Fundamental groups and Diophantine geometry

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References:

The motivic fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel. Invent. Math. 161 (2005), no. 3

The unipotent Albanese map and Selmer varieties for curves
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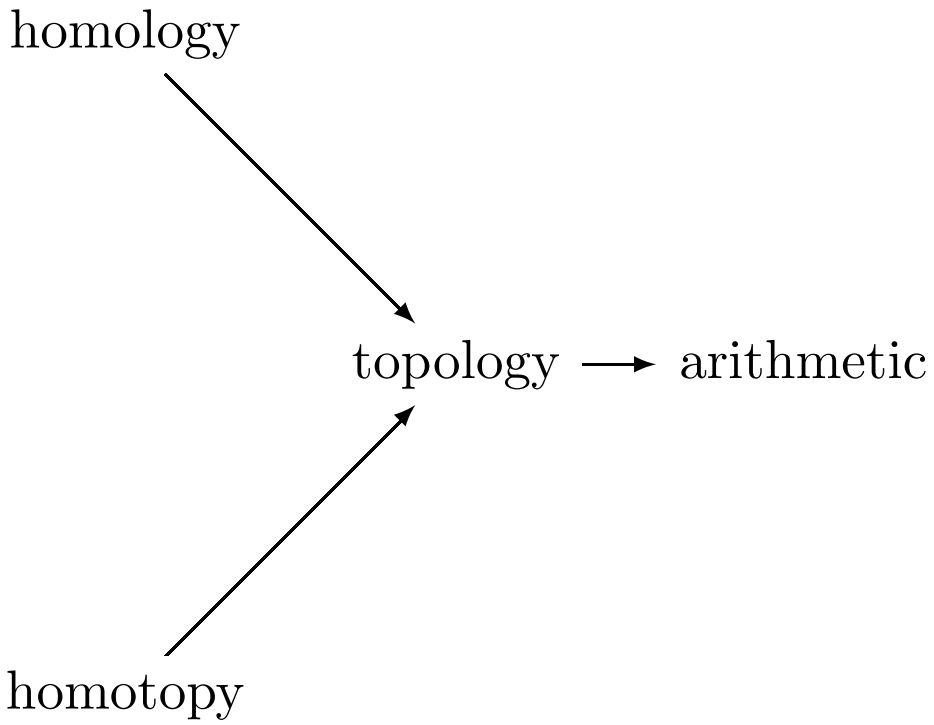

homology
topology
arithmetic
homotopy

$X/\mathbb{Q}$ smooth curve with good model

$$\mathcal{X}/R$$

for $R = \mathrm{Spec}(\mathbb{Z}[1/S])$, S a finite set of primes.

$$X(R)$$

the R-points of X.

Would like to understand its structure.

Theorem of Siegel and Faltings:

$$X(R)$$

finite if X is hyperbolic.

For example, generic equation $f(Z_0, Z_1, Z_2) = 0$ of degree ≥ 4 has only finitely many R-solutions.

Basic difficulty: *no* apparent sophisticated structure on $X(R)$.

Obviously in contrast to the case of an *elliptic curve* E where $E(R) = E(\mathbb{Q})$ ends up being a finitely-generated abelian group, in fact, a lattice.

Abelianize X?

Classical construction:

$$i_b : X \hookrightarrow J_X$$

where J_X is the *Jacobian* if X.

J_X is a natural sub-quotient of the free abelian group generated by X. In this context,

$$i_b(x) = [x] - [b].$$

Initial construction an analytic one via Hodge theory:

$$J_X := H_1(X,\mathbb{Z})\backslash H^0(\Omega_X)^* = H_1(X,\mathbb{Z})\backslash H_1(X,\mathbb{C})/F^0$$

i_b depends on choice of base-point b and sends x to

$$\alpha \mapsto \int_b^x \alpha$$

Algebraic version was constructed by Weil

in order to study $X(R)$.

In fact, J_X is a projective variety defined over $\mathbb{Q}$ and has a good model over R. i_b is defined over $\mathbb{Q}$ if $b \in X(\mathbb{Q})$.

