le \binom{(r+1)(r+2)}{r+2}$, using $p \le r$. In either case, setting $\mathcal{C} = \mathcal{B}_i$ we have

$$|\mathcal{C}| > \binom{(r+1)(r+2)}{r+2}^{-1} (r+1)!^{-4}|\mathcal{A}| > 6rm^{r-1}.$$

Here we have used the well-known inequality $\binom{a}{b} \le \left(\frac{ea}{b}\right)^b$, where e is the base of natural logarithms, to obtain $\binom{(r+1)(r+2)}{r+2}(r+1)!^4 \le (e(r+1))^{r+2}(r+1)^{4(r+1)} < (3(r+1))^{5r+6} \le (6r)^{5r+6}$. Since $|\mathcal{A}| \ge (6r)^{5r+7}m^{r-1}$ we have the required bound for the first step of the proof.

Possibly by enlarging $\mathcal{C}$, we may suppose $\mathcal{C} = \mathcal{A} \cap \mathcal{F}^k_{r-p,p}$, with $\mathcal{F}^k_{r-p,p}$ defined on some sets $([m]\backslash G, G)$, where $|G| = k + 2p - r$. We claim that for any $A \in \mathcal{A}\backslash\mathcal{C}$ we have $|G\backslash A| \le p - 1$. For consider any $A \in \mathcal{A}\backslash\mathcal{C}$ with $A = (G\backslash X) \cup Y$, where $X \subset G$, $Y \subset [m]\backslash G$. Either $|X| \ge p+1$ or $|X| \le p - 1$, since if $|X| = p$ then $A \in \mathcal{C}$. Assume $|X| \ge p+1$. Now for any $C_i \in \mathcal{C}$ we have $|C_i \cap A| \ge k - r$. Writing $C_i = (G\backslash X_i) \cup Y_i$ with $X_i \subset G$, $|X_i| = p$, $Y_i \subset [m]\backslash G$, $|Y_i| = r - p$ we have

$$\begin{aligned} k - r \le |C_i \cap A| &= |G| - |X \cup X_i| + |Y \cap Y_i| \\ &= k + 2p - r - |X| - p + |X \cap X_i| + |Y \cap Y_i|. \end{aligned}$$

This implies $|X \cap X_i| + |Y \cap Y_i| \ge |X| - p \ge 1$.

Now there are $\binom{|G|}{p} - \binom{|G|-|X|}{p} \le |X|(|G| - 1)^{p-1}$ subsets of G of size p that intersect X and $\binom{m-|G|}{r-p} - \binom{m-|G|-|Y|}{r-p} \le |Y|(m - |G| - 1)^{r-p-1}$ subsets of $[m]\backslash G$ of size $r - p$ that intersect Y. We deduce that

$$\begin{aligned} |\mathcal{C}| &\le |X|(|G| - 1)^{p-1} \cdot \binom{m-|G|}{r-p} + \binom{|G|}{p} \cdot |Y|(m - |G| - 1)^{r-p-1} \\ &< (|X| + |Y|)m^{r-1}. \end{aligned}$$

But $|X| - p \le |X \cap X_i| + |Y \cap Y_i| \le |X_i| + |Y_i| = r$, so $|X| \le r + p$, and $|Y| = k - |G| + |X| \le k - (k + 2p - r) + r + p = 2r - p$, so $|\mathcal{C}| < 3rm^{r-1}$. This contradicts our earlier lower bound, so establishes the claim that for any $A \in \mathcal{A}\backslash\mathcal{C}$ we have $|G\backslash A| \le p - 1$. This shows that $\mathcal{A} \subset \mathcal{I}^k_{r-p,p}$ defined on the sets $([m]\backslash G, G)$, which proves the theorem. □

5. The main upper bounds

Here we prove two important cases of Theorem 1.1, which will form the basis of an inductive proof of the full theorem given in the next section.

