The unipotent Albanese map and Diophantine geometry

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Given a compact smooth hyperbolic curve X over a number field F, we can consider its points $X(F_v)$ in some non-archimedean completion F_v of F. Depending on some mild hypotheses one can define a $p-$adic logarithmic Albanese map

$$j_1 : X(F_v) \to H_1(X_v)/F^0$$

where the H_1 refers to De Rham homology and we are taking the quotient by the Hodge filtration. This map can be used (Chabauty) to prove Faltings' theorem on finiteness of $X(F)$ in certain circumstances, for example, if the Mordell-Weil rank of the Jacobian is strictly less than the genus of X. One proves this by showing that the pull-back via j_1 of some non-trivial linear function on H_1/F^0 has to vanish on the global points. On the other hand, such a pull-back lies inside the ring of Coleman functions, and hence, has only finitely many zeros on $X(F_v)$.

Our program is to generalize this technique by lifting j_1 to

$$j_n : X(F_v) \to U_n/F^0$$

where $U = \pi_1(X_v, x)$ is the pro-unipotent De Rham fundamental group of X_v and U_n is the quotient via the $n+1$-th level of the descending central series. This lift is constructed by using the crystalline structure on the De Rham fundamental group. The idea then is that the coordinate ring of U_n/F^0 provides many more functions ($p-$adic multiple polylogarithms) with which one attempts to annihilate the global points. At the moment, this program is not realized because of a lack of control in global Galois cohomology. However, there are interesting relations to vanishing conjectures of Jannsen, and the finiteness theorem of Siegel on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ can be proved using this idea and Soulé's vanishing theorem for Tate twists. Eventually, a careful analysis of the function spaces involved should give some effectivity (on numbers of points) and results in higher-dimensions as well.