Galois theory and Diophantine geometry

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X smooth compact hyperbolic curve over F.

$\Gamma = \mathrm{Gal}(\bar{F}/F)$

Grothendieck's section conjecture:

Splittings of

$$0 \to \widehat{\pi}_1(\bar{X}, b) \to \widehat{\pi}_1(X, b) \to \Gamma \to 0$$

up to $\widehat{\pi}_1(\bar{X}, b)$-conjugacy are in bijection with sections of

$$\begin{array}{c} X \\ \downarrow \\ \mathrm{Spec}(F) \end{array}$$

Applications?

Section conjecture $\overset{?}{\Rightarrow}$ Mordell conjecture. Expected by Grothendieck.

Initial reasoning was erroneous.

Note: Two separate problems under consideration. Wish to address the second, i.e., application of π_1 to Diophantine finiteness theorems.

Warning: Very little definite progress.

Choose basepoint $b \in X(F)$. Then

$$\text{Splittings} \leftrightarrow H^1(\Gamma, \widehat{\pi}_1(\bar{X}, b))$$

Canonical description of the map

$$\delta : X(F) \rightarrow H^1(\Gamma, \widehat{\pi}_1(\bar{X}, b))$$

$$x \mapsto [\widehat{\pi}_1(\bar{X}; b, x)]$$

Recall

$$\widehat{\pi}_1(X, b) := \mathrm{Aut}(f_b)$$

and

$$\widehat{\pi}_1(X; b, x) := \mathrm{Isom}(f_b, f_x)$$

where

$$f_b : Cov(X) \rightarrow \text{finite sets}$$

$$\begin{array}{ccc} Y & & Y_b \\ \downarrow & \mapsto & \downarrow \\ X & & b \end{array}$$

$\widehat{\pi}_1(X; b, x)$ is a torsor for $\widehat{\pi}_1(X, b)$ with a compatible Γ-action. Can study variation of this torsor with x.

To apply to Diophantine problems, helpful to recall the genus 1 situation. In this case,

$$\delta : X(F) \to H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

is obtained from Kummer theory.

Image lies in a subgroup

$$H^1_f(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

satisfying Selmer conditions.

Finiteness of $Sha[p^\infty]$ for every p is equivalent to

$$\delta : \widehat{X(F)} \simeq H^1_f(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

Well-known connections to Diophantine problems.

Higher genus: Chabauty's method. Assume

$$rkJ_X(F) < \text{dim} J_X \qquad (*)_2$$

Then $X(F)$ is finite.

$$\begin{array}{ccccc}
X(F) & \hookrightarrow & X(F_v) & & \\
\downarrow & & \downarrow & \searrow & \\
J_X(F) & \hookrightarrow & J_X(F_v) & \overset{log}{\rightarrow} & T_eJ_X(F_v) \\
 & & & & \downarrow\alpha \\
 & & & & F_v
\end{array}$$

α: linear function on the g-dimensional $\mathbb{Q}_p$-vector space T_eJ_X such that $\alpha \circ log$ vanishes on $J_X(\mathbb{Q})$.

Re-interpret Chabauty using p-adic Hodge theory and ideas of Bloch-Kato-Kolyvagin.

$X/\mathbb{Q}$ genus 1.

Kato produces $c \in H^1(\Gamma, H_1(\bar{X}, \mathbb{Q}_p))$ such that the map

$$H^1(\Gamma, H^1(\bar{X}, \mathbb{Q}_p)(1)) \rightarrow$$

$$\rightarrow H^1(G_p, H^1(\bar{X}, \mathbb{Q}_p)(1)) \stackrel{exp*}{\rightarrow} F^0 H_1^{DR}(X_p)$$

takes

$$c \mapsto L_X(1)\alpha$$

α a global 1-form. Using it to annihilate points

$$x \in X(\mathbb{Q}) \subset X(\mathbb{Q}_p) \subset T_e X = H_1^{DR}/F^0$$

gives finiteness of $X(\mathbb{Q})$ if $L(1) \neq 0$.

Chabauty's diagram can also be replaced by

$$\begin{array}{ccccc} X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) & & \\ \downarrow & & \downarrow & \searrow & \\ H^1_f(\Gamma, H_1(\bar{X}, \mathbb{Q}_p)) & \hookrightarrow & H^1_f(\Gamma, H_1(\bar{X}, \mathbb{Q}_p)) & \stackrel{log}{\to} & T_e J_X \end{array}$$

Finiteness follows whenever

$$Im(H^1_f(\Gamma, H_1(\bar{X}, \mathbb{Q}_p)))$$

is not Zariski dense.

According to this diagram, Chabauty's method is an *imprecise* higher genus analogue of Kolyvagin-Kato.

But an extension of the method unique to higher genus arises from promoting the above to a whole sequence of diagrams:

$$\begin{array}{ccccc} X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) & & \\ \downarrow & & \downarrow & \searrow & \\ H^1_f(\Gamma, U_n^{et}) & \rightarrow & H^1_f(G_p, U_n^{et}) & \stackrel{D}{\rightarrow} & U_n^{DR}/F^0 \\ & & & & \downarrow \alpha \\ & & & & \mathbb{Q}_p \end{array}$$

U's are different components of the *motivic fundamental group* of X.

