A p-adic non-abelian criterion for good reduction of curves

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1 Introduction

Let K be a complete discrete valuation field of characteristic 0, with valuation ring $\mathcal{O}_K$ and perfect residue field k of positive characteristic p. We fix an algebraic closure $\overline{K}$ of K and denote by G_K the Galois group of $\overline{K}$ over K.

Let X_K denote a smooth, proper, geometrically irreducible scheme over $\mathrm{Spec}(K)$. An interesting question in Arithmetic Geometry is the question of deciding if X_K has or has not good reduction. For example, if A_K is an abelian scheme over $\mathrm{Spec}(K)$ then we have:

Theorem 1.1 (Néron, Ogg, Shafarevich, Serre-Tate). *A_K has good reduction if and only if for some (all) prime integer $\ell \neq p$ the ℓ-adic G_K-representation $T_\ell(A_K)$ is unramified.*

and

Theorem 1.2 (Fontaine, Mokrane, Coleman-Iovita, Breuil). *A_K has good reduction if and only if the p-adic G_K-representation $T_p(A_K)$ is crystalline.*

It is not expected that such theorems hold in general. For example if X_K is a smooth, proper, geometrically irreducible curve over $\mathrm{Spec}(K)$, then for all prime integers $\ell \neq p$, the ℓ-adic G_K-representations $\mathrm{H}_i\big(X_{\overline{K}}, \mathbb{Z}_\ell\big)$ are unramified and the p-adic G_K-representations $\mathrm{H}_i\big(X_{\overline{K}}, \mathbb{Z}_p\big)$ are crystalline for $i = 0, 1, 2$, but the converse is not always true.

It is known that a criterion for good reduction of the curve X_K has to be non-abelian. In order to be more precise let us first assume that X_K has semi-stable reduction, i.e. there is a regular scheme X, proper and flat of relative dimension 1 over $\mathcal{O}_K$ whose generic fiber is X_K. Let us also assume that the genus of X_K is larger or equal to 2. We fix a geometric point b of X_K and for every prime integer ℓ denote by $\pi_1^{(\ell)}$ the maximal pro-ℓ quotient of the geometric, étale fundamental group $\pi_1(X_{\overline{K}}, b)$ of X_K. We denote by $\{\pi_1^{(\ell)}[n]\}_{n\geq 1}$ the lower central series of $\pi_1^{(\ell)}$ and let us recall that for each $n \geq 1$ we have natural, outer representations of G_K on the quotients $\pi_1^{(\ell)}/\pi_1^{(\ell)}[n]$.

Theorem 1.3 (Takayuki Oda). *X_K has good reduction if and only if for some (all) prime integer $\ell \neq p$ the outer representations $\pi_1^{(\ell)}/\pi_1^{(\ell)}[n]$ are unramified for all $n \geq 1$. In fact it is sufficient that this happens for all $1 \leq n \leq 3$ for X_K to have good reduction.*

The main purpose of this article is to state and prove the p-adic analogue of theorem 1.3. At a first glance our theorem would read: *X_K has good reduction if and only if $\pi_1^{(p)}/\pi_1^{(p)}[n]$ is crystalline for every $n \geq 1$,* but a quick analysis shows that this statement does not make sense.

In fact this problem has partially been investigated in [Vo], [Ol], [Ha] and it has become clear that instead of working with $\pi_1^{(p)}$ one should work with the p-adic unipotent fundamental group of X_K. Let us briefly explain the setting. We denote by K_0 the maximal unramified subfield of K, i.e. the fraction field of $\mathbb{W}(k)$ in K, assume that there exists a point $b \in X(\mathcal{O}_K)$ and denote by $b_{\overline{K}}$ and b_K the corresponding points of $X_K(\overline{K})$, respectively $X_K(K)$. We then denote by $G^{\mathrm{et}} := G^{\mathrm{et}}\big(X_{\overline{K}}, b_{\overline{K}}\big)$ and $G^{\mathrm{dR}} := G^{\mathrm{dR}}\big(X_K, b_K\big)$ the unipotent p-adic étale, respectively the unipotent de Rham fundamental groups. They are characterized by the property that their algebraic representations on finite dimensional $\mathbb{Q}_p$-vector spaces (resp. K-vector spaces) classify unipotent lisse $\mathbb{Q}_p$-adic étale sheaves on $X_{\overline{K}}$ (resp. vector bundles on X_K endowed with integrable connections). The first important property of these is that they are pro-algebraic groups over

$\mathbb{Q}_p$, respectively over K, with extra structure for example G^{et} has a natural action of G_K by automorphisms.
We denote by $B_{\mathrm{cris,K}}$ and $B_{\mathrm{st,K}}$ the base changes to K of Fontaine's rings B_{cris} and B_{st}, respectively. Then our result could be simply formulated as the following sequence of statements.

Theorem 1.4. *G^{et} is semi-stable i.e. we have a natural isomorphism as group-schemes over $B_{\mathrm{st,K}}$, G_K-equivariant:*

$$G^{\mathrm{et}} \times_{\mathbb{Q}_p} B_{\mathrm{st,K}} \cong G^{\mathrm{dR}} \times_K B_{\mathrm{st,K}}.$$

Definition 1.5. We say that G^{et} is crystalline if the above isomorphism holds for the base changes to $B_{\mathrm{cris,K}} \subset B_{\mathrm{st,K}}$, i.e. we have a canonical isomorphism as group-schemes over $B_{\mathrm{cris,K}}$, G_K-equivariant

$$G^{\mathrm{et}} \times_{\mathbb{Q}_p} B_{\mathrm{cris,K}} \cong G^{\mathrm{dR}} \times_K B_{\mathrm{cris,K}},$$

whose base change to $B_{\mathrm{st,K}}$ is the one in theorem 1.4.

Now we can formulate our main result as:

Theorem 1.6. *X_K has good reduction if and only if G^{et} is crystalline.*

We'll now be more precise and formulate the sequence of stamenets above in the language of p-adic Hodge theory. As G^{et} and G^{dR} are pro-algebraic groups we may write them as $G^{\mathrm{et}}\big(X_{\overline{K}}, b_{\overline{K}}\big) = \mathrm{Spec}\big(A_\infty^{\mathrm{et},\vee}\big)$ and $G^{\mathrm{dR}}\big(X_K, b_K\big) := \mathrm{Spec}\big(A_\infty^{\mathrm{dR},\vee}\big)$, where $A_\infty^{\mathrm{et},\vee}$ and $A_\infty^{\mathrm{dR},\vee}$ are Hopf-algebras over $\mathbb{Q}_p$ and K-respectively. Then, theorem 1.4 can be expressed in a more precise way as

Theorem 1.7. *(1) The $\mathbb{Q}_p$-algebra $A_\infty^{\mathrm{et},\vee}$ is the direct limit $\lim\limits_{n\to\infty} \mathcal{E}_{n,b}^{\mathrm{et},\vee}$ of finite dimensional $\mathbb{Q}_p$-representations of G_K such that each $\mathcal{E}_{n,b}^{\mathrm{et},\vee}$ is semistable in the sense of Fontaine and $\mathcal{E}_{1,b}^{\mathrm{et},\vee} = \mathbb{Q}_p$ provides the structure as $\mathbb{Q}_p$-algebra;*

(2) the K-algebra $A_\infty^{\mathrm{dR},\vee}$ is the direct limit $\lim\limits_{n\to\infty} \mathcal{E}_{n,b}^{\mathrm{dR},\vee}$ where each $\mathcal{E}_{n,b}^{\mathrm{dR},\vee}$ is a filtered K-vector space and $\mathcal{E}_{1,b}^{\mathrm{dR},\vee} = K$ (with trivial filtration $\mathrm{Fil}^0 K = K$, $\mathrm{Fil}^1 K = 0$) provides the structure as K-algebra;

