Non-linearity, fundamental groups, and Diophantine geometry

April 27, 2005

X: smooth variety over $\mathbb{Q}$. So X defined by polynomial equations with rational coefficients.

Arithmetic of X

$\nearrow$ $\nwarrow$

Topology Geometry

From 50's to 80's applications of topology to arithmetic came primarily from *homology*:

-Basic language of homological algebra;

-Arithmetic cohomology theories.

These days, immediately associate to X at least four different cohomology groups:

-$H^i(X(\mathbb{C}), \mathbb{Q})$: Singular cohomology of topological space given by the complex points of X.

-$H^i(\bar{X}, \mathbb{Q}_p)$: Étale cohomology with p-adic coefficients.

-$H^i_{DR}(X) = H^i(X, \Omega^\bullet_X)$: The algebraic De Rham cohomology of X.

-$H^i_{cr}(X \bmod p, \mathbb{Q}_p)$: The crystalline cohomology of X mod p.

All have ‘formally similar’ linear structures. Also, ‘compatible’ in many ways, e.g.

$$H^i(\bar{X}, \mathbb{Q}_p) \simeq H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_p$$

Supposedly accounted for by a theory of ‘motives’:

$$\begin{array}{ccc} \text{Varieties} & \longrightarrow & \text{Motives} \\ & \searrow & \downarrow \\ & & \text{Vector Spaces} \end{array}$$

A focal point of this lecture: In the homological approach, main ideas and techniques are *linear*.

Important contrast:

Linear $\leftrightarrow$ Non-linear

In topology,

Homology $\leftrightarrow$ Homotopy

But also, the linear theory provides input into the non-linear theory via linearization.

Topological version,

$$\pi_1/[\pi_1,\pi_1] = H_1$$

However, one expects a fully non-linear theory, when fully understood, to be more powerful than a linearized version.

The techniques of homotopy as applied to arithmetic should yield information not accessible to homology.

Homology $\longrightarrow$ study of L-functions.

Homotopy $\longrightarrow$ study of Diophantine sets.

Note: L-functions also yield information about Diophantine sets. But actually, linearized Diophantine sets. For example, might be interested in

$$X(\mathbb{Q})$$

the set of rational points. But L-functions give information on something like $\mathbb{Z}[X(\mathbb{Q})]$.

Cannot yield, in any obvious way, Faltings' theorem (Mordell conjecture), for example.

Classically, a basic 'homological' construction used in Diophantine geometry of a hyperbolic curve X is the *Jacobian* J_X. A very linear-object:

$$J_X :=$$

(Free abelian group generated by X)/(eq. rel.)

Itself an abelian group variety.

Homological nature: Over $\mathbb{C}$

$$J_X = H_1(X, \mathbb{Z}) \backslash H_1(X, \mathbb{C}) / F^0$$

If we fix a base-point $x \in X$, can map the curve into the Jacobian using the Albanese map:

$$X \to J_X$$

$$y \mapsto [x - y]$$

Can also express this map using abelian integrals

$$y \mapsto [\omega \mapsto \int_x^y \omega]$$

Jacobian can also be thought of as a parameter space for line bundles of degree zero on X. Thereby, we can view X as parametrizing line bundles on itself.

Ends up being a key idea: parametrize other geometric objects using points of X to somehow enlarge the information in the point.

Weil tried to use this parametrization to prove the Mordell conjecture. Doesn't quite work because J has too many points: also related to linearity.

The eventual proof of Mordell conjecture involves a non-linear parametrization. A family of curves (Kodaira-Parshin construction):

$$\begin{array}{c} Z \\ \downarrow \\ X \end{array}$$

$$y \mapsto Z_y$$

$$(\mapsto H^1_{et}(\bar{Z}))$$

Similar parametrization occurs in Wiles' theorem (Frey-Hellegouarch correspondence).

Somewhat ad hoc. Desirable to have a non-linear version of Weil's construction.

The idea of using homotopy in this regard stems in part from Grothendieck's 'anabelian' philosophy. Specifically, the *section conjecture*.

Relates $X(\mathbb{Q})$ to the pro-finite fundamental group.

General idea is that for a class of schemes maps between pro-finite fundamental group should be induced by a map of schemes.

However, very difficult!