Lemma 5.1. *$f_m(0, b + 1, b, 0)$ is $O(m^b)$.*

Proof. Trivially $f_m(0, 1, 0, 0) = 1$. Assume $b > 0$. If we delete a row from $F_{0,b+1,b,0}$ we obtain either the matrix $F_{0,b,b,0}$ or the matrix $F_{0,b+1,b-1,0}$. From the following Lemma 5.2, with $r = b - 1$, we see that

$$\text{forb}(m, \{F_{0,b,b,0}, F_{0,b+1,b-1,0}\}) = O(m^{b-1}).$$

Applying Lemma 3.1 completes the proof. □

Lemma 5.2. *Suppose $\mathcal{F}$ is a set system on $[m]$ and that no pair $A, B \in \mathcal{F}$ satisfies $|\bar{A} \cap B| \geq r+1$ and $|A \cap \bar{B}| \geq r+1$ or satisfies $|\bar{A} \cap B| \geq r+2$ and $|A \cap \bar{B}| \geq r$. Then $|\mathcal{F}|$ is $O(m^r)$.*

For the convenience of the reader we will first outline the idea of the proof, before supplying the formal details. Let $\mathcal{F}^k$ denote the sets in $\mathcal{F}$ of size k. Our first step is to show that by discarding not too many sets from $\mathcal{F}$ we can assume it to have a convenient structure; namely, that $\mathcal{F}^k$ is either empty, or reasonably large and contained in some system $\mathcal{F}^k_{r_1,r_2}$ defined on some sets (H_k, G_k). Here we use Theorem 1.2, as the first hypothesis implies that $\mathcal{F}^k$ is $(k-r)$-intersecting. Next we consider any pair $t < k$ with $\mathcal{F}^t$ and $\mathcal{F}^k$ non-empty. In the second step we use the second hypothesis to show that any set $B \in \mathcal{F}^t$ satisfies $|B \cap H_k| < r_1$. Then in the third step we show that $|\mathcal{F}^t| < (k-t+r)m^{r-1}$; this is achieved by (i) showing that any $B \in \mathcal{F}^t$ intersects $H_t \cap G_k$, which follows from $|B \cap H_t| = r_1$ (by definition), $|B \cap H_k| < r_1$ (the second step), and (ii) the estimate $|H_t \cap G_k| < k-t+r$ (which also follows from the second step). Finally, in the fourth step we note that summing the estimates of the third step gives a telescoping sum of order m^r.

Proof of Lemma 5.2. We divide the proof into the steps described in the previous paragraph.

Step 1. Let $\mathcal{F}^k$ denote the sets in $\mathcal{F}$ of size k. For any k with $|\mathcal{F}^k| < (6r)^{5r+7}m^{r-1}$ we delete $\mathcal{F}^k$ from $\mathcal{F}$, thus deleting at most $(6r)^{5r+7}m^r$ sets from $\mathcal{F}$. Now $\mathcal{F}^k$ is $(k-r)$-intersecting, since if $A, B \in \mathcal{F}^k$ and $|A \cap B| < k-r$ then, using $|A| = |B| = k$, we have $|\bar{A} \cap B| \geq r+1$ and $|A \cap \bar{B}| \geq r+1$, which contradicts our first hypothesis. Thus for k with $|\mathcal{F}^k| \geq (6r)^{5r+7}m^{r-1}$ we can apply Theorem 1.2 and deduce that there are constants r_1, r_2 with $r_1 + r_2 = r$ and disjoint sets H_k, G_k with $H_k \cup G_k = [m]$, $|G_k| = k - r_1 + r_2$ so that $\mathcal{F}^k$ is contained in $\mathcal{I}^k_{r_1,r_2}$ on the ground sets (H_k, G_k). Since $|\mathcal{I}^k_{r_1,r_2} \backslash \mathcal{F}^k_{r_1,r_2}| < m^{r-2}$, by discarding at most m^{r-1} sets, we may suppose that $\mathcal{F}^k \subset \mathcal{F}^k_{r_1,r_2}$ for all k. Next we may delete all $\mathcal{F}^k$ except for those contained in $\mathcal{F}^k_{r_1,r_2}$ for some fixed r_1, r_2 with $r_1 + r_2 = r$; the resulting bound must be multiplied by the number of possible choices for r_1, r_2, which is $r+1$. Finally, we may assume that all k with $|\mathcal{F}^k| > 0$ have the same parity, at the cost of a factor of two in the bound. It suffices to obtain an $O(m^r)$ bound for $|\mathcal{F}|$, even after these reductions.