(3) there exist isomorphisms $\psi_n\colon \mathcal{E}_{n,b}^{\mathrm{dR},\vee} \cong D_{\mathrm{st}}\big(\mathcal{E}_{n,b}^{\mathrm{et},\vee}\big) \otimes_{K_0} K$ as filtered K-vector spaces, compatibly for varying n
so that

(i) the induced isomorphism

$$\psi_\infty\colon A_\infty^{\mathrm{dR},\vee} \cong D_{\mathrm{st}}\big(A_\infty^{\mathrm{et},\vee}\big) \otimes_{K_0} K$$

is an isomorphism of Hopf algebras over K;

(ii) for $n = 2$ the dual of the isomorphism

$$\big(\mathcal{E}_{2,b}^{\mathrm{dR},\vee} / \mathcal{E}_{1,b}^{\mathrm{dR},\vee}\big) \cong \big(D_{\mathrm{st}}\big(\mathcal{E}_{2,b}^{\mathrm{et},\vee} / \mathcal{E}_{1,b}^{\mathrm{et},\vee}\big) \otimes_{K_0} K$$

induced by ψ_2 is the p-adic comparison isomorphism (see [AI]) of filtered K-vector spaces

$$\mathrm{H}^1_{\mathrm{dR}}\big(X_K/K\big) \cong D_{\mathrm{st}}\big(\mathrm{H}^1_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)\big) \otimes_{K_0} K.$$

Let us explain the notations in theorem 1.7. If we denote by $\mathcal{E}^{\rm et}_{n,b}$, respectively by $\mathcal{E}^{\rm dR}_{n,b}$ the $\mathbb{Q}_p$, respectively K duals of $\mathcal{E}^{\rm et,\vee}_{n,b}$ and $\mathcal{E}^{\rm dR,\vee}_{n,b}$ then these are naturally represenatations of $G^{\rm et}$, respectively $G^{\rm dR}$ and therefore there are unipotent étale local systems $\mathcal{E}^{\rm et}_n$ and universal unipotent $\mathcal{O}_{X_K}$-modules endowed with integrable connections $\mathcal{E}^{\rm dR}_n$ such that $\mathcal{E}^{\rm et}_{n,b}$ is the fiber of $\mathcal{E}^{\rm et}_n$ at $b_{\overline{K}}$ and $\mathcal{E}^{\rm dR}_{n,b}$ is the fiber of $\mathcal{E}^{\rm dR}_n$ at b_K. Moreover, these sheaves have very interesting universal properties which are characterized in section 3.5 and in section 3.6 we show how they can be inductively constructed.

Theorem 1.7 is proven via a p-adic comparison isomorphism between these two systems of objects. In fact, we prove a finer result. Write $\mathbb{W} := \mathbb{W}(k)$ and denote by $\mathcal{O} := \mathbb{W}[\![Z]\!]$ and by $\mathcal{O} \longrightarrow \mathcal{O}_K$ the $\mathbb{W}$-algebra homomorphism sending Z to π. Let $P_\pi(Z)$ be the minimal polynomial of π over $\mathbb{W}$. Let $\mathcal{O}_{\rm cris}$ be the p-adic completion of the DP envelope of $\mathcal{O}$ with respect to the ideal $\big(p, P_\pi(Z)\big)$. We define Frobenius extending the Frobenius on $\mathbb{W}$ by requiring that $Z \mapsto Z^p$. We now consider the crystalline (log crystalline would have been a more appropriate but too long name for it) unipotent fundamental group $G^{\rm cris}\big(X, \widetilde{b}\big) := \mathrm{Spec}\big(A^{\rm cris,\vee}_\infty\big)$ associated to the category of unipotent log isocrystals on the mod p reduction X_0 of X relative to the thickening $\mathrm{Spec}(\mathcal{O}_K/p\mathcal{O}_K) \subset \mathrm{Spf}(\mathcal{O}_{\rm cris})$. Here we endow $\mathcal{O}$ with the log structure defined by Z and $\mathcal{O}_{\rm cris}$ with the induced log structure. Then,

Theorem 1.8. *(1) The $\mathcal{O}_{\rm cris}\big[p^{-1}\big]$-algebra $A^{\rm cris,\vee}_\infty$ is the direct limit $\lim\limits_{n\to\infty} \mathcal{E}^{\rm crys,\vee}_{n,b}$ of free $\mathcal{O}_{\rm cris}\big[p^{-1}\big]$-modules, endowed with logarithmic connections relative to $\mathbb{W}(k)$, horizontal and étale Frobenius linear operators, descending exhaustive filtrations satisfying Griffiths' transversality and $\mathcal{E}^{\rm cris,\vee}_{1,b} = \mathcal{O}_{\rm cris}\big[p^{-1}\big]$ with the standard derivation, Frobenius, DP filtration provides the structure as $\mathcal{O}_{\rm cris}\big[p^{-1}\big]$-algebra;*

(2) using the map of $\mathbb{W}(k)$-algebras $\mathcal{O}_{\rm cris} \to \mathcal{O}_K$, sending $Z \mapsto \pi$, there exist isomorphisms $t^\vee_n \colon \mathcal{E}^{\rm cris,\vee}_{n,b} \otimes_{\mathcal{O}_{\rm cris}} K \longrightarrow \mathcal{E}^{\rm dR,\vee}_{n,b}$ as filtered K-vector spaces, where we endow the LHS with the image filtration, compatibly for varying n;

(3) there exist G_K-equivariant isomorphisms $\rho^\vee_n \colon \mathcal{E}^{\rm cris,\vee}_{n,b} \widehat{\otimes}_{\mathcal{O}_{\rm cris}} B_{\log} \cong \mathcal{E}^{\rm et,\vee}_{n,b} \otimes_{\mathbb{Q}_p} B_{\log}$ as filtered $B_{\log}$-modules, compatibly with connections and Frobenii and compatibly for varying n. Here, $B_{\log}$ is a variant of Fontaine's period ring $B_{\rm st}$ and carries G_K-action, filtration, connection, Frobenius;

such that

(i) the induced isomorphism

$$t^\vee_\infty \colon A^{\rm cris,\vee}_\infty \otimes_{\mathcal{O}_{\rm cris}} K \longrightarrow A^{\rm dR,\vee}_\infty$$

is an isomorphism of Hopf algebras over K;

(ii) the induced isomorphism

$$\rho^\vee_\infty \colon A^{\rm cris,\vee}_\infty \widehat{\otimes}_{\mathcal{O}_{\rm cris}} B_{\log} \longrightarrow A^{\rm et,\vee}_\infty \otimes_{\mathbb{Q}_p} B_{\log}$$

is an isomorphism of Hopf algebras over $B_{\log}$;

(iii) using the map of $\mathbb{W}(k)$-algebras $\mathcal{O}_{\rm cris} \to K_0$, sending $Z \to 0$, and taking G_K-invariants, the isomorphism $\rho^\vee_\infty$ produces an isomorphism

$$A^{\rm cris,\vee}_\infty \otimes_{\mathcal{O}_{\rm cris}} K_0 \cong D_{\rm st}\big(A^{\rm et,\vee}_\infty\big)$$

of Hopf algebras over K_0, compatible with Frobenius and monodromy operator where on the LHS we take the residue of the connection on $A_\infty^{\mathrm{cris},\vee}$ at $Z = 0$;

(iv) for $n = 2$ the dual of the isomorphisms $t_n^\vee$ and $\rho_n^\vee$ produce the p-adic comparison isomorphism as filtered $B_{\log}$-modules, compatible with G_K-action, Frobenius, connection and filtrations:

$$\mathrm{H}^1_{\mathrm{logcris}}\big(X_0/\mathcal{O}_{\mathrm{cris}}\big)\widehat{\otimes}_{\mathcal{O}_{\mathrm{cris}}}B_{\log} \cong \mathrm{H}^1_{\mathrm{et}}\big(X_{\overline{K}}, \mathbb{Q}_p)\big) \otimes_{\mathbb{Q}_p} B_{\log}$$

and of filtered K-vector spaces

$$\mathrm{H}^1_{\mathrm{logcris}}\big(X_0/\mathcal{O}_{\mathrm{cris}}\big)\widehat{\otimes}_{\mathcal{O}_{\mathrm{cris}}}K \cong \mathrm{H}^1_{\mathrm{dR}}\big(X_K/K\big)$$

and of K_0-vector spaces, compatibly with monodromy operators and Frobenius,

$$\mathrm{H}^1_{\mathrm{logcris}}\big(X_0/\mathcal{O}_{\mathrm{cris}}\big) \otimes_{\mathcal{O}_{\mathrm{cris}}} K_0 \cong D_{\mathrm{st}}\big(\mathrm{H}^1_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)\big).$$

