Galois groups and fundamental groups

Minhyong Kim

In classical Galois theory, one starts with a field F, a separable closure $\bar{F}$, and considers the Galois group $G = \mathrm{Gal}(\bar{F}/F)$ consisting of the field automorphisms of $\bar{F}$ that act trivially on F. The Galois correspondence matches up the closed subgroups $H \subset G$ and the intermediate fields

$$\begin{array}{c} \bar{F} \\ | \\ L \\ | \\ F \end{array}$$

In topology, we are given a path-connected topological space X and a point $x \in X$. This data gives rise to the fundamental group $\pi_1(X, x)$ consisting of the homotopy classes of loops based at X. If one fixes a universal covering space $\tilde{X} \to X$, then there is a correspondence between subgroups of $\pi_1(X, x)$ and intermediate covering spaces

$$\begin{array}{c} \tilde{X} \\ | \\ X' \\ | \\ X \end{array}$$

In this series of lectures, we will describe Grothendieck's theory of the fundamental group [1] that provides a general formalism for unifying the examples above.

Starting from this basic theory, we will go on to outline a few related construction and applications:

(1) Actions of Galois groups of number fields on the pro-finite fundamental groups of varieties ([2], [3]).

(2) Anabelian geometry and the characterization of varieties by their fundamental groups [4].

(3) Pro-unipotent completions of fundamental groups and applications to Diophantine geometry ([6], [7], [8]).

References

[1] Oort, Frans The algebraic fundamental group. Geometric Galois actions, 1, 67–83, London Math. Soc. Lecture Note Ser., 242, Cambridge Univ. Press, Cambridge, 1997.

[2] Ihara, Yasutaka Braids, Galois groups, and some arithmetic functions. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 99–120, Math. Soc. Japan, Tokyo, 1991.

[3] Ihara, Yasutaka Some arithmetic aspects of Galois actions in the pro-p fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999), 247–273, Proc. Sympos. Pure Math., 70, Amer. Math. Soc., Providence, RI, 2002.

[4] Nakamura, Hiroaki; Tamagawa, Akio; Mochizuki, Shinichi The Grothendieck conjecture on the fundamental groups of algebraic curves. Sugaku Expositions 14 (2001), no. 1, 31–53.

[5] Amorós, J.; Burger, M.; Corlette, K.; Kotschick, D.; Toledo, D. Fundamental groups of compact Khler manifolds. Mathematical Surveys and Monographs, 44. American Mathematical Society, Providence, RI, 1996. xii+140 pp. (Especially the appendix on completions.)

[6] Deligne, P. Le groupe fondamental de la droite projective moins trois points. (French) Galois groups over Q (Berkeley, CA, 1987), 79–297, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.

[7] Coleman, Robert F. Effective Chabauty. Duke Math. J. 52 (1985), no. 3, 765–770.

[8] Kim, M. The motivic fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel. Preprint available at http://front.math.ucdavis.edu math.NT/0409456.