Isn’t it strange that 5 penguins, 5 apples, and 5 spoons have something in common, in spite of the wild differences between a penguin, an apple, and a spoon? An attempt to explain exactly what it is the collections share drives us inexorably towards the metaphysical question concerning the nature of a number. We will mostly avoid that question today. Scientists, on the whole, like to stay away from questions of the ‘what is’ sort. They find it fruitful most of the time to concentrate rather on ‘how,’ or, more precisely, on how something works. So a physicist may not be much better than the
man in the street when called upon to define force, mass, position, or time, but he has a pretty good understanding of relations like

\[ m \frac{d^2x}{dt^2} = F, \]

or

\[ F = G \frac{mM}{R^2}, \]

telling him how a massive object moves, that is, changes position and velocity with time, when subject to a force (gravity, for example). In a similar vein, we can, in pure mathematics, correctly perform operations like

\[ 5 + 3 = 8 \]

or even

\[ 1729 \times 691 = 1104739 \]

without pausing overly long to consider the meaning of it all. Furthermore, whether the objects we combine are penguins or apples, we are sure of the remarkable phenomenon that the abstract calculation done once will allow us to use the result uniformly: five penguins when combined with three penguins will definitely yield eight penguins, and the same for apples, spoons, or human beings. I believe some philosophers have attempted a definition to the effect that ‘five is what is common to all collections of five objects,’ or something equally unsatisfactory. Like it or not, it is reasonable that this definition encourages us to consider the number 5 as a variable of sorts, which can be assigned specific values like 5 apples or 5 penguins in accordance with the needs of the situation. Schoolchildren these days are able to take the abstraction one step further, and casually consider equalities like

\[ (x + y)^2 = x^2 + 2xy + y^2. \]

The meaning of this can be explained now in terms of definite but arbitrary numbers plugged into the places of \( x \) and \( y \):

\[ (5 + 3)^2 = 5^2 + 2 \times 5 \times 3 + 3^2; \quad (1729 + 691)^2 = 1729^2 + 2 \times 1729 \times 691 + 691^2; \quad \ldots \]

It is hard not to be amazed that operations on numbers, once the height of abstraction, can now be considered as concrete interpretation.

Pythagoras (ca. 580–500 BC) is famous for the dictum that ‘all is number,’ possibly inspired by the rational relationship between the lengths of strings that produce pleasant sounds when vibrating
in unison. The scope of this example is ridiculously narrow when compared to the imagination of the modern physicist, who hesitates not to model all of reality in terms of far more complex superpositions of vibrational modes. But Pythagoras thought not just of things in terms of numbers, but also numbers in terms of things. One finds reference to puzzling dictionaries of the following sort:

1=Number of reason;
2=First female number;
3=First male number;
4=The number of just revenge;
5=Marriage;
6=Creation;

and so on. In his Metaphysics, Aristotle appears to ascribe such a correspondence to conceptual immaturity. That is, living in pre-Platonic times, Pythagoras is supposed still to have had great difficulty in thinking of a number its own, not in association with material things. Curiously enough, this is the plight of the modern arithmetic geometer as well, the enormous advances in human conceptual sophistication notwithstanding. It is surprisingly helpful to think of numbers in terms of things, preferably spaces, when trying to solve algebraic equations in integers.

A case in point is the proof of Fermat’s last theorem. There, you’ll recall one starts from the observation that it’s quite easy to add up two square numbers and find another square number as a result:

\[3^2 + 4^2 = 9 + 16 = 25 = 5^2;\]
\[5^2 + 12^2 = 25 + 144 = 169 = 13^2;\]
\[8^2 + 15^2 = 64 + 225 = 289 = 17^2.\]

However, it turns out to be impossible to add up two (non-zero) cubes and obtain another cube, or to add up two fourth powers and obtain another fourth power. In general, the theorem states the impossibility of finding non-zero triples of integers \(a, b,\) and \(c\) satisfying the relation

\[a^n + b^n = c^n\]

for any \(n\) bigger than 2. In the elaborate proof, the beginning of the long story associates to such a triple of numbers a geometric object called an elliptic curve, which has the shape of a torus:

![Torus](image)

One then proceeds to show that the resulting torus has very special internal properties (depending on \(a^n, b^n,\) and \(c^n\)), eventually rendering it an impossible object. Hence, the original triple must have been impossible.