That is, we have

$$X/R \hookrightarrow J_X/R$$

underlying

$$X(\mathbb{C}) \hookrightarrow J_X(\mathbb{C})$$

Hence,

$$X(R) \hookrightarrow J_X(R)$$

Advantage is that $J_X(R)$ is an abelian group.

Disadvantage is that $J_X(R)$ is an abelian group.

Added structure obliterates information about $X(R)$. In particular, J_X usually cannot be used for proving finiteness of $X(R)$.

Problem is that

$$J_X \otimes \mathbb{Q} = Ext^1_{MM_{\mathbb{Z}}}(\mathbb{Q}, H_1(X))$$

so abelian nature is intrinsic to the category of *motives*.

In fact, for a general variety V, the category of motives, being of homological nature, destroys information about

$$V(\mathbb{Q})$$

How to remedy this?

Weil (1938), 'Généralisation des fonctions abélienne': *non-abelian fantasy*

-'A text presented as analysis, whose significance is essentially algebraic, but whose motivation is arithmetic.' (Serre)

-Discusses need to move beyond abelian objects in study of arithmetic.

-Initiates study of vector bundles in this context to generalize the Jacobian.

-Homological nature of the Jacobian is emphasized and footnote contains allusion to importance of non-abelian π_1.

From current day perspective, moduli of semi-stable bundles corresponds to the theory of *reductive completions* of π_1.

However, at the time of Weil, no serious *arithmetic theory* of π_1.

Grothendieck (60's): Pro-finite π_1.

V variety.

$\mathrm{Cov}(V)$ category of finite étale coverings of V.

$b : \mathrm{Spec}(K) \to V$ geometric point.

Determines a fiber functor

$F_b : \mathrm{Cov}(V) \to$ Finite Sets

$$\begin{array}{ccc} Y & & Y_b \\ \downarrow & \mapsto & \downarrow \\ V & & b \end{array}$$

$$\hat{\pi}_1(V, b) := Aut(F_b)$$

Remarks:

-Isomorphic to a Galois group, but *not canonically*.

-Galois group case corresponds to b a separable closure of generic point:

$$b : \mathrm{Spec}(\overline{K(V)}) \to V$$

-Significantly, definition allows 'small' base-points naturally, as well as variation in x.

Applications slow to come in the manner envisaged by Weil, that is, over number fields.

Used for Diophantine problems over *finite fields* via the Weil conjectures, in a *horizontal* rather than *vertical* direction.

V/S scheme with geometrically connected fibers. $s \in S$ a geometric point. V_s fiber over s. $b \in V_s$ geometric point.

Then

$$\begin{array}{ccccccc} \hat{\pi}_1(V_s, b) & \to & \hat{\pi}_1(V, b) & \to & \hat{\pi}_1(S, s) & \to 0 \\ \uparrow & & & & \uparrow & \\ \text{vertical} & & & & \text{horizontal} & \end{array}$$

In Weil conjecture, study action of $\hat{\pi}_1(S, s)$ on homology of V_s.

But even here, naturality of small base-point was useful.

Substantive Diophantine work on the vertical fundamental group starts again with Grothendieck in the 80's.

In essence, involves considering the whole *fundamental groupoid* coming from

$$\hat{\pi}_1(V; b_1, b_2) := \mathrm{Isom}(F_{b_1}, F_{b_2})$$

for varying points b_1, b_2 together with composition

$$\hat{\pi}_1(V; b_2, b_3) \times \hat{\pi}_1(V; b_1, b_2) \rightarrow \hat{\pi}_1(V; b_1, b_3)$$

In fact, already sufficient to consider

$$\hat{\pi}_1(V; b, v)$$

for fixed b and varying v as *torsors* for

$$\hat{\pi}_1(V, b)$$

$$\hat{\pi}_1(V; b, v) \times \hat{\pi}_1(V, b) \to \hat{\pi}_1(V; b, v)$$

Return to the case of a compact hyperbolic curve $X/\mathbb{Q}$. Then $\hat{\pi}_1(\bar{X}, b)$ and $\hat{\pi}_1(\bar{X}; b, x)$ for $b, x \in X(\mathbb{Q})$ carry actions of the Galois group

$$\Gamma := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

Difficult arithmetic analogues of Hodge structure.