Step 2. Now consider any pair $t < k$ with $\mathcal{F}^t$ and $\mathcal{F}^k$ non-empty. By the parity assumption we have $t \leq k-2$. Consider any $B \in \mathcal{F}^t$. Note that for a $C \in \mathcal{F}^k$ we cannot have $|B \cap \bar{C}| \geq r$, as then $|C \cap \bar{B}| = |C| - |B| + |B \cap \bar{C}| \geq r+2$, which contradicts our assumptions. Now

$$r > |B \cap \bar{C}| \geq |B \cap H_k \cap \bar{C}| \geq |B \cap H_k| - |C \cap H_k| = |B \cap H_k| - r_1,$$

so $|B \cap H_k| < r + r_1$. In fact, we even have $|B \cap H_k| < r_1$. For suppose $r_1 \leq |B \cap H_k| < r + r_1$. Then

$$\begin{aligned} r > |B \cap \bar{C}| = |B \cap \bar{C} \cap H_k| + |B \cap \bar{C} \cap G_k| = |B \cap H_k| - |C \cap B \cap H_k| \\ + |\bar{C} \cap G_k| - |\bar{C} \cap \bar{B} \cap G_k|, \end{aligned}$$

and since $|\bar{C} \cap G_k| = r_2$ we deduce that

$$|C \cap B \cap H_k| + |\bar{C} \cap \bar{B} \cap G_k| > 0.$$

Now $|C \cap B \cap H_k| > 0$ is satisfied by at most $|B \cap H_k|m^{r-1} < 2rm^{r-1}$ sets $C \in \mathcal{F}^k_{r_1,r_2}$, and $|\bar{C} \cap \bar{B} \cap G_k| > 0$ is satisfied by at most $|\bar{B} \cap G_k|m^{r-1} < 2rm^{r-1}$ sets $C \in \mathcal{F}^k_{r_1,r_2}$, using the inequality $|\bar{B} \cap G_k| = |G_k| - |B \cap G_k| = k - r_1 + r_2 - (k - |B \cap H_k|) < -r_1 + r_2 + r + r_1 < 2r$. Thus we obtain the contradiction $|\mathcal{F}^k| < 4rm^{r-1}$, so we see indeed that $|B \cap H_k| < r_1$.

Step 3. It follows that $r_1 > 0$. Also, we have

$$r_1 > |B \cap H_k| \geq |B \cap H_k \cap G_t| \geq |H_k \cap G_t| - |\bar{B} \cap G_t| = |H_k \cap G_t| - r_2,$$

so $|H_k \cap G_t| < r$. Then

$$|H_t \cap G_k| = |G_k| - |G_t \cap G_k| = |G_k| - |G_t| + |G_t \cap H_k| < k - t + r.$$

Now

$$|B \cap H_t \cap G_k| = |B \cap H_t| - |B \cap H_t \cap H_k| \geq r_1 - |B \cap H_k| > 0.$$

Therefore we have

$$|\mathcal{F}^t| < |H_t \cap G_k| m^{r-1} < (k - t + r) m^{r-1}.$$

Step 4. Finally we can make the following estimate for the size of $\mathcal{F}$. Let $k_1 < k_2 < \cdots < k_z$ be those values for which $|\mathcal{F}^{k_i}| > 0$. By the above reasoning we have $|\mathcal{F}^{k_i}| < (k_{i+1} - k_i + r) m^{r-1}$, so

$$|\mathcal{F}| = \sum_{i=1}^{z} |\mathcal{F}^{k_i}| < |\mathcal{F}^{k_1}| + \sum_{i=1}^{z-1} (k_{i+1} - k_i + r) m^{r-1} < |\mathcal{F}^{k_1}_{r_1,r_2}| + (k_z + (z-1)r) m^{r-1}$$
$$= O(m^r).$$