Let us remark that results similar to theorem 1.8 have been proved using different methods in [Ol] in the case that X has good reduction and $\mathcal{O}_K = \mathbb{W}(k)$, but with no restriction on the dimension of X_K. The results also follow from [Vo] using a relative version of the theory of Fontaine-Lafaille, for curves with good reduction, but assuming that $\mathcal{O}_K$ has ramification index $\leq p - 1$ and for $n \leq \frac{p-1}{2}$. More recently such a result was proved in [Ha] for affine curves with good reduction. Our approach is based on [AI]. The comparison is provided after fixing a log deformation $\widetilde{X}$ of X to $\mathcal{O}$. It has the advantage of describing the monodromy operator on log crystalline objects in a very geometric way which in the end allows us to prove our main result. Let us recall that for every $n \geq 1$, $\mathcal{E}_{n,b}^{\mathrm{et}}$ is a p-adic representation of G_K. We then have the following explicit version of theorem 1.6.

Theorem 1.9. *The curve X_K has good reduction if and only if the G_K-representations $\mathcal{E}_{n,b}^{\mathrm{et}}$ are crystalline for every $n \geq 1$.*

We remark that for $n = 2$ the hypothesis and theorem 1.7(ii) imply that the p-adic Tate module of the Jacobian $J(X_K)$ of X_K is a crystalline G_K-representation. This is known to be equivalent to the fact that the Jacobian of XK, $J(X_K)$ has good reduction, see theorem 1.2 (from [CI]).

Let us explain our startegy for proving theorem 1.9.

• Theorem 1.8 implies that for every $n \geq 1$, $\mathcal{E}_n^{\mathrm{et}}$ is an arithmetically semi-stable étale local system on $X_{\overline{K}}$, which implies that $\mathcal{E}_{n,b}^{\mathrm{et}}$ is a semi-stable p-adic G_K-representation.

• Theorem 1.8 also implies that $\mathbb{D}^{\mathrm{ar}}_{\log}\big(\mathcal{E}_n^{\mathrm{et}}\big) \cong \big(\mathcal{E}_n^{\mathrm{cris}}, \nabla_n\big)$, which implies that $D_{\mathrm{st}}\big(\mathcal{E}_{n,b}^{\mathrm{et}}\big) \cong \mathcal{E}_{n,b}^{\mathrm{cris}}$, as filtered, Frobenius monodromy modules. Specially, the monodromy operator on $\mathcal{E}_{n,b}^{\mathrm{cris}}$ can be identified with the residue of the connection ∇_n at $Z = 0$.

• Finally we choose an embedding of $K \hookrightarrow \mathbb{C}$, as fields and use it to base-change $\widetilde{X}$ and $\big(\mathcal{E}_n^{\mathrm{cris}}, \nabla_n\big)$. We obtain a family of curves $\widetilde{X}_{\mathbb{C}}$ over the complex open disk D, smooth and proper over $D^* := D - \{0\}$ and semi-stable at $Z = 0$ and a locally free $\mathcal{O}_{\widetilde{X}_{\mathbb{C}}}$-module $\mathcal{E}_{n,\mathbb{C}}^{\mathrm{cris}}$ with an integrable, log connection $\nabla_{n,\mathbb{C}}$. By identifying the p-adic and complex monodromy operators, theorem 1.9 then follows by applying T. Oda's proposition 10 in [O].

Remark 1.10. The Theorems 1.7 and 1.8 are proven in a more general context, as we allow the case of open curves with good compactifcations.

2 Notations

Let K be a complete discrete valuation field of characteristic 0, with valuation ring $\mathcal{O}_K$ and perfect residue field k of positive characteristic p. Fix a uniformizer π of $\mathcal{O}_K$. We endow $S := \mathrm{Spec}(\mathcal{O}_K)$ with the log structure M defined by the pre log structure $\mathbb{N} \longrightarrow \mathcal{O}_K$ sending $n \in \mathbb{N}$ to $\pi^n \in \mathcal{O}_K$. We let $\big(\widehat{S}, \widehat{M}\big)$ denote the associated p-adic log formal scheme.

Write $\mathbb{W} := \mathbb{W}(k)$ and we denote by $\mathcal{O} := \mathbb{W}[\![Z]\!]$ and by $\mathcal{O} \longrightarrow \mathcal{O}_K$ the $\mathbb{W}$-algebra homomorphism sending Z to π. Its kernel is generated by an Eisnstein polynomial $P_\pi(Z)$, the minimal polynomial of π over $\mathbb{W}$. We define Frobenius on $\mathcal{O}$ to be the homomorphism given by the usual Frobenius on $\mathbb{W}(k)$ and $Z \mapsto Z^p$. We write $P_\pi(Z) \in \mathbb{W}[Z]$ for the monic minimal polynomial of π over $\mathbb{W}$. It is a generator of $\mathrm{Ker}\big(\mathcal{O} \longrightarrow \mathcal{O}_K\big)$. We denote by $\widetilde{S} := \mathrm{Spf}(\mathcal{O})$ the associated formal scheme for the (p, Z)-adic topology and by $\widetilde{M}$ the log structure on $\widetilde{S}$ associated to the prelog structure $\mathbb{N} \longrightarrow \mathcal{O}$ sending $n \in \mathbb{N}$ to $Z^n \in \mathcal{O}$. The natural closed immersion of formal schemes $\widehat{S} \hookrightarrow \widetilde{S}$ is exact with respect to the given log structures. Denote by $\mathcal{O}_{\mathrm{cris}}$ the p-adic completion of the DP envelope of $\mathcal{O}$ with respect to the ideal $\big(p, P_\pi(Z)\big)$. We denote by $\omega^1_{\mathcal{O}_{\mathrm{cris}}/\mathbb{W}} \cong \mathcal{O}_{\mathrm{cris}} \frac{dZ}{Z}$ the continuous log 1-differential forms of $\mathcal{O}_{\mathrm{cris}}$ relative to $\mathbb{W}$

Let X be a proper curve over $\mathcal{O}_K$, i.e., a proper and flat scheme of relative dimension 1. Assume that the generic fiber X_K is geometrically irreducible and smooth over K and that $X \to \mathrm{Spec}(\mathcal{O}_K)$ is semistable. In particular, we endow X with a log structure N defined by (i) its special fiber and (ii) finitely many disjoint sections $s_i \colon \mathrm{Spec}(\mathcal{O}_{K_i}) \to X$ for $i = 1, \ldots, n$ defined over unramified extensions $\mathcal{O}_K \subset \mathcal{O}_{K_i}$. We assume that if g is the genus of X_K and $\deg s_i = [K_i : K]$ then

$$g - 3 + \sum_{i=1}^{n} \deg s_i \geq 0 \tag{1}$$

The morphism $f \colon X \to S$ induces a log smooth morphism $f \colon (X, N) \longrightarrow (S, M)$. We let $\big(\widehat{X}, \widehat{N}\big)$ be the associated p-adic log formal scheme and $\widehat{f} \colon \big(\widehat{X}, \widehat{N}\big) \longrightarrow \big(\widehat{S}, \widehat{M}\big)$ the associated morphism of p-adic log formal schemes.