What is equally impossible is to give you here any non-trivial sense of this procedure. Suffice it to say that it is still remarkably useful to think of numbers in terms of visible objects.
Nevertheless, on the whole, we are now pretty comfortable with recursive levels of abstraction\textsuperscript{1}:

- five penguins, five apples, five spoons $\rightarrow 5$;
- $3, 5, 10, ..., x$.

And then, large segments of the population are able to thoughtlessly perform quite complex operations like multiplication and division with a facility that would have been utterly unimaginable to Pythagoras or Plato. The best mathematics proves its worth by becoming commonplace over the period of a few thousand years. It is an excellent thing that the question ‘What is number?’ is usually regarded as absurd these days.

After numbers were allowed independent existence, it seems many an entity struggled over the intervening millenia to be accorded the status of numberhood. You have certainly heard of the convulsions surrounding zero, $-1$, or perhaps even of its imaginary square root. Our own age has seen the ascendance of binary digits, like $\text{10010111000}$

as one might find in some form inside a DVD or a message from the Cassini probe. Their initial occurrence had very little to do with the general concept of a number, however, in that such digits were simply regarded as a convenient form for representing words and images, reflecting well the mechanics of the machines that used them. But now, we like to perform rather sophisticated operations on these things, such as

$$111 + 100 = 011,$$

or

$$111 \times 100 = 001,$$

in coherent ways that must look utterly mysterious to outsiders.

A child in a British primary school once responded to my query with the sensible opinion that a number is something you can add and multiply. Indeed, without these operations, we may as well have written $\text{baababbbaaa}$ instead of the long number above\textsuperscript{2}. In any case, such strange operations have become over the last few decades the most applicable portion of mathematics, essentially because of the advantage they provide for storing and correcting large quantities of data.

Frequently at the university, I teach a course on algebraic number theory, which, by and large, is concerned with tightly structured rules for multiplying pairs or triples of integers in reasonable ways. For example, we might consider rules like

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

or

$$(a, b)(c, d) = (ac - 10bd, ad + bc).$$

For the second rule, concretely (!), we are saying something like

$$(2, 3)(5, 4) = (-110, 35).$$

What is the meaning of this? Theoretically, the reasons are similar to the situation of binary digits in that the pairs become, thereby, algebraic structures in their own right rather than symbols (merely identifying a hotel room, for example). Here is a nice law of multiplication for triples of integers:

$$(a, b, c)(d, e, f) = (ad + 13bf + 13ce, ae + bd + 13cf, af + be + cd),$$

\textsuperscript{1}Here, we indicate two. Even a moment’s reflection should convince you this is far from the whole picture. As atoms are the building blocks of molecules that come together to form cells, which, in turn, agglomerate into an organism, a variable of abstract algebra has in reality evolved through many layers of composite complexity.

\textsuperscript{2}Some philosophers have undoubtedly noted the importance of this additional structure as well in the perennial discussion about essence. My ignorance prevents me from more respectable citation.
and here is a case of it worked out with specific numbers

\[(1, 2, 3)(4, 5, 6) = (355, 247, 28).\]

A somewhat more complicated way to multiply is

\[(a, b, c)(d, e, f) = (ad + 7bf + 7ce, ae + bd + 7cf, af + be + cd).\]

The sense in which the first one is simpler is rather hard to explain in a few words, but I will remark that it yields a system much closer to the usual whole numbers. The key point is that the first system has a good notion of prime numbers similar to the familiar one, while the second adds a layer of complexity.

At the moment, such operations are probably known only to arithmeticians, interested to classify them and derive consequences important to the study of a wide range of structures in pure mathematics. But I am fond sometimes of speculating about vast cities in space built in layers each one above another, wherein the typical house address will involve such triples of integers, long after the inefficient system of naming streets with words will have been forgotten. As the addresses are stored and transmitted in abundance along channels of futuristic communication, it is not hard to imagine a demand for good systems of multiplication that will facilitate the process. Eventually, a fluid enough notation for such multi-numerals will render them fully accessible to the public, and may appear above doorways in place of these unwieldy triples. I may be forgiven for predicting their acquisition at the level of primary school, in some form or another, three thousand years hence.