This completes the proof of the lemma. □

Lemma 5.3. *$f_m(1, b, b, 1)$ is $O(m^b)$ for $b \geq 1$.*

Proof. Let M be a matrix with no F_{1bb1} configuration. Note that interchanging 0's for 1's (i.e. taking complements of the corresponding set system) gives a matrix M' which also has no F_{1bb1} configuration. Therefore, by deleting at most half of the columns we may suppose that all columns have a 0 in the first row. Next we may assume there are no columns with at most b 1's, by deleting at most $O(m^b)$ such columns. Let M'' be the resulting matrix and let $\mathcal{M}$ be the set system on $[m]$ of which M'' is the incidence matrix. Then for every $A \in \mathcal{M}$ we have $1 \notin A$, so for every pair $A, B \in \mathcal{M}$ we have $1 \notin A \cap B$. Hence, to avoid F_{1bb1}, we either have $A \cap B = \emptyset$ or $|A \cap \bar{B}| \leq b - 1$ or $|B \cap \bar{A}| \leq b - 1$.

In the case $b = 1$, for every pair $A, B \in \mathcal{M}$ we have $A \cap B = \emptyset$, $A \subset B$ or $B \subset A$. In this case we claim that $|\mathcal{M}| \leq 2m$. To establish this, we consider $\mathcal{M}^* = \{A \in \mathcal{M} : \emptyset \subsetneq A \subsetneq [m]\}$ and show that $|\mathcal{M}^*| \leq 2(m-1)$ by induction on m. The base case of $m = 1$ is trivial. Now suppose that $M_1, \ldots, M_t$ is a list of the elements of $\mathcal{M}^*$ that are not contained in any other elements of $\mathcal{M}^*$. Then $M_1, \ldots, M_t$ are pairwise disjoint. Also, $\mathcal{M}_i^* = \{A \in \mathcal{M} : \emptyset \subsetneq A \subsetneq M_i\}$ satisfies the same conditions as $\mathcal{M}^*$ on the ground set M_i, so by induction $|\mathcal{M}_i^*| \leq 2(|M_i| - 1)$ for $1 \leq i \leq t$. Now if $t = 1$ we have $|\mathcal{M}^*| = 1 + |\mathcal{M}_1^*| \leq 1 + 2(|M_1| - 1) < 2(m-1)$, since $M_1 \subsetneq [m]$. Also, if $t > 1$ then $|\mathcal{M}^*| = t + \sum_{i=1}^{t} |\mathcal{M}_i^*| \leq t + \sum_{i=1}^{t} 2(|M_i| - 1) \leq 2m - t \leq 2(m-1)$. Either way $|\mathcal{M}^*| \leq 2(m-1)$, and so $|\mathcal{M}| \leq 2m$.

Now suppose $k > b \geq 2$ and let $\mathcal{M}^k$ be the sets in $\mathcal{M}$ in size k. Then for every pair $A, B \in \mathcal{M}^k$ we either have $A \cap B = \emptyset$ or $|A \cap B| \geq k - (b-1)$. The following Lemma 5.4 shows that $|\mathcal{M}^k| = O(m^{b-1})$, and so $|\mathcal{M}| = O(m^b)$, as required. □

Lemma 5.4. *Suppose $r \geq 1$ and $\mathcal{F}$ is a k-uniform family of subsets of $[m]$, with $k \geq r + 2$, so that every pair $A, B \in \mathcal{F}$ is either disjoint or intersects in at least $k - r$ points, and for every $A \in \mathcal{F}$ we have $1 \notin A$. Then $|\mathcal{F}|$ is $O(m^r)$.*

Remark. For fixed k and sufficiently large $m > m(k)$ this follows from a result of Frankl and Füredi [9]. However, in our application r is fixed and k is arbitrary subject to the sole condition $k < m$.