As the deformation theory of $\widehat{f}$ is unobstructed by [K2], there exists a deformation $\tilde{f} \colon \big(\widetilde{X}, \widetilde{N}\big) \longrightarrow \big(\widetilde{S}, \widetilde{M}\big)$ of $\widehat{f}$ with the property that $\widehat{f} : (\widehat{X}, \widehat{N}) \longrightarrow (\widehat{S}, \widehat{M})$ is the fiber of $\tilde{f}$ for the map $Z \to \pi$. In particular, for every singular point S of X_k if the local structure of (X, N) at S is $\mathcal{O}_K[\![x, y]\!]/(xy - \pi)$ then the local structure of $\big(\widetilde{X}, \widetilde{N}\big)$ at S is of the form $\mathcal{O}[\![Z, x, y]\!]/(xy - Z)$.

We also fix a base point $b \colon S \to X$ factoring through the smooth locus of X and disjoint from the sections $s_1, \ldots, s_m$ and a lift $\widetilde{b} \colon \mathrm{Spf}(\mathcal{O}) \to \widetilde{X}$ of the section $\widehat{b} \colon \widehat{S} \to \widehat{X}$ defined by b.

2.1 Rings of p-adic periods

We recall the definition of the crystalline period ring A_{cris} defined in [Fo, §2.3] and of the semistable period ring $A_{\log}$ defined in [K1, §3].

Choose a compatible system of $n!$–roots $\pi^{\frac{1}{n!}}$ of π in $\overline{K}$ and a compatible system of primitive n–roots ϵ_n of 1 for varying $n \in \mathbb{N}$. Consider the ring

$$\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}} := \lim_{\leftarrow} \widehat{\mathcal{O}}_{\overline{K}},$$

where the transition maps are given by raising to the p-th power. Define the elements $\overline{p} := \big(p, p^{\frac{1}{p}}, \ldots\big)$, $\overline{\pi} := \big(\pi, \pi^{\frac{1}{p}}, \cdots\big)$ and $\varepsilon := \big(1, \epsilon_p, \cdots\big)$. The set $\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}$ has a natural ring structure [Fo, §1.2.2] in which $p \equiv 0$ and a log structure associated to the morphism of monoids $\mathbb{N} \to \widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}$ given by $1 \mapsto \overline{\pi}$. Write $A_{\mathrm{inf}}\left(\mathcal{O}_{\overline{K}}\right)$, or simply A_{inf}, for the Witt ring $\mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}\big)$. It is endowed with the log structure associated to the morphism of monoids $\mathbb{N} \to \mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}\big)$ given by $1 \mapsto \big[\overline{\pi}\big]$. There is a natural ring homomorphism $\theta\colon \mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}\big) \longrightarrow \widehat{\mathcal{O}}_{\overline{K}}$ [Fo, §1.2.2] such that $\theta\big(\big[\overline{\pi}\big]\big) = \pi$. In particular, it is surjective and strict considering on $\widehat{\mathcal{O}}_{\overline{K}}$ the log structure associated to $\mathbb{N} \to \widehat{\mathcal{O}}_{\overline{K}}$ given by $1 \mapsto \pi$. Its kernel is principal and generated by $P_\pi\big(\big[\overline{\pi}\big]\big)$ or by the element $\xi := \big[\overline{p}\big] - p$.

(i) We define A_{cris} as the p-adic completion of the DP envelope of $\mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}\big)$ with respect to the ideal generated by p and the kernel of θ.

(ii) We also define $A_{\log}$ as the p-adic completion of the log DP envelope of the morphism $\mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}\big) \otimes_{\mathbb{W}(k)} \mathcal{O}$ with respect to the morphism $\theta \otimes \theta_{\mathcal{O}}\colon \mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}\big) \otimes_{\mathbb{W}(k)} \mathcal{O} \longrightarrow \widehat{\mathcal{O}}_{\overline{K}}$.

In particular,

$$A_{\log} \cong A_{\mathrm{cris}}\left\{\langle u-1\rangle\right\}.$$

More precisely, there exists an isomorphism of A_{cris}-algebras from the p-adic completion $A_{\mathrm{cris}}\left\{\langle V\rangle\right\}$ of the DP polynomial ring over A_{cris} in the variable V and $A_{\log}$ sending V to $u-1$ with $u := \frac{[\overline{\pi}]}{Z}$; cf. [K1, Prop. 3.3] and [Bre, §2] where the same ring is denoted $\widehat{A}_{\mathrm{st}}$. We endow A_{cris} and $A_{\log}$ with the p-adic topology and the divided power filtration. We write $B_{\mathrm{cris}} := A_{\mathrm{cris}}\big[t^{-1}\big]$ and $B_{\log} := A_{\log}\big[t^{-1}\big]$, where $t := \log\big([\varepsilon]\big)$, with the inductive limit topology and the filtration $\mathrm{Fil}^n B_{\mathrm{cris}} := \sum_{m\in\mathbb{N}} \mathrm{Fil}^{n+m} A_{\mathrm{cris}} t^{-m}$ and $\mathrm{Fil}^n B_{\log} := \sum_{m\in\mathbb{N}} \mathrm{Fil}^{n+m} A_{\log} t^{-m}$.

All these period rings are endowed with a Frobenius, compatible with the Frobenius on $\mathbb{W}$ and on $\mathcal{O}$ introduced above, and having the property that $\varphi(u) = u^p$ and $\varphi(t) = pt$. They are also endowed with a continuous action of the Galois group G_K, acting trivially on $\mathbb{W}(k)$ and on $\mathcal{O}$ and acting on $\mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}\big)$ through its action on $\widehat{\mathcal{O}}_{\overline{K}}$. Moreover, there is a derivation

$$d\colon B_{\log} \longrightarrow B_{\log}\frac{dZ}{Z}$$

which is B_{cris} linear and satisfies $d\big((u-1)^{[n]}\big) = (u-1)^{[n-1]} u \frac{dZ}{Z}$; see [K1, Prop. 3.3] and [Bre, Lemma 7.1]. We let

$$N\colon B_{\log} \longrightarrow B_{\log}$$

be the operator such that $d(f) = N(f)\frac{dZ}{Z}$. It is proven in [K1, Thm. 3.7] that Fontaine's period ring B_{st}, see [Fo, §3.1.6], is isomorphic to the largest subring of $B_{\log}$ on which N acts nilpotently.

$B_{\log}$-*admissible representations:* Following [Bre, Def. 3.2] we call a $\mathbb{Q}_p$-adic representation V of G_K, $B_{\log}$-admissible if

(1) $D_{\log}(V) := \big(B_{\log} \otimes_{\mathbb{Q}_p} V\big)^{G_K}$ is a free $B_{\log}^{G_K}$-module;

(2) the morphism $B_{\log} \otimes_{B_{\log}^{G_K}} D(V) \longrightarrow B_{\log} \otimes_{\mathbb{Q}_p} V$ is an isomorphism strictly compatible with the filtrations.

We denote by $\mathcal{MF}_{B_{\log}^{G_K}}(\varphi, N)$ the category of finite and free $B_{\log}^{G_K}$-modules M, endowed with (i) a monodromy operator N_M compatible via Leibniz rule with the one on $B_{\log}^{G_K}$, (ii) a decreasing exhaustive filtration $\mathrm{Fil}^n M$ which satisfies Griffiths' transversality with respect to N_M and such that the multiplication map $B_{\log}^{G_K} \times M \to M$ is compatible with the filtrations, (iii) a semilinear Frobenius morphism $\varphi_M\colon M \to M$ such that $N_M \circ \varphi_M = p\varphi_M \circ N_M$ and $\det \varphi_M$ is invertible in $B_{\log}^{G_K}$. If V is $B_{\log}$-admissible it is proven [Bre, §6.1] that $D_{\log}(V)$ is an object of $\mathcal{MF}_{B_{\log}^{G_K}}(\varphi, N)$.