Partially understood only in special cases, say

$X = \mathbf{A}^1 \setminus \{0\}$ (cyclotomic character);

$X = E$ an elliptic curve (voluminous literature);

or $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ (Ihara's theory).

In any case, $\hat{\pi}_1(\bar{X}; b, x)$ are now Γ-equivariant torsors for $\hat{\pi}_1(\bar{X}, b)$ classified by non-abelian continuous cohomology set

$$H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

That is, given a torsor P, choosing an element $p \in P$ and measuring its lack of Γ-invariance determines a function

$$g \in \Gamma \mapsto c(g) \in \hat{\pi}_1(\bar{X}, b)$$

characterized by

$$g(p) = pc(g)$$

Thus, we have a map

$$\hat{\kappa} : X(\mathbb{Q}) \to H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

$$x \mapsto [\hat{\pi}_1(\bar{X}; b, x)]$$

Grothendieck's section conjecture:

$\hat{\kappa}$ is a bijection.

Remarks:

-In original formulation, $H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$ is in bijection with splittings of exact sequence

$$0 \to \hat{\pi}_1(\bar{X}, b) \to \hat{\pi}_1(X, b) \to \Gamma \to 0$$

and conjecture says every splitting is geometric. In this form, part of *anabelian geometry*.

-Injectivity known (Mordell-Weil $\Rightarrow$ non-abelian Mordell-Weil). Surjectivity appears very difficult.

-Many maps in arithmetic geometry of nature

'scheme-theoretic objects $\rightarrow$ Galois-theoretic objects.'

Suitable surjectivity statement key issue in problems of central interest. (Conjectures of Birch and Swinnerton-Dyer, Beilinson, Bloch-Kato.)

-Grothendieck and Deligne expected the section conjecture to be relevant to Diophantine geometry, especially the theorem of Faltings. Initial expectation appears to have been erroneous.

-Two separate deep problems:

(1) section conjecture itself;

(2) 'section conjecture implies Mordell conjecture.'

Wish to explore (2), again allowing X to be non-compact.

Main ingredient is the *motivic* fundamental group U^M (Deligne).

$$\hat{\pi}_1(\bar{X}, b)$$

$$|$$

$$U^M$$

$$|$$

$$H_1(\bar{X})_{\mathbb{Q}}$$

U^M consists of 'realizations' that are pro-unipotent pro-algebraic groups. Definition of each obtained by replacing the Galois category of coverings $\mathrm{Cov}(\bar{X})$ with various *Tannakian* categories. Results in an easier structure than $\hat{\pi}_1(\bar{X}, b)$.

Over $\mathbb{C}$, can take discrete group $\pi_1(X(\mathbb{C}), b)$ and consider $\mathbb{Q}$-unipotent completion,

$$U^B = \pi_1 \otimes \mathbb{Q}$$

defined as the group-like elements in completed Hopf algebra

$$\varprojlim_n \mathbb{Q}[\pi_1]/I^n$$

where $I \subset \mathbb{Q}[\pi_1]$ is the augmentation ideal.

Also, view as Tannaka dual to category

$$\mathrm{Un}(X(\mathbb{C}), \mathbb{Q})$$

of unipotent $\mathbb{Q}$-local systems on $X(\mathbb{C})$.

That is,

$$U^B = Aut^{\otimes}(F_b)$$

where

$$F_b : \mathrm{Un}(X(\mathbb{C}), \mathbb{Q}) \mapsto Vect_{\mathbb{Q}}$$

$$\mathcal{L} \mapsto \mathcal{L}_b$$

Well-known machinery extends this definition to many different settings including étale, De Rham, and crystalline. Just need the right category to play role of $\mathrm{Un}(X(\mathbb{C}), \mathbb{Q})$.