Again, we preface the formal proof of this lemma with an informal sketch of the idea. Here we will follow the idea of Lemma 3.2, dividing our family as $\mathcal{F}\backslash\emptyset = \cup_{p\geq 2}\mathcal{F}_p$, where $\mathcal{F}_p$ is the set of all subsets $A \in \mathcal{F}$ such that p is the smallest element of A. For any $A, B \in \mathcal{F}_p$, we have $p \in A \cap B$ and $1 \in \bar{A} \cap \bar{B}$. Thus we deduce that $|A\backslash B| \leq r$ and hence $\mathcal{F}_p$ is $(k-r)$-intersecting. As in the proof of Lemma 5.2, we start with a structural step, using Theorem 1.2 to show that by discarding not too many sets from $\mathcal{F}$ we can assume that each $\mathcal{F}_p$ has a convenient structure: it is either empty, or reasonably large and contained in some system $\mathcal{F}^k_{r_1,r_2}$ defined on some sets (H_p, G_p).

The bulk of the proof is devoted to showing, for every pair p, q with $\mathcal{F}_p, \mathcal{F}_q$ non-empty, that G_p and G_q are disjoint, and H_p and H_q are disjoint (after a certain reduction to be described). Now, since $|\mathcal{F}_p| \leq |H_p \cup G_p|^r$, it will follow that

$$|\mathcal{F}| = \sum_p |\mathcal{F}_p| \leq \sum_p |H_p \cup G_p|^r \leq \left(\sum_p |H_p \cup G_q|\right)^r \leq (2m)^r,$$

which is the required bound.

There are two essentially different cases, each having its own subtleties. The first case is when $r_2 = 0$. In this case, G_p is a set of size $k-r$ and every set in $\mathcal{F}_p$ contains G_p. Here the first step of showing that G_p and G_q are disjoint is achieved by exhibiting a set in $\mathcal{F}_q$ which is disjoint from G_p. The second step is to delete all sets in $\mathcal{F}$ containing $G_p \cup G_q$ for every pair p, q: from the first step it follows that we delete at most m^r sets. The third step is a reduction of each H_p, deleting all points that belong to few sets of $\mathcal{F}_p$ and the corresponding sets from $\mathcal{F}_p$. The fourth and final step is to see that H_p and H_q are disjoint. This is achieved by showing that the existence of a single set in $\mathcal{F}_p$ containing a particular point $x \notin G_p$ restricts the possible number of sets in $\mathcal{F}_q$ containing x to less than that achieved in the third step.

The second case, when $r_2 \geq 1$, follows the same plan, but the details are rather different. One simplifying feature is that it is easy to see that k must be very large compared to r in this case. On the other hand, it becomes much harder to show that G_p and G_q are disjoint, as the sets in $\mathcal{F}_p$ do not contain all of G_p. We structure this case as follows. The first step is to show $q \in G_q\backslash G_p$. The second step is that show that every set in $\mathcal{F}_q$ is disjoint from G_p. The third step is to show that G_p and G_q are disjoint. For the fourth step, since k is large, a much simpler reduction of H_p than before, namely that every $x \in H_p$ belongs to at least one set of $\mathcal{F}_p$, is sufficient for deducing that H_p and H_q are disjoint.

Proof of Lemma 5.4. We will structure the proof as described in the preceding paragraphs.

Structural step. For each p with $2 \leq p \leq m$ we let $\mathcal{F}_p$ be the set of all subsets $A \in \mathcal{F}$ such that p is the smallest element of A. Then $\mathcal{F}\backslash\emptyset = \cup_p \mathcal{F}_p$. For any $A, B \in \mathcal{F}_p$, we have $p \in A \cap B$ and $1 \in \bar{A} \cap \bar{B}$. Thus we deduce that $|A\backslash B| \leq r$ and hence $\mathcal{F}_p$ is $(k-r)$-intersecting. Following the same reduction as in Lemma 5.2 we will assume that there are fixed r_1, r_2 with $r = r_1 + r_2$ so that whenever $\mathcal{F}_p$ is non-empty we have $|\mathcal{F}_p| \geq (6r)^{5r+7}m^{r-1}$, and $\mathcal{F}_p \subseteq \mathcal{F}_{r_1,r_2}$ on ground sets (H_p, G_p). We recall that this simplification of only considering those $\mathcal{F}_p \subseteq \mathcal{F}_{r_1,r_2}$ for a fixed choice r_1, r_2 only affects any bound obtained by a factor $r+1$, the number of possible choices for the pair r_1, r_2, and so does not affect the conclusion. We separate the remainder of the proof into two cases.