Another approach: There is a non-linear Albanese map

$$X \to \text{classifying space}$$

$$y \mapsto [\pi_1(X; x, y)]$$

Actual object involved is the De Rham fundamental group (Chen, Hain, Morgan,. . .).

$$\pi_1^{DR}(X, x)$$

Unipotent completion of usual fundamental group.

Obtained by taking complex linear combinations $\Sigma c_l[l]$ of paths and regarding two such as equivalent if they have the same ‘parallel transport’ action on vector bundles with unipotent connection.

Actually a pro-algebraic group. Each quotient

$$Z^{n+1}\backslash\pi_1^{DR}(X,x)$$

by a subgroup in the descending central series is a finite-dimensional unipotent algebraic group.

Can also construct De Rham path spaces

$$\pi_1^{DR}(X; x, y)$$

which are *torsors* for $\pi_1^{DR}(X, x)$.

These groups and path spaces have a Hodge filtration

$$\pi_1^{DR}(X; x, y) \cdots \supset F^{-2} \supset F^{-1} \supset F^0$$

and an integral lattice $L_y \subset \pi_1^{DR}(X; x, y)$ coming from the image of the topological paths. The torsor structure is compatible with the extra structures. These torsors end up being classified by

$$L \backslash \pi_1^{DR}(X, x) / F^0$$

Non-linear Albanese map (Hain)

$$X \longrightarrow L\backslash\pi_1^{DR}(X,x)/F^0$$

is given by

$$y \mapsto [\pi_1^{DR}(X;x,y)]$$

Note: The ‘abelianization’ gives us the usual Albanese map:

$$y \mapsto [Z^2\backslash\pi_1^{DR}(X;x,y)]$$

The most important point for Diophantine applications is that one can use this map to pull natural functions back from the classifying space to X.

For example, when $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$, the algebraic functions on $\pi_1^{DR}(X, x)/F^0$ form a vector space

$$\mathbb{C}[\alpha_w]$$

spanned by functions α_w, indexed by words w in a two letter alphabet $\{A, B\}$. The function α_w corresponding to the word

$$w = A^{k_1-1}BA^{k_2-1}B \cdots A^{k_m-1}B$$

gives rise to the function

$$L_w(z) = \Sigma_{n_1>n_2>\cdots>n_m} \frac{z^{n_1}}{n_1^{k_1} \cdots n_m^{k_m}},$$

a *multiple polylogarithm*. Ubiquitous in arithmetic, geometry, and physics.

More generally, the non-linear Albanese map can be expressed using iterated integrals.

For arithmetic applications, need a p-adic version of the construction above: Theory of the crystalline fundamental group and p-adic Hodge theory. Leads to p-adic multiple polylogarithms and p-adic iterated integrals. Important distinction from the archimedean theory is that the functions we end up with are single-valued.

In the case of $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$, applies to the finiteness of $X(\mathbb{Z}[1/S])$ the set of S-integers. (Theorem of Siegel.)

Theorem: On $X(\mathbb{Z}_p)$ there exists a non-trivial linear combination

$$\Sigma_w c_w P_w$$

of p-adic multiple polylogarithms that vanishes on

$$X(\mathbb{Z}[1/S]) \subset X(\mathbb{Z}_p).$$

For compact curves of higher genus, reduces Faltings theorem to conjectures of Beilinson.

These results come from the theory of the motivic fundamental group, motivic non-linear Albanese map, (Deligne) and 'linearization'.

Linearization:

$$0 \to Z^n/Z^{n+1} \to \pi_1^M/Z^{n+1} \to \pi_1^M/Z^n \to 0$$

Use of the motivic fundamental group and motivic Albanese map:

$$\begin{array}{ccc} X(\mathbb{Z}[1/S]) & \rightarrow & \pi_1^{DR}(X \otimes \mathbb{Q}_p, x) \\ \downarrow & & \uparrow \\ H^1_f(\Gamma_S, \pi_1^{et}(\bar{X}, x) \otimes \mathbb{Q}_p) & \rightarrow & H^1_f(G_p, \pi_1^{et}(\bar{X}, x) \otimes \mathbb{Q}_p) \end{array}$$

Perhaps can view as a partial vindication of Grothendieck's 'anabelian' vision. Especially mysterious in that the anabelian philosophy gets connected in the process to the theory of (mixed) motives.