Étale realization:

$$\mathrm{Un}^{et}(\bar{X}, \mathbb{Q}_p)$$

is the category of unipotent $\mathbb{Q}_p$-lisse sheaves on $\bar{X}_{et}$ and

$$U^{et} := Aut^{\otimes}(F_b)$$

where

$$F_b : \mathrm{Un}^{et}(\bar{X}, \mathbb{Q}_p) \to \mathrm{Vect}_{\mathbb{Q}_p}$$

$$\mathcal{L} \mapsto \mathcal{L}_b$$

De Rham realization:

$$\mathrm{Un}^{dr}(X \otimes \mathbb{Q}_p)$$

category of of unipotent vector bundles with flat connection on $X \otimes \mathbb{Q}_p$.

$$U^{dr} := Aut^{\otimes}(F_b)$$

$$F_b : \mathrm{Un}^{dr}(X \otimes \mathbb{Q}_p) \to \mathrm{Vect}_{\mathbb{Q}_p}$$

$$(V, \nabla) \mapsto V_b$$

Crystalline realization: p prime of good reduction and $Y = X \mathrm{mod}\ p$. $\bar{b} \in Y(\mathbb{F}_p)$.

$$\mathrm{Un}^{cr}(Y)$$

is then the category of unipotent over-convergent isocrystals, thought of as connections on $X \otimes \mathbb{Q}_p$.

$$U^{cr} := Aut^{\otimes}(F_{\bar{b}})$$

$$F_{\bar{b}} : \mathrm{Un}^{cr}(Y) \rightarrow \mathrm{Vect}_{\mathbb{Q}_p}$$

$$(V, \nabla) \mapsto V(]\bar{b}[)^{\nabla=0}$$

In all realizations, also have path spaces

$$P(x) = \mathrm{Isom}^{\otimes}(F_b, F_x)$$

for points $x \in X$ which are torsors for U. Can study their variation as x varies. In contrast to pro-finite theory, variation has an analytic structure, rendering it easier to study.

All the groups and torsors carry extra structures. Most importantly, action of $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on U^{et} and $P^{et}(x)$. These structures are compatible under comparison isomorphisms.

Taken together, they constitute the motivic fundamental group U^M and motivic path torsors $P^M(x)$.

In various settings need to consider finite-dimensional quotients

$$U_n = (U)^n \backslash U$$

where the descending central series on U is given by $U^1 = U$, $U^{n+1} = [U, U^n]$. Get thereby an inductive structure

$$0 \to U^{n+1} \backslash U^n \to U_{n+1} \to U_n \to 0$$

that is important for reducing the study of U^M to vector groups, i.e., $(U^M)^{n+1} \backslash (U^M)^n$.

The basic tool for studying points is a map

$$\kappa^M : x \mapsto [P^M(x)]$$

from X to a *classifying space*

$$\mathcal{D}$$

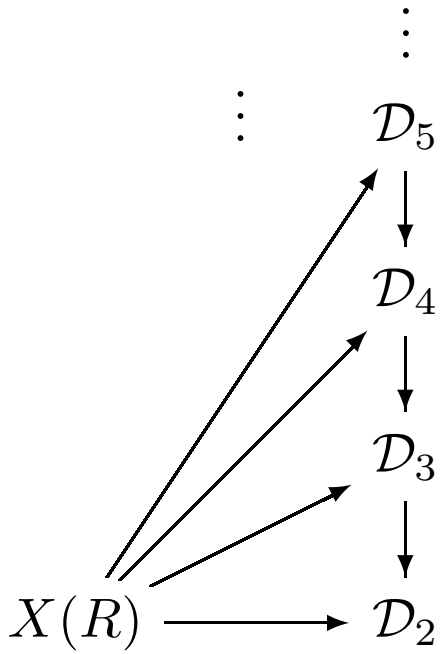
for motivic torsors.

Considerably easier than

$$\hat{\kappa} : X(R) \to H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

but refines Albanese maps.

In fact, they constitute a tower:



where the bottom map consists of Albanese maps in various realizations.