Case 1. $r_2 = 0, r_1 = r \geq 1$.

Step 1.1. For every s, G_s is a set of size $k - r \geq 2$, and every set $C \in \mathcal{F}_s$ contains G_s. Let p, q be two integers with $2 \leq q < p \leq m$ and with $\mathcal{F}_p$ and $\mathcal{F}_q$ being non-empty. Consider a set B in $\mathcal{F}_q$. Suppose that $B \cap G_p$ is non-empty. Then B intersects every set $C \in \mathcal{F}_p$, so by assumption $|B \cap C| \geq k - r$ for every $C \in \mathcal{F}_p$. Now $k - r \leq |B \cap C| \leq |B \cap G_p| + |C \cap H_p| = |B \cap G_p| + r$, so $|B \cap G_p| \geq k - 2r$, i.e. $|B \backslash G_p| \leq 2r$. If B does not contain G_p but $B \cap G_p$ is non-empty then every set $C \in \mathcal{F}_p$ intersects $B \backslash G_p$ in order to have $|B \cap C| \geq k - r$. This implies $|\mathcal{F}_p| \leq |B \backslash G_p| m^{r-1} \leq 2rm^{r-1}$, a contradiction. Therefore, if B intersects G_p it must contain G_p. Similarly if C intersects G_q it must contain G_q. Now suppose $q < p$. Since any $B \in \mathcal{F}_q$ contains the point q, and $q \notin G_p$ as $q < p$, there are at most m^{r-1} sets $B \in \mathcal{F}_q$ that contain G_p. Since $|\mathcal{F}_q| \geq (6r)^{5r+7} m^{r-1}$, there exists a set $B \in \mathcal{F}_q$ that is disjoint from G_p. In particular we see that G_p and G_q are disjoint.

Step 1.2. The number of sets in $\mathcal{F}$ containing $G_p \cup G_q$ is at most $m^{k-2(k-r)} = m^{2r-k} \leq m^{r-2}$ (since $k \geq r + 2$) so by deleting at most m^r sets we may assume that there are no such sets for any pair p, q. By the reasoning in the previous paragraph it follows that every set $B \in \mathcal{F}_q$ is disjoint from G_p and every set $C \in \mathcal{F}_p$ is disjoint from G_q.

Step 1.3. For $r \geq 2$ we may assume that for each $x \in H_p$, there are at least $2^r m^{r-2}$ sets $C \in \mathcal{F}_p$ with $x \in C$. Otherwise we can remove such sets from $\mathcal{F}_p$ and x from H_p losing at most $2^r m^{r-1}$ sets for a given p, and so at most $2^r m^r$ in total. For $r = 1$ we can assume that every $x \in H_p$ is in at least one set $C \in \mathcal{F}_p$, or we delete x from H_p without affecting any sets of $\mathcal{F}_p$.

Step 1.4. Now suppose that there exists $x \in H_p \cap H_q$. By assumption there are sets $C \in \mathcal{F}_p$ and $B \in \mathcal{F}_q$ both containing x. Then $|B \cap C| \geq k - r$ by assumption. Since B is disjoint from G_p we have $k - r \leq |B \cap C| \leq |C \backslash G_p| = r$, i.e. $k \leq 2r$. This is already a contradiction when $r = 1$, since $k \geq r + 2$, so suppose $r > 1$. Any set $C \in \mathcal{F}_q$ that contains x must contain some other set of $k - r - 1$ points in B. We have remarked above that C is disjoint from G_q and so C contains $k - r - 1$ points other than x in $B \backslash G_q$, which is a set of size r. Since C also contains G_p it contains $k - 2(k - r) = 2r - k$ other points, and so the number of such sets C is at most $\binom{r}{k-r-1} m^{2r-k} < 2^r m^{r-2}$. However we arranged above that at least $2^r m^{r-2}$ sets $C \in \mathcal{F}_p$ contain x, so for some C we have a contradiction. Therefore H_p and H_q are disjoint, as required.

Case 2. $r_2 \geq 1$.