*The astute reader will have noticed already an obvious connection between number and space. To describe points in the plane, one needs two numbers (or a street name and a number), but for points in space, three numbers are natural. Pairs appear in the latitude and longitude for positions on the face of the earth, or when streets intersect,
while a pilot in flight will be keenly aware of the altitude in addition to the coordinates of the point below him.

A methodology for keeping track of positions with collections of numbers in this way is called a coordinate system by learned people. Quite curiously, such a natural process of attaching numbers to objects, a direction of correspondence opposite to that of Pythagoras, appears to have required a far more precise command of scientific thinking than that of the ancient mystics. There are subtleties: you may have noticed with irritation, for example, that one cannot assign a longitude to the poles. That there can be no reasonable way to assign it can be seen by imagining yourself sitting right on the south pole and observing all the longitudinal lines coming towards you. You will then see that there will be lines arbitrarily close to you with all possible longitudes. One can change the coordinates in an attempt to avoid this. For example, a person living in Antarctica may well have covered that continent with a nice grid similar to what we usually apply to points in the plane. Of course, the grid will be cut off at some point, and the Antarctican might not particularly care about this any more than a person in Manhattan minds that 125th street comes to an end. However, if he ends up exploring the seas beyond the boundaries of his world he will wish to extend his coordinates, probably in a manner resembling the current system, but with different pairs of numbers attached to places. But the point is that no matter how he does this, he will always run into the same problem: his new system of number pairs cannot avoid collision somewhere (actually, at least two locations), and it will never be possible to assign coordinates in a nice continuous fashion to the entire surface of the earth. In a similar vein, one necessarily makes an abrupt jump from positive to negative longitude when going around the the equator (or any other latitude) once.
It is a quite general fact that the collection of numbers we use for describing locations on a geometric figure will in some complex way reflect the shape under consideration. When properly systematized into an algebra of coordinates, these observations evolve into many different branches of geometry, eventually contributing to truly profound discoveries relating number and space. In three–dimensions, for example, it is a source of great mystery that there are portions of the universe, say the interior of a black hole, that we cannot map in any way at all with numbers, even little bits at a time.

\[(A \times B) \times C = A \times (B \times C).\]
These pictures are intended to convey a visual sense that that this algebraic property of multiplication is better modeled by the interaction of circles coming together in three-dimensional space than that of points colliding in the plane. The key point is that the transition from one bracketing to the other is visibly more smooth in three dimensions than in two.

In fact, we are currently witnessing the construction of a sophisticated algebra of space underlying almost all the algebraic structures of interest to mathematicians, and in which the precise nature of multiplication is the most important point to consider. Why multiplication? One reason comes from the study of physical systems, where the position of, say, a small particle in some multi-dimensional space might be approximately represented by a sequence of numbers like

$$\psi = (2, 3, 2, 5, 6, 8, 1, 9).$$

If there is another one in state

$$\phi = (4, 9, 5, 3, 5, 8, 3, 1),$$

then the two particles in conjunction is specified, as if by black magic, using a multiplication of particles:

$$\psi \otimes \phi = (2 \times 4, 2 \times 9, 2 \times 5, 2 \times 3, 2 \times 5, 2 \times 8, 2 \times 3, 2 \times 1, 3 \times 4, 3 \times 9, 3 \times 5, 3 \times 3 \times 5, 3 \times 8, 3 \times 3 \times 1, \ldots$$

$$\ldots, 9 \times 4, 9 \times 9, 9 \times 5, 9 \times 3, 9 \times 5, 9 \times 8, 9 \times 3, 9 \times 1),$$

different again from any of those previously considered. This strange state of affairs might even constitute the ultimate origin of multiplication per se, used though we are to its being built out of addition in an elementary fashion for usual numbers. Here are some pictures illustrating a simple version of spatial algebra, which I will leave you to contemplate for a moment without further explanation:
My chief ambition as a young man reading science was to discover the basic constituents of space itself. What this could possibly mean I of course didn’t know at the time. It was something of an embarrassing curiosity that couldn’t have been divulged even to the sympathetic ear of a close friend. Perhaps I can attempt a few words now, having had a modicum of success in the classical path of scholarship, whereby the gradual gathering of precision at the core of an inchoate question is the beginning of discovery.

First let us note that space itself has a definite shape. To appreciate this, we need to consider for a moment that the evidence for the shape of anything at all is gleaned through a process of interaction and resistance. I may press against this table, for example, and deduce a contour from the gradual realization that my hand will not move further in certain directions. Even without touching an object, our most immediate sense of shape is gathered from certain particles of light that experience similar resistance, a sequence of bounces along polygonal paths, before popping into our eye. What then is the resistance of space? You may not think there is any, until you are asked anew why an attempt to jump off the ground under our feet will almost always end in a return. If I were to step out of a spaceship somewhere near the orbit of Mars, but far away from the planet itself, I may well feel nothing at all, hence, a sensation of ‘floating.’ On the other hand, to my friends in the ship, I will be seen as trapped in a specific trajectory, quite likely terminating with a plunge into the sun.
Thus, space appears to resist our pressing and flailing in a very persistent manner indeed. Among the several explanations possible, Einstein is famous for having proposed the shape of space itself as the medium of this resistance, as the rocky slopes of a canyon might direct a stream along a meandering but well-defined path. Once we become fully aware that this shape is genuinely there, it becomes imperative to ask how it may be broken down into constituent pieces, as a building into bricks, granite into atoms, or a beam of light into the colors of a rainbow when passing through a prism. The answer will quite likely elude the best minds in the world for many years to come, and certainly no such mystery is likely to yield to my own modest capabilities. Nevertheless, this is the question of questions that was uncomfortably present at the outset of my foray into the universe (of science), that led me to ponder the Pythagorean harmony (or struggle) of number and space, and eventually directs me to the algebra of space itself, of which the pale system of numbers making up quotidian concern is perhaps but a poor projection of a shadow onto the rough walls of a cave. Passing now the middle of my life, I aspire to return at some point to this beginning.

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...With equal passion I have sought knowledge. I have wished to understand the hearts of men. I have wished to know why the stars shine. And I have tried to apprehend the Pythagorean power by which number holds sway above the flux. . . Love and knowledge, so far as they were possible, led upward toward the heavens. But always pity brought me back to earth. Echoes of cries of pain reverberate in my heart. Children in famine, victims tortured by oppressors, helpless old people a burden to their sons, and the whole world of loneliness, poverty, and pain make a mockery of what human life should be...

(Bertrand Russell, autobiography)

Who among us has not burnt with this passion or suffered not from like despair? If I were to be roundly scolded for such self-indulgent distractions as the nature of number or the atoms of space, and told instead to till the fields or sweep rubbish in public avenues, perhaps my heart would know no rejoinder. But enough of humanity, faithful souls like the late Lee Won-Kook, tolerate our glassy pursuits and even help us along in them with generous gifts built up from the kind of hard work of a lifetime that puts my daily excesses to complete shame. Since I am not good with ceremonious words, I may be forgiven a small personal anecdote to help convey my thanks. My son started secondary school on Friday of last week and, as may be common among boys of that age, expressed trepidation and dislike around a new beginning. Since this lecture was on my mind, I came to tell him about the gift of Mr. Lee, from which I benefit and he as well through me. I supposed I should have urged him to count his blessings or some such thing. But I have never been a good father in that way, able to impart the proper lessons of moral sensibility in a convincing manner. Instead, I think the point of my sermon was the wonder of it all, the good will, action, and sense of purpose that manages to spread from one person to another through incidental encounters and infect the world like an anti-disease. A relevant story comes from my own father, who is a literary man himself, but who has urged me to study science starting at a young age. He remembered his science teacher from a primary school in Gwangju, gifted in the classroom, but endowed also with the kind of gentle passion for knowledge that endears in the midst of instructing. In those slower times, he would oftentimes gather the children on a hilltop near school in the dark of night to explain the planets and the galaxy, the mythology of the constellations. I heard this story around the time I was finishing university, and the conjecture that this teacher was probably the reason I had ended up a scientist. Thus, A teaches B, who influences D, E, F, slightly or profoundly. They may then come together to do something useful or truthful, or one or two may have children, who then, god willing, will carry with them some bodily memory of the good that went through A. The wonderful environment for study and research one finds at this university through the generosity of people like Mr. Lee is certainly reason enough for deep gratitude. But perhaps even more comforting on this day is our remembrance of that organic algebra of the faithful, extending its vast operations across generations and eons, keeping our migrant souls turned towards dim horizons of meaning and truth, mindful even amid our illusions of solitude.