Over $\mathbb{C}$, map *higher Albanese maps*

$$\kappa_n^B : X(\mathbb{C}) \to L_n \backslash U_n^B \otimes \mathbb{C}/F^0$$

defined by Hain. Coordinates given by *iterated integrals*

$$x \mapsto \int_b^x \alpha_1 \alpha_2 \cdots \alpha_i$$

Over $\mathbb{Q}_p$, have p-adic unipotent Albanese maps

$$\kappa_n^{dr/cr} : X(\mathbb{Z}_p) \to U_n^{dr}/F^0$$

defined using p-adic iterated integrals.

For example, when $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ components of $\kappa_n^{dr/cr}$ consist of p-adic multiple polylogarithms (Furusho).

These are the local Archimedean and non-Archimedean components of κ^M that give explicit form to the maps

$$x \mapsto [P^B(x)]$$

and

$$x \mapsto [P^{dr}(x) \simeq P^{cr}(\bar{x})]$$

Global component:

$$\kappa_n^{et,glob} : X(R) \to H^1_f(\Gamma, U_n^{et})$$

$$x \mapsto P^{et}(x)$$

with target in unipotent Selmer variety

$$H^1_f(\Gamma, U_n^{et})$$

Continuous non-abelian cohomology generalizing pro-p Selmer groups

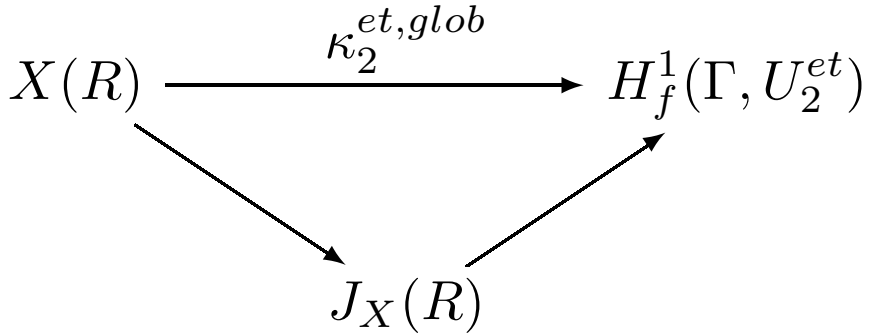
$$H^1_f(\Gamma, T_p J_X \otimes \mathbb{Q}_p)$$

occurring in BSD conjecture. Subscript refers to natural local conditions on cohomology classes.

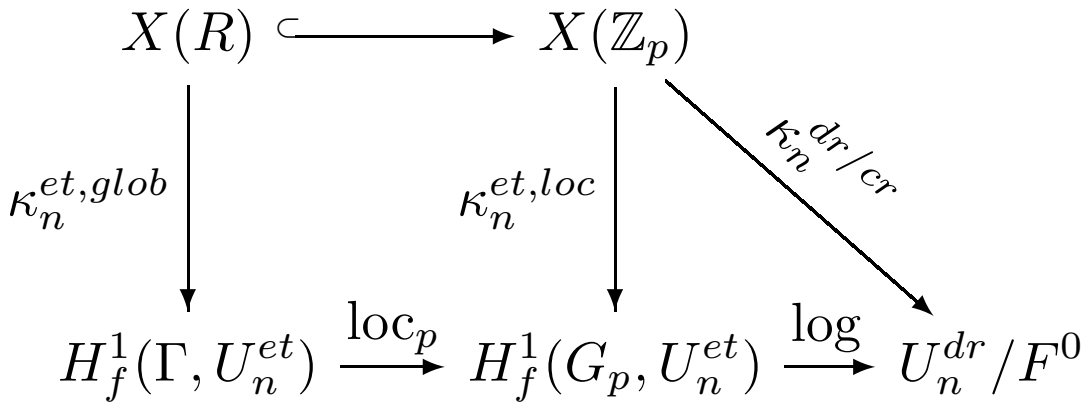
Has the natural structure of an affine algebraic variety.

But *not* abelian groups for $n > 2$. Hence preserves more of the structure of $X(R)$.

Remark: Factorization at bottom level.



To use these constructions for Diophantine geometry, have to combine into one big diagram.