We can assume that some $\mathcal{F}_p$ is non-empty. Then $(6r)^{5r+7} m^{r-1} \leq |\mathcal{F}_p| \leq \binom{k-r_1+r_2}{r_2} \binom{m}{r_1}$ and so $k - r_1 + r_2 \geq (6r)^{5r+7}$.

Step 2.1. For every p, G_p is a set of size $k - r_1 + r_2$, and every set $C \in \mathcal{F}_p$ is obtained from G_p by deleting a set $\bar{C} \cap G_p$ of size r_2 and adding a set $C \cap H_p$ of size r_1. Suppose $\mathcal{F}_p$ and $\mathcal{F}_q$ are non-empty with $q < p$. Observe that $q \in G_q \backslash G_p$. For we must have $q \in G_q$, otherwise $B \cap H_q$ contains q for every $B \in \mathcal{F}_q$, giving $|\mathcal{F}_q| < m^{r-1}$, a contradiction. Similarly we cannot have $q \in G_p$, since, by the definition of $\mathcal{F}_p$, for every $C \in \mathcal{F}_p$ we have $q \notin C$. But then $\bar{C} \cap G_p$ contains q for every $C \in \mathcal{F}_p$, giving $|\mathcal{F}_p| < m^{r-1}$, a contradiction.

Step 2.2. Consider a set B in $\mathcal{F}_q$. Suppose for the sake of contradiction that $B \cap G_p$ is non-empty. First we will deduce that $|B \cap G_p| \geq k - r$. If $|B \cap G_p| > r_2$ then B intersects every set $C \in \mathcal{F}_p$. Otherwise, there may be sets $C \in \mathcal{F}_p$ disjoint from B, but for any such set $\bar{C} \cap G_p$ intersects $B \cap G_p$, so there are at most $|B \cap G_p| m^{r-1} \leq rm^{r-1}$ such sets $C \in \mathcal{F}_p$ which are disjoint from B. In particular there is some $C \in \mathcal{F}_p$ that intersects B. Then by assumption $k - r \leq |B \cap C| \leq |B \cap G_p| + |C \cap H_p| = |B \cap G_p| + r_1$, so $|B \cap G_p| \geq k - 2r$, i.e. $|B\backslash G_p| \le 2r$. Now suppose that $|B \cap G_p| < k - r$. If $C \in \mathcal{F}_p$ is not disjoint from B then by our assumptions $|B \cap C| \ge k - r$, and since $|B \cap G_p| < k - r$ at least one point of $B \cap C$ is in $B\backslash G_p$. However there are most $|B\backslash G_p|m^{r-1} \le 2rm^{r-1}$ such C. We saw above that at most rm^{r-1} sets of $\mathcal{F}_p$ are disjoint from B, so $|\mathcal{F}_p| \le 3rm^{r-1}$, a contradiction. We deduce that $|B \cap G_p| \ge k - r$.

Continuing towards the contradiction, we use the previous paragraph to obtain $k - r \le |B \cap G_p| \le |G_q \cap G_p| + |B \cap H_q| = |G_q \cap G_p| + r_1$, so $|G_p \cap G_q| \ge k - r - r_1$, i.e. $|G_p\backslash G_q| = |G_q\backslash G_p| \le r + r_2 \le 2r$. Note now that every pair $B \in \mathcal{F}_q$, $C \in \mathcal{F}_p$ intersect, as $|B \cap C| \ge |G_p \cap G_q| - 2r_2 \ge k - 3r > 0$, since $k - r_1 + r_2 \ge (6r)^{5r+7}$. There are at most $|G_q\backslash G_p|m^{r-1} \le 2rm^{r-1}$ sets $B \in \mathcal{F}_q$ for which $\bar{B} \cap G_q$ intersects $G_q\backslash G_p$, so there is some set B for which $\bar{B} \cap G_q \subset G_p \cap G_q$. Similarly there is some set $C \in \mathcal{F}_p$ for which $\bar{C}\cap G_p \subset G_p\cap G_q$. Therefore $0 < |B\cap C| \le |G_p\cap G_q| - |\bar{B}\cap G_q| - |\bar{C}\cap G_p| = |G_p\cap G_q| - 2r_2$. Recalling that $q \in G_q\backslash G_p$ we obtain $|B \cap C| \le |G_p| - 1 - 2r_2 = k - r - 1$, a contradiction. We deduce that B is disjoint from G_p for every $B \in \mathcal{F}_q$.