Comparison with semistable representations: Consider the category $\mathcal{MF}_K(\varphi, N)$ of finite dimensional K_0-vector spaces D endowed with (i) a monodromy operator N_D, (ii) a descending and exhaustive filtration $\mathrm{Fil}^n D_K$ on $D_K := D \otimes_{K_0} K$, (iii) a Frobenius φ_D such that $\det\varphi_D \neq 0$ and $N_D \circ \varphi_D = p\varphi_D \circ N_D$; see [CF]. Such a module is called B_{st}-admissible if there exists a $\mathbb{Q}_p$-representation V of G_K such that $D_{\mathrm{st}}(V) := \big(V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}\big)^{G_K}$ is isomorphic to D compatibly with monodromy operator, Frobenius and filtration after extending scalars to K. Consider the functor

$$T\colon \mathcal{MF}_K(\varphi, N) \longrightarrow \mathcal{MF}_{B_{\log}^{G_K}}(\varphi, N)$$

sending $D \mapsto T(D) := D \otimes_{K_0} B_{\log}^{G_K}$ with monodromy operator $N_D \otimes 1 + 1 \otimes N$, Frobenius $\varphi_D \otimes \varphi$ and filtration defined on [Bre, p. 201] using the filtration on D_K and the monodromy operator. More precisely, there is a natural map $B_{\log}^{G_K} \longrightarrow K$, sending Z to π, providing a morphism $\rho\colon T(D) \to D_K$. Then, $\mathrm{Fil}^n T(D)$ is defined inductively on n by setting $\mathrm{Fil}^n T(D) := \big\{x \in T(D) | \rho(x) \in \mathrm{Fil}^n D_K, \quad N(x) \in \mathrm{Fil}^{n-1} T(D)\big\}$. There is also a natural map $\iota_0\colon B_{\log}^{G_K} \longrightarrow K_0$ sending $Z \mapsto 0$.

Proposition 2.1. [Bre] *The notions of $B_{\log}$-admissible representations and of B_{st}-admissible representations are equivalent. For any such V, we have an identification $T\big(D_{\mathrm{st}}(V)\big) \cong D_{\log}(V)$ such that*

(i) $D_{\mathrm{st}}(V) \otimes_{K_0} K \cong D_{\log}(V) \otimes_{B_{\log}^{G_K}} K$ as filtered K-vector spaces considering on the RHS the image filtration.

(ii) $D_{\mathrm{st}}(V) \cong D_{\log}(V) \otimes^{\iota_0}_{\mathcal{O}_{\mathrm{cris}}} K_0$ as K_0-vector space so that the monodromy operator on the LHS is the residue of the monodromy operator on the RHS.

Proof. The first claim is proven in [Bre, Thm. 3.3]. One knows that T is in fact an equivalence due to [Bre, Thm. 6.1.1]. From the the proof of loc. cit. one deduces also the claimed compatibility of filtrations and the relation between the monodromy operators. □

3 Universal unipotent objects

3.1 The Kummer étale site

We define the Kummer étale site X^{ket} associated to $\big(X, N\big)$ as follows. The objects are Kummer étale morphisms $g\colon \big(Y, N_Y\big) \to (X, N)$ in the sense of [Il, §2.1]. The morphisms from an object $(Y, N_Y) \to (X, N)$ to an object $(Z, N_Z) \to (X, N)$ are morphisms $t\colon \big(Y, N_Y\big) \to \big(Z, N_Z\big)$ of log schemes over (X, N). The coverings are collections of Kummer étale morphisms

$\{(Y_i, N_i) \to (Y, N_Y)\}_i$ such that Y is the set theoretic union of the images of the Y_i's. This defines a site; see loc. cit.

An object U of $X^{\rm ket}$ is called *small* if it is affine, connencted and there exists an étale morphism (i) $U \to \mathrm{Spec}\big(\mathcal{O}_K\big[T, T^{-1}\big]\big)$ which is a chart for the log structure on U considering on $\mathrm{Spec}\big(\mathcal{O}_K\big[T, T^{-1}\big]\big)$ either the log structure defined by the special fiber or the log structure defined by the special fiber and by the section $T = 1$ or (ii) $U \to \mathrm{Spec}\big(\mathcal{O}_K[S, T]/(ST - \pi)\big)$ which is a chart for the log structure on U considering on $\mathrm{Spec}\big(\mathcal{O}_K\big[T, S]/(ST - \pi)\big]\big)$ the log structure defined by its special fiber.

In the following we will consider the following categories:

3.2 The étale category

Denote $\mathrm{Uni}_{\mathbb{Q}_p}\big(X_{\overline{K}}^{\rm ket}\big)$ the category of $\mathbb{Q}_p$-unipotent local systems on $X_{\overline{K}}$ for the Kummer étale topology. This is the full tensor subcategory of $\mathbb{Q}_p$-sheaves L on $X_{\overline{K}}^{\rm ket}$ with the property that L admits a filtration

$$L = L^1 \supset L^2 \supset \cdots \supset L^n \supset L_{n+1} = 0$$

such that

$$L^i/L^{i+1} \simeq \mathbb{Q}_p^{r_i}$$

for each i. We say that the index of unipotency of L is $\leq n$. Note that we use $\mathbf{1} := \mathbb{Q}_p$ here to denote the constant sheaf on $X_{\overline{K}}$.

We let $b_{\overline{K}}^*\colon \mathrm{Uni}_{\mathbb{Q}_p}\big(X_{\overline{K}}^{\rm ket}\big) \longrightarrow \mathrm{Vect}_{\mathbb{Q}_p}$ be the functor associating to L the $\mathbb{Q}_p$-vector space $b_{\overline{K}}^*(L) = L(\overline{K})$. It is exact and it commutes with tensor products and duals. Moreover $\underline{1} = b_{\overline{K}}^*(\mathbf{1}) = \mathbb{Q}_p$

3.3 The de Rham category

Write $\mathrm{Uni}_{\rm dR}\big(X_K, N_K\big)$ for the full subcategory of the category of locally free $\mathcal{O}_{X_K}$-modules M, endowed with an integrable log connection ∇ with respect to the log structure N_K, which are unipotent. Namely we require that (M, ∇) admits a filtration by $\mathcal{O}_{X_K}$-modules

$$M = M^1 \supset M^2 \supset \cdots \supset M^n \supset M_{n+1} = 0$$

such that each M^i is preserved by the connection ∇ and M^i/M^{i+1}, with the induced connection, is isomorphic to $\mathbf{1}^{m_i}$ with $\mathbf{1} := \big(\mathcal{O}_{X_K}, d\big)$, for each i. We also say that the index of unipotency of (M, ∇) is $\leq n$. The category $\mathrm{Uni}_{\rm dR}\big(X_K, N_K\big)$ admits tensor products and duals.

We let $b_K^*\colon \mathrm{Uni}_{\rm dR}\big(X_K, N_K\big) \longrightarrow \mathrm{Vect}_K$ be the functor associating to (M, ∇) the K-vector space defined by the pull back of M via b_K. It is exact and it commutes with taking tensor products and duals and it sends $\mathbf{1}$ to $\underline{1} = K$.

3.4 The crystalline category

Let X_0 be the reduction of X modulo p and let us recall that we denoted by $\mathcal{O}_{\rm cris}$ the p-adic completion of the DP envelope of $\mathcal{O} := \mathbb{W}[[Z]]$ with respect to the kernel of the map $\mathcal{O} \to \mathcal{O}_K/p\mathcal{O}_K$ defined by $Z \mapsto 0$. $\mathcal{O}_{\rm cris}$ is endowed with the log structure induced from the one on $\mathcal{O}$. Following [K2, §5], consider the site $\big(X_0/\mathcal{O}_{\rm cris}\big)^{\rm cris}_{\rm log}$, consisting of quintuples $\big(U, T, M_T, \iota, \delta\big)$ where

(a) $U \to X_0$ is Kummer étale,

(b) $\big(T, M_T\big)$ is a fine log scheme over $\mathcal{O}_{\rm cris}$ (with its log structure) in which p is locally nilpotent,

(c) $\iota\colon U \to T$ is an exact closed immersion over $\mathcal{O}_{\rm cris}$,

(d) δ is DP structure on the ideal defining the closed immersion $U \subset T$, compatible with the DP structure on $\mathcal{O}_{\rm cris}$.