The map log associates to a crystalline U^{et}-torsor $T = \mathrm{Spec}(\mathcal{T})$, the U^{dr}-torsor

$$\log(T) = \mathrm{Spec}([\mathcal{T} \otimes B_{cr}]^{G_p})$$

Commutativity of triangle comes from non-abelian p-adic comparison isomorphism (Shiho, Vologodsky, Olsson, Faltings)

$$\log(\pi^{et}(\bar{X}; b, x)) = \pi^{dr}(X \otimes \mathbb{Q}_p; b, x)$$

Proposition

$$X(\mathbb{Z}_p) \xrightarrow{\kappa_n^{dr/cr}} U_n^{dr}/F^0$$

has Zariski dense image.

Proposition

When $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$, the image of

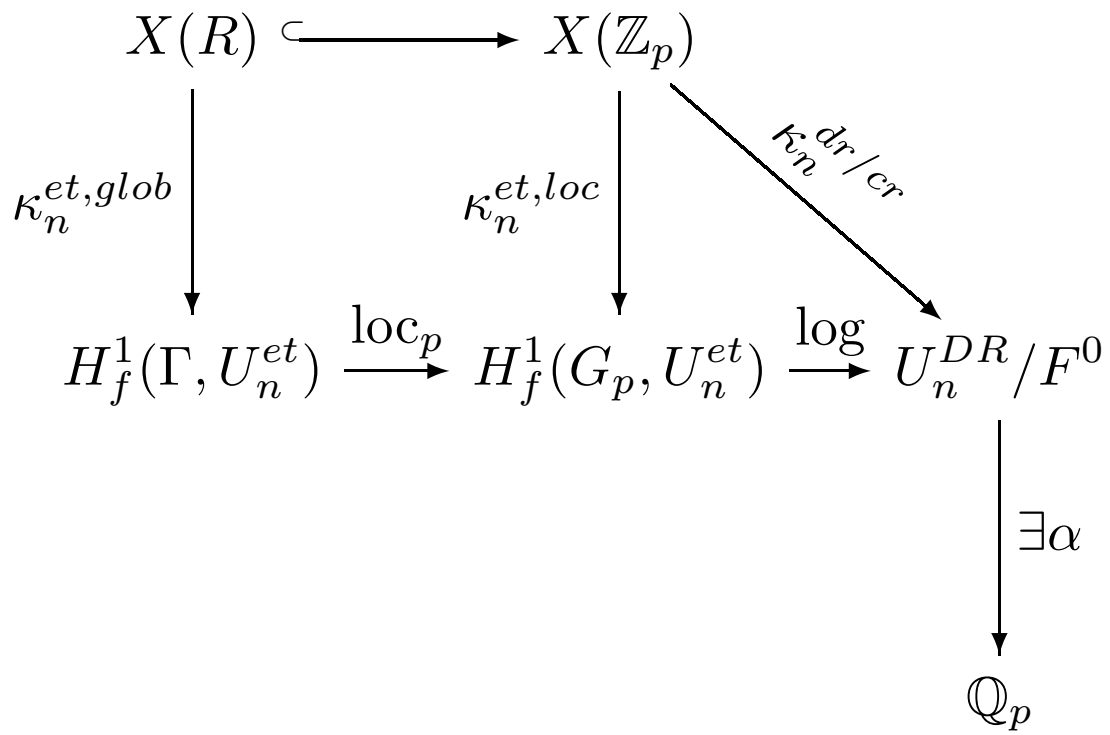
$$H^1_f(\Gamma, U_n^{et}) \xrightarrow{\mathrm{loc}_p} H^1_f(G_p, U_n^{et}) \xrightarrow{\log} U_n^{dr}/F^0$$

is *not* Zariski dense for $n >> 0$.

Idea: As n grows, both local and global dimension grows, but global dimension grows more slowly than local dimension.

Corollary

$X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ has finitely many $\mathbb{Z}[1/S]$ points.