One component, the De Rham fundamental group of $X_{\mathbb{Q}_p}$, uses the category

$$\mathrm{Un}(X_{\mathbb{Q}_p})$$

of unipotent vector bundles with flat connection.

That is, the objects are $(\mathcal{V}, \nabla)$, vector bundles $\mathcal{V}$ on $X_{\mathbb{Q}_p}$ equipped with flat connections

$$\nabla : \mathcal{V} \rightarrow \Omega_{X/S} \otimes \mathcal{V}$$

that admit a filtration

$$\mathcal{V} = \mathcal{V}_n \supset \mathcal{V}_{n-1} \supset \cdots \supset \mathcal{V}_1 \supset \mathcal{V}_0 = 0$$

by sub-bundles stabilized by the connection, such that

$$(\mathcal{V}_{i+1}/\mathcal{V}_i, \nabla) \simeq (\mathcal{O}^r_{X_{\mathbb{Q}_p}}, d)$$

Associated to $b \in X$ get

$$e_b : \mathrm{Un}(X_{\mathbb{Q}_p}) \rightarrow \mathrm{Vect}_{\mathbb{Q}_p}$$

The *De Rham fundamental group*

$$U^{DR} := \pi_{1,DR}(X_{\mathbb{Q}_p}, b)$$

is the pro-unipotent pro-algebraic group that represents

$$\mathrm{Aut}^{\otimes}(e_b)$$

(Tannaka dual) and the path space

$$P^{DR}(x) := \pi_{1,DR}(X; b, x)$$

represents

$$\mathrm{Isom}^{\otimes}(e_b, e_x)$$

The pro-unipotent p-adic étale fundamental group

$$U^{et}$$

and étale path spaces

$$P^{et}(x)$$

defined in the same way using the category of unipotent $\mathbb{Q}_p$ local systems.

$Z^i \subset U$ defined by descending central series.

$$U_i = Z^i \backslash U$$

Can push out torsors as well to get

$$P_i = Z^i \backslash P \times U$$

Extra structures:

U^{et}, P^{et}: Γ-action.

U^{DR}, P^{DR}: Hodge filtrations and Frobenius-actions.

$H^1_f(\Gamma, U_n)$ Selmer varieties classifying torsors that satisfying natural local conditions. Most important one: Restriction to G_p trivializes over B_{cr}.

U^{DR}/F^0 classifies U^{DR}-torsors with Frobenius action and Hodge filtration. Map

$$X(\mathbb{Q}) \rightarrow H^1_f(\Gamma, U_n)$$

associates to a point the torsor $P_n^{et}(x)$. Similarly

$$X(\mathbb{Q}_p) \rightarrow U_n^{DR}/F^0$$

uses torsor $P_n^{DR}(x)$. Compatibility provided by non-abelian p-adic comparison isomorphism.

Return to diagram.

$$\begin{array}{ccccc} X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) & & \\ \downarrow & & \downarrow & \searrow & \\ H^1_f(\Gamma, U_n^{et}) & \rightarrow & H^1_f(G_p, U_n^{et}) & \stackrel{D}{\rightarrow} & U_n^{DR}/F^0 \\ & & & & \downarrow\alpha \\ & & & & \mathbb{Q}_p \end{array}$$

Assume $(*)_n$:

$$Im(H^1_f(\Gamma, U_n^{et}) \subset U_n^{DR}/F^0$$

not Zariski dense.

$(*)_n$ implies finiteness of integral or rational points.

α algebraic function that vanishes on global points. Can be expressed in terms of p-adic iterated integrals, e.g., p-adic multiple polylogarithms in the case of $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$.

Note: Special values of such functions related to L-values. However, α here not precise enough to have such specific relations as in genus 1 case.

$(*)_n$ for $n >> 0$ implied by various motivic conjectures.

-Bloch-Kato conjecture on image of p-adic Chern class map.

-Fontaine-Mazur conjecture on geometric Galois representations.

-X affine. Jannsen's conjecture on vanishing of

$$H^2_f(\Gamma, H^n(\bar{V}, \mathbb{Q}_p(r))$$

for large r.

All provide bounds on dimensions of

$$H^1_f(\Gamma, U_n)$$

Precise form: All classes in

$$H^1_f(\Gamma, H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1))$$

are motivic. That is,

Motives $\rightarrow H^1_f(\Gamma, H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1))$

surjective.

Analogous to

$$X(F) \rightarrow H^1(\Gamma, \hat{\pi}_1(\bar{X}, b)$$

surjective.

Sort of substitute for

‘Section conjecture $\Rightarrow$ Mordell conjecture.’

To improve the situation, much more precise study of

$$H^1_f(\Gamma, U_n^{et}) \rightarrow H^1_f(G_p, U_n^{et}) \stackrel{D}{\rightarrow} U_n^{DR}/F^0$$

related to p-adic L-functions desirable, with the aim of arriving at a *precise non-abelian analogue* of the Kolyvagin-Kato method.

Closing questions for anabelian geometers:

-Can one understand precisely the lack (or necessity?) of local conditions in the section conjecture? Note that local conditions are natural in the case of $H^1_f(\Gamma, U^{et})$.

-In which circumstances is the localization map

$$H^1(\Gamma, \widehat{\pi}_1) \rightarrow H^1(G_p, \widehat{\pi}_1)$$

injective? (This should be true for hyperbolic curves according to the section conjecture.)