We let $\mathrm{Cris}(X_0/\mathcal{O})$ be the category of crystals of finitely presented $\mathcal{O}_{X_0/\mathcal{O}_{\rm cris}}$-modules on $\big(X_0/\mathcal{O}_{\rm cris}\big)^{\rm cris}_{\rm log}$, cf. [K2, Def 6.1]. By [Be, Prop. IV.1.7.6] it is an abelian category. Given a crystal $\mathcal{E}$ let $\mathcal{E}_n$ be the crystal $\mathcal{E}_n := \mathcal{E}/p^n\mathcal{E}$. It defines a $\mathcal{O}^{\rm DP}_{\widetilde{X}}/p^n\mathcal{O}^{\rm DP}_{\widetilde{X}}$-module, endowed with integrable logarithmic connection ∇_n relative to $\mathcal{O}_{\rm cris}/p^n\mathcal{O}_{\rm cris}$; see [K2, Thm. 6.2]. Here $\mathcal{O}^{\rm DP}_{\widetilde{X}} := \mathcal{O}_{\widetilde{X}}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_{\rm cris}$. Let $\mathcal{E}_{\widetilde{X}} := \lim\limits_{\infty\leftarrow n} \mathcal{E}_n$ be the associated sheaf of $\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{X}}$-modules on $X_0^{\rm ket}$ with log connection $\nabla_{\mathcal{E}_{\widetilde{X}}}$ relative to $\mathcal{O}_{\rm cris}$. It follows from [Be, Prop. IV.1.1.3] that this crystal is finitely presented if and only if $\mathcal{E}_{\widetilde{X}}$ is finitely presented as $\mathcal{O}^{\rm DP}_{\widetilde{X}}$-module. By [Be, Cor. IV.1.7.7] a sequence of crystals is exact if and only if the associated sequence of $\mathcal{O}^{\rm DP}_{\widetilde{X}}$-modules is exact.

Let $\mathrm{Ind}\big(\mathrm{Cris}(X_0/\mathcal{O})\big)$ be the abelian category of inductive systems consisting of the inductive system $\mathcal{E} \to \mathcal{E} \to \mathcal{E} \to \cdots$ where $\mathcal{E}$ is a crystal of finitely presented $\mathcal{O}_{X_0/\mathcal{O}_{\rm cris}}$-modules and the transition maps $\mathcal{E} \to \mathcal{E}$ are multiplication by p. We denote by $\mathbf{1}$ the structure sheaf isocrystal.

Let $\mathrm{Uni}_{\log}\big(X_0/\mathcal{O}_{\rm cris}\big)$ be the full subcategory of $\mathrm{Ind}\big(\mathrm{Cris}(X_0/\mathcal{O})\big)$ consisting of isocrystals $\mathcal{E}$, which are unipotent. More precisely, we require that $\mathcal{E}$ admits a filtration

$$\mathcal{E} = \mathcal{E}^1 \supset \mathcal{E}^2 \supset \cdots \supset \mathcal{E}^n \supset \mathcal{E}_{n+1} = 0$$

such that each $\mathcal{E}^i$ is a log isocrystal of X_0 with respect to $\mathcal{O}_{\rm cris}$ and for every i the quotient $\mathcal{E}^i/\mathcal{E}^{i+1}$ is isomorphic to $\mathbf{1}^{m_i}$. Also this case, we say the index of unipotency is $\leq n$. The category $\mathrm{Uni}_{\log}\big(X_0/\mathcal{O}_{\rm cris}\big)$ is closed under tensor products and duals.

We let $\widetilde{b}^*\colon \mathrm{Uni}_{\log}\big(X_0/\mathcal{O}_{\rm cris}\big) \longrightarrow \mathrm{Vect}_{\mathcal{O}_{\rm cris}[p^{-1}]}$ be the functor associating to $\mathcal{E}$ the $\mathcal{O}_{\rm cris}\big[p^{-1}\big]$-module defined by pull back of $\mathcal{E}_{\widetilde{X}}\big[p^{-1}\big]$ via $\widetilde{b}$. Here $\mathrm{Vect}_{\mathcal{O}_{\rm cris}[p^{-1}]}$ is the category of finite and free $\mathcal{O}_{\rm cris}\big[p^{-1}\big]$-modules. The functor $\widetilde{b}^*$ is exact and it commutes with taking duals and tensor products. It sends $\mathbf{1}$ to $\underline{\mathbf{1}} := \mathcal{O}_{\rm cris}\big[p^{-1}\big]$.

3.5 Axiomatic characterization and properties of the universal unipotent objects

Let Uni be any one of the categories above. Let $\mathcal{C}$ be the category of $\mathrm{Vect}_{\mathbb{Q}_p}$ in the étale case, Vect_K in the de Rham case and $\mathrm{Vect}_{\mathcal{O}_{\rm cris}[p^{-1}]}$ in the crystalline case. We call an object of Uni constant if it is of the form $T \otimes_{\underline{1}} \mathbf{1}$ for some T in $\mathcal{C}$. We simply write $F\colon \mathrm{Uni} \to \mathcal{C}$, in short $L \to L_b$, for the functor defined in each case. It sends $\mathbf{1}$ to $\underline{\mathbf{1}}$, it is exact and it commutes with duals and tensor products.

Let $\mathcal{A}$ be the category of $\mathbb{Q}_p$-adic sheaves on $X_{\overline{K}}^{\mathrm{ket}}$ in the étale case, the category of quasi-coherent $\mathcal{O}_{X_K}$-modules with integrable logarithmic connection in the de Rham case and the category $\mathrm{Ind}\big(\mathrm{Cris}(X_0/\mathcal{O})\big)$ in the log crystalline case. Then $\mathcal{A}$ is an abelian category with enough injectives and $\mathrm{Uni} \subset \mathcal{A}$ is a full sub-category. Given two objects $\mathcal{E}$ and $\mathcal{F}$ in Uni, we write $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$ for the i-th derived functor of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$. We denote by $\mathrm{H}^i(\mathcal{E}) := \mathrm{H}^i_{\mathcal{A}}(\mathcal{E})$ the following: in the étale case $\mathrm{H}^i\big(X_{\overline{K}}^{\mathrm{ket}}, \mathcal{E}\big)$, in the de Rham case $\mathbb{H}^i_{\mathrm{dR}}\big(X_K, \mathcal{E}\big)$ and in the crystalline case $\mathrm{H}^i_{\mathrm{logcris}}\big((X_0/\mathcal{O}_{\mathrm{cris}}), \mathcal{E}\big)$. Note that in each case

$$\underline{\mathbf{1}} \cong \mathrm{Hom}\big(\mathbf{1}, \mathbf{1}\big).$$

Moreover using that for $\mathcal{E}$ in Uni the functor $\mathcal{E}^\vee \otimes \_$ is exact, we get that

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \cong \mathrm{H}^i(\mathcal{E}^\vee \otimes \mathcal{F}).$$

We define the category Uni^* taking for objects pairs $(\mathcal{V}, v)$, where $\mathcal{E}$ is an object of Uni and $v \in \mathcal{V}_b$. A morphism $(\mathcal{V}, v) \to (\mathcal{W}, w)$ is a morphism $g \colon \mathcal{V} \to \mathcal{W}$ in Uni such that $g_9 V 0 = u$. Thus, Uni^* is the category of 'pointed objects' in Uni.

Let Uni be any one of the categories of unipotent étale, de Rham or crystalline sheaves attached to X.

Definition 3.1. A projective system of objects $\{(\mathcal{E}_n, e_n)\}_{n\geq 1}$ in Uni^* such that $\mathcal{E}_n$ has index of unipotency $\leq n$ for every $n \geq 1$ will be called **universal** if for every $(\mathcal{V}, v)$ object in Uni^* with index of unipotency $\leq n$ there is a unique morphism in Uni^*, $g : (\mathcal{E}_n, e_n) \longrightarrow (\mathcal{V}, v)$.