$\alpha|\mathrm{Im}_n[H^1_f(\Gamma, U_n^{et})] = 0$

$\alpha|\mathrm{Im}_n[X(\mathbb{Z}_p)] \neq 0$

Even in general, whenever we can find n such that

$$H^1_f(\Gamma, U^{et}_n) \xrightarrow{\mathrm{loc}_p} H^1_f(G_p, U^{et}_n) \xrightarrow{\log} U^{dr}_n/F^0$$

has non-dense image, can prove finiteness of $X(R)$.

Appears to be a good framework for π_1-proof of Diophantine finiteness.

-Classical case, $n = 2$ works when

$$\mathrm{rank} J_X(R) < \dim T_e J_X$$

Method of Chabauty.

-$n = 3$ works for $X = E \setminus \{e\}$ when (E, e) is an elliptic curve of Mordell-weil rank 1. (Joint work with Tamagawa.)

-Predicted to work in general by

standard conjectures on mixed motives

such as conjecture of Bloch and Kato on surjectivity of p-adic Chern class map.

Bloch-Kato $\Rightarrow$ Mordell.

viewed as substitute for

Section conjecture $\Rightarrow$ Mordell.

Note that many ingredients predicted by Weil's non-abelian fantasy have appeared in rather concrete form.

However, unipotent completion rather than reductive completion.

Bloch-Kato+section conjecture+ϵ provides an effective algorithm for computing all rational points via *non-abelian descent*.

See slides from Tokyo seminar, October 16, 2006.

Key point is (given the other two portions)

Section conjecture $\Rightarrow$ Termination of non-abelian descent

'The Chinese character *Ki* has manifold definite meanings.'

-Tetsuji Shioda

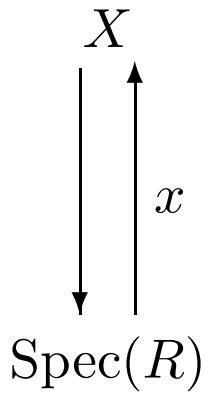
'Poetry begins to atrophy when it strays too far from music. Music begins to atrophy when it strays too far from dance.'

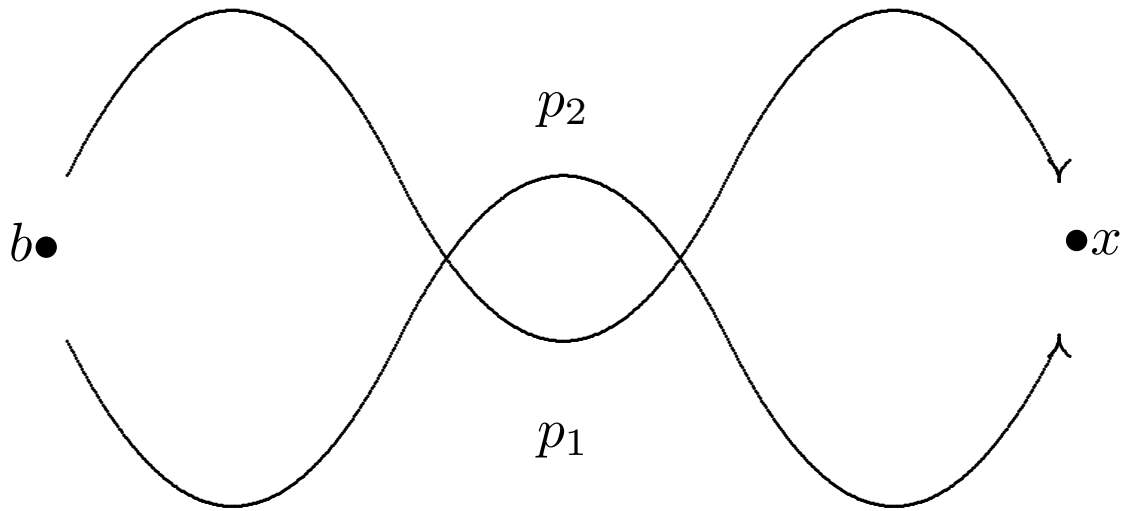
-Ezra Pound.

'Those are good words for young algebraic geometers.' (in reference to Pound)

-Shun-ichi Kimura

$$f(x_1, x_2) = 0$$


X
x
$\mathrm{Spec}(R)$


p_2
b
x
p_1