Step 2.3. Now suppose that $G_p\cap G_q$ is non-empty. Since $0 = |B\cap G_p| \ge |G_p\cap G_q| - |\bar{B}\cap G_q| = |G_p \cap G_q| - r_2$ we have $|G_p \cap G_q| \le r_2$. Also $\bar{B} \cap G_q$ must contain $G_p \cap G_q$ for every $B \in \mathcal{F}_q$, so $|\mathcal{F}_q| < |G_p \cap G_q|m^{r-1} < rm^{r-1}$, a contradiction. Therefore G_p and G_q are disjoint.

Step 2.4. We can assume that every $x \in H_p$ is in at least one set $C \in \mathcal{F}_p$, or we delete x from H_p without affecting any sets of $\mathcal{F}_p$. It now follows that H_p and H_q are disjoint. For suppose there is some $x \in H_p \cap H_q$. By assumption there are sets $C \in \mathcal{F}_p$ and $B \in \mathcal{F}_q$ both containing x. Since B is disjoint from G_p we have $k - r \le |B \cap C| \le |C\backslash G_p| = r_1 \le r$, i.e. $k \le 2r$. This contradicts the inequality $k - r_1 + r_2 \ge (6r)^{5r+7}$, so H_p and H_q are disjoint.

In both Case 1 and Case 2 we proved that $H_p \cap H_q = G_p \cap G_q = \emptyset$ for every pair p, q with $\mathcal{F}_p, \mathcal{F}_q$ non-empty. It follows that $|\mathcal{F}| \le \sum_p |H_p \cup G_p|^r \le (2m)^r$, which is the required bound. □

6. Proof of Theorem 1.1

Finally we complete the proof of our main theorem. The lower bounds were given in Section 2, so it remains to establish the upper bounds.

Case 1. $a \ge d$ and $b > c$.

From Lemma 5.1 we have $f_m(0, b, b - 1, 0) = O(m^{b-1})$. Applying Lemma 3.2 inductively gives $f_m(a, b, b - 1, a) = O(m^{a+b-1})$. Since $f_m(a, b, c, d) \le f_m(a, b, b - 1, a)$ we have the required bound.

Case 2. $a \ge d$, $a \ge 1$ and $b = c \ge 1$.

From Lemma 5.3 we have $f_m(1, b, b, 1) = O(m^b)$. Now $f_m(a, b, b, a) = O(m^{a+b-1})$, again applying Lemma 3.2 inductively. Since $f_m(a, b, c, d) \le f_m(a, b, b, a)$ we have the required bound.

Case 3. $a \ge d$, $b = c = 0$.

It is not hard to see that $f_m(a, 0, 0, 0) = \sum_{i=0}^{a} \binom{m}{i}$ and $f_m(1, 0, 0, 1) = m+2$ (see [4]). Since $f_m(a - d, 0, 0, 0) = O(m^{a-d})$ it follows inductively from Lemma 3.2 that $f_m(a, 0, 0, d) = O(m^a)$.

Case 4. $a = d = 0$, $b = c \ge 1$.

From Lemma 5.1 we have $f_m(0, b, b, 0) \le f_m(0, b + 1, b, 0) = O(m^b)$. □

Remark. Noga Alon pointed out that an upper bound $f_m(0, b, b, 0) = O(m^b)$ also follows from a result of Kleitman [16], who showed that the maximum number of columns in an m-row $(0, 1)$-matrix in which every pair of columns differ in at most $2t$ places is $\sum_{i=0}^{t} \binom{m}{i}$. If M is a matrix with no F_{0bb0} configuration then for each k, any two columns with k 1's differ in at most $2(b-1)$ places. Summing over all k there can be at most $(m + 1) \sum_{i=0}^{b-1} \binom{m}{i} = O(m^b)$ columns.

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