One easily sees from the universal property that if a universal projective system exists in Uni^* then it is unique up to unique isomorphism.

For the rest of this section we present an axiomatic characterization of universal projective systems and in the next section we'll give an inductive construction which will show that such systems exist. For affine curves with good reduction this was accomplished in [Ha].

Consider a system $\big\{(\mathcal{E}_n, e_n)\big\}_{n\in\mathbb{N}}$ in Uni^* with transition morphisms $f_n \colon \big(\mathcal{E}_{n+1}, e_{n+1}\big) \to \big(\mathcal{E}_n, e_n\big)$ such that

(i) $\mathcal{E}_1 = \mathbf{1}$ and $e_1 = 1$,

(ii) $f_n : \mathcal{E}_{n+1} \to \mathcal{E}_n$ is surjective (as a morphism in $\mathcal{A}$) and has constant kernel $\mathcal{T}_n \cong T_n \otimes \mathbf{1}$ for every $n \in \mathbb{N}$,

(iii) the coboundary map $T_n^\vee \cong \mathrm{Hom}\big(\mathcal{T}_n, \mathbf{1}\big) \to \mathrm{Ext}^1\big(\mathcal{E}_n, \mathbf{1}\big)$, defined by the sequence of Ext-groups associated to the short exact sequence $0 \to \mathcal{T}_n \to \mathcal{E}_{n+1} \to \mathcal{E}_n \to 0$ in $\mathcal{A}$, is an isomorphism.

From (iii) we immediately get

Lemma 3.2. *For every n the map $\mathrm{Hom}\big(\mathcal{E}_n, \mathbf{1}\big) \to \mathrm{Hom}\big(\mathcal{E}_{n+1}, \mathbf{1}\big)$ is an isomorphism. In particular, $\underline{1} \cong \mathrm{End}(\mathbf{1}) \cong \mathrm{Hom}\big(\mathcal{E}_n, \mathbf{1}\big)$.*

For every n the map $\mathrm{Ext}^1\big(\mathcal{E}_n, \mathbf{1}\big) \to \mathrm{Ext}^1\big(\mathcal{E}_{n+1}, \mathbf{1}\big)$ is the zero map.

We prove the analogue of [Ha, Prop. 2.6]:

Proposition 3.3. *A projective system* $\{((\mathcal{E}_n, e_n), f_n)\}_{n\geq 1}$ *satisfying the properties i), ii), iii) above is universal.*

Proof. We have to prove that if $(\mathcal{V}, v)$ is an object of Uni* with index of unipotency $\leq n$, then there is a unique morphism $g\colon (\mathcal{E}_n, e_n) \to (\mathcal{V}, v)$ such that $g_b(e_n) = v$.

We proceed by induction on n. Let $n = 1$. Then $\mathcal{V} \cong \mathbf{1}^r$ is constant and there is a unique map $\mathcal{E}_1 = \mathbb{Q}_p \to \mathcal{V}$ that takes $e_1 = 1 \in \mathcal{E}_{1,b}$ to $v \in V_b$. Assume that statement true for n and let $\mathcal{V}$ have index $\leq n+1$. We know that $\mathcal{V}$ admits a filtration

$$V = \mathcal{V}^1 \supset \mathcal{V}^2 \supset \cdots \mathcal{V}^{n+1} \supset \mathcal{V}^{n+2} = 0$$

such that $\mathcal{V}^i/\mathcal{V}^{i+1} \simeq \mathbf{1}^{r_i}$. Consider the extension

$$(S) \qquad 0 \longrightarrow \mathcal{V}^{n+1} \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}/\mathcal{V}^{n+1} \to 0,$$

where $\mathcal{V}/\mathcal{V}^{n+1}$ now has index $\leq n$. Let $\bar{v} \in \left(\mathcal{V}/\mathcal{V}^{n+1}\right)_b$ be the image of v. Thus, by the inductive hypothesis there is a unique morphism

$$\phi_n\colon (\mathcal{E}_n, e_n) \longrightarrow (\mathcal{V}/\mathcal{V}^{n+1}, \bar{v}).$$

We use it to pull-back the extension S, then $\phi_n^*(S)$ is an extension of $\mathcal{E}_n$ by a constant sheaf $\mathcal{V}^{n+1}$. We pull-back this extension to $\mathcal{E}_{n+1}$ via the projection $f_n\colon \mathcal{E}_{n+1} \to \mathcal{E}_n$ and notice that the new extension must split (by Lemma 3.2). Therefore, we get a morphism of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{T}_n & \longrightarrow & \mathcal{E}_{n+1} & \longrightarrow & \mathcal{E}_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\psi & & \downarrow\phi_n & & \\
0 & \longrightarrow & \mathcal{V}^{n+1} & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{V}/\mathcal{V}^{n+1} & \longrightarrow & 0
\end{array}$$

We have $v - \psi(e_{n+1}) \in \mathcal{V}_b^{n+1}$. Since $\mathcal{V}^{n+1}$ is constant, there exists a global section s such that $s_b = v - \psi(e_{n+1})$. Via the constant quotient $\mathcal{E}_{n+1} \longrightarrow \mathcal{E}_1 = \mathbb{Q}_p$, this then gives us a map h from $\mathcal{E}_{n+1}$ to $\mathcal{V}^{n+1}$ that takes e_{n+1} to $v - \psi(e_{n+1})$. If we set $\phi_{n+1} = \psi + h$, then $\phi_{n+1}(e_{n+1}) = v$. Suppose ϕ'_{n+1} is another lifting of ϕ_n. Then

$$\alpha := \phi_{n+1} - \phi'_{n+1}\colon \mathcal{E}_{n+1} \longrightarrow \mathcal{V}^{n+1}$$

and $\alpha(e_{n+1}) = 0$. Since $\mathcal{V}^{n+1}$ is constant and $\mathrm{Hom}(\mathcal{E}_{n+1}, \mathbf{1}) = \mathrm{End}(\mathbf{1}) = \underline{1}$ by Lemma 3.2, the map α factors through a quotient map $\mathcal{E}_1 = \mathbf{1} \longrightarrow \mathcal{V}^{n+1}$ that takes e_1 to 0. Thus, by the uniqueness for $n = 1$, we have $\alpha = 0$, and $\phi_{n+1} = \phi'_{n+1}$. $\square$

3.6 Existence of universal projective systems

As in the previous section we let Uni* denote any one of the categories of pointed unipotent étale, de Rham, respectively crystalline sheaves associated to X and we construct a universal pointed projective system $\left\{\left(\mathcal{E}_n, e_n\right)\right\}_{n\geq 1}$ in Uni* with the properties of proposition 3.3. We'd like to point out that in all cases (i.e. étale, de Rham and crystalline) $\mathrm{H}^i\big(\mathbf{1}\big)$ is a free $\underline{\mathbf{1}}$ module of finite rank for $i = 0, 1, 2$ and we have an alternating pairing called "cup product" $\cup : \mathrm{H}^1\big(\mathbf{1}\big) \times \mathrm{H}^1\big(\mathbf{1}\big) \longrightarrow \mathrm{H}^2\big(\mathbf{1}\big)$. We denote by $\left\{\left(\mathcal{E}_n^{\mathrm{et}}, e_n^{\mathrm{et}}\right)\right\}_{n\in\mathbb{N}}$ the universal projective system in the étale case,

$\{(\mathcal{E}_n^{\mathrm{dR}}, e_n^{\mathrm{dR}})\}_{n\in\mathbb{N}}$ the universal projective system in the de Rham case and $\{(\mathcal{E}_n^{\mathrm{cris}}, e_n^{\mathrm{cris}})\}_{n\in\mathbb{N}}$ the universal projective system in the crystalline case.

First of all we define $\underline{\mathbf{1}}$-submodules $R^n \subset \big(\mathrm{H}^1(\mathbf{1})\big)^{\otimes n}$, inductively on $n \geq 1$. The inductive definition will give us sub-spaces

$$\iota_n\colon R^{n+1} \hookrightarrow R^n \otimes_{\underline{\mathbf{1}}} \mathrm{H}^1(\mathbf{1}).$$

Define

$$R^1 := \mathrm{H}^1(\mathbf{1}), \qquad R^2 = \mathrm{Ker}(\gamma_1),$$

with

$$\gamma_1 = \cup\colon \mathrm{H}^1(\mathbf{1}) \otimes \mathrm{H}^1(\mathbf{1}) \longrightarrow \mathrm{H}^2(\mathbf{1})$$

the cup product. For $n \geq 2$ set

$$\gamma_n\colon R^n \otimes_{\underline{\mathbf{1}}} \mathrm{H}^1(\mathbf{1}) \xrightarrow{\iota_{n-1}\otimes 1} R^{n-1} \otimes_{\underline{1}} \mathrm{H}^1(\mathbf{1}) \otimes_{\underline{\mathbf{1}}} \mathrm{H}^1(\mathbf{1}) \xrightarrow{1\otimes\cup} R^{n-1} \otimes_{\underline{1}} \mathrm{H}^2(\mathbf{1}),$$

put $R^{n+1} = \mathrm{Ker}\,(\gamma_n)$ and define $\iota_n\colon R^{n+1} \hookrightarrow R^n \otimes_{\underline{\mathbf{1}}} \mathrm{H}^1(\mathbf{1})$ to be the natural inclusion.

Proposition 3.4. *There exists a pointed system $\{(\mathcal{E}_n, e_n)\}_{n\in\mathbb{N}}$ such that $\mathcal{E}_1 = \mathbf{1}$ and $e_1 = 1$ and we have exact sequences*

$$0 \longrightarrow \mathcal{T}_n \longrightarrow \mathcal{E}_{n+1} \longrightarrow \mathcal{E}_n \longrightarrow 0$$

with the following properties

(i) $\mathrm{Ext}^j(\mathcal{E}_n, \mathbf{1})$ *is a finite and projective $\underline{\mathbf{1}}$-module for $j = 0$, 1 and 2. It is non zero for $j = 1$;*

(ii) $\mathcal{T}_n = T_n \otimes_{\underline{1}} \mathbf{1}$ *is a constant object and* $T_n \cong R^{n,\vee} := \mathrm{Hom}_{\underline{\mathbf{1}}}(R^n, \underline{1})$*;*

(iii) the map $T_n^\vee = \mathrm{Hom}(\mathcal{T}_n, \mathbf{1}) \to \mathrm{Ext}^1(\mathcal{E}_n, \mathbf{1})$*, induced on* Ext*-groups by the above short exact sequence, is an isomorphism;*

(iv) the map $\mathrm{Ext}^2(\mathcal{E}_{n+1}, \mathbf{1}) \to \mathrm{Ext}^2(\mathcal{T}_n, \mathbf{1}) = T_n^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{H}^2(\mathbf{1})$ *is an isomorphism;*

(v) the sequence

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{E}_{n+1}, \mathbf{1}) \xrightarrow{\alpha_n} \mathrm{Ext}^1(\mathcal{T}_n, \mathbf{1}) \xrightarrow{\beta_n} \mathrm{Ext}^2(\mathcal{E}_n, \mathbf{1}) \longrightarrow 0$$

is an isomorphism;

(vi) identifying $R^{n+1} \cong T_{n+1}^\vee = \mathrm{Hom}(\mathcal{T}_{n+1}, \mathbf{1}) \cong \mathrm{Ext}^1(\mathcal{E}_{n+1}, \mathbf{1})$ *via (ii) and (iii) and* $\mathrm{Ext}^1(\mathcal{T}_n, \mathbf{1}) \cong T_n^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{Ext}^1(\mathbf{q}, \mathbf{1}) \cong R^n \otimes_{\underline{1}} \mathrm{H}^1(\mathbf{1})$ *the map α_n in (v) is the map ι_n.*

In particular, such a system is universal due to Proposition 3.3.

Remark 3.5. The reader may have noticed that the properties i) to vi) in proposition 3.4 are not independent. For example v) is a consequence of iii) and iv) etc. We prefer to list them all as we did for they all appear in the proof of the proposition.

Proof. We proceed by induction on n. For $n=1$ notice that $\mathrm{Ext}^j\big(\mathbf{1},\mathbf{1}\big)\cong \mathrm{H}^j\big(\mathbf{1}\big)$, $j=0$, 1 and 2, are the cohomology groups of the structure sheaf. Recall that $X\to S$ is a geometrically connected semistable curve. Thus for $j=0$ the group $\mathrm{H}^0\big(\mathbf{1}\big)$ coincides with $\underline{\mathbf{1}}$. For $j=1$ it is a free $\underline{\mathbf{1}}$-module in the étale, de Rham and crystalline cases of rank $\geq 2g+r-1\geq 2$, with $r=\sum_{i=1}^n \deg s_i$, due to assumption (1). Eventually $\mathrm{H}^2\big(\mathbf{1}\big)$ is a free $\underline{\mathbf{1}}$-module of rank 1 if $r=0$ and it is trivial if $r\neq 0$ by Poincaré duality.

Assume that $\big(\mathcal{E}_n,e_n\big)_{n\leq N}$ has been constructed so that (i)–(ii) of the proposition hold for all $1\leq n\leq N$, (iii)–(vi) hold for $n<N$ and (vii) $\mathrm{rk}T_n\geq \mathrm{rk}T_{n-1}$ for every $n\leq N$. Set $T_N:=\mathrm{Ext}^1\big(\mathcal{E}_N,\mathbf{1}\big)^\vee$. By assumption it is a non-zero, free $\underline{\mathbf{1}}$-module of finite rank. Put $\mathcal{T}_N:=T_N\otimes\mathbf{1}$. It then follows that

$$\mathrm{Ext}^1\big(\mathcal{E}_N,\mathcal{T}_N\big)\cong T_N\otimes \mathrm{Ext}^1\big(\mathcal{E}_N,\mathbf{1}\big)\cong T_N\otimes T_N^\vee\cong \mathrm{End}(T_N).$$

Consider the extension

$$0\longrightarrow \mathcal{T}_N\longrightarrow \mathcal{E}_{N+1}\longrightarrow \mathcal{E}_N\longrightarrow 0$$

defined by the image of the identity map $\mathrm{Id}\in\mathrm{End}(T_N)$. Let $e_{N+1}\in\mathcal{E}_{N+1,b}$ be any element mapping to e_N. The coboundary map $T_N^\vee=\mathrm{Hom}\big(\mathcal{T}_N,\mathbf{1}\big)\to \mathrm{Ext}^1\big(\mathcal{E}_N,\mathbf{1}\big)$ is the isomorphism $\mathrm{Ext}^1\big(\mathcal{E}_N,\mathcal{T}_N\big)\cong T_N\otimes T_N^\vee$ described above and, as $T_N\neq 0$, it is an isomorphism. This proves the inductive step in (ii) and (iii) except for the identification $T_N^\vee\cong R^N$. Using the long exact sequence in cohomology associated to

$$0\longrightarrow \mathcal{T}_N\longrightarrow \mathcal{E}_{N+1}\longrightarrow \mathcal{E}_N\longrightarrow 0$$

we also deduce that the map $\mathrm{Ext}^1\big(\mathcal{E}_N,\mathbf{1}\big)\to\mathrm{Ext}^1\big(\mathcal{E}_{N+1},\mathbf{1}\big)$ is 0. In particular we have the exact sequence

$$0\longrightarrow \mathrm{Ext}^1\big(\mathcal{E}_{N+1},\mathbf{1}\big)\xrightarrow{\alpha_N}\mathrm{Ext}^1\big(\mathcal{T}_N,\mathbf{1}\big)\xrightarrow{\beta_N}\mathrm{Ext}^2\big(\mathcal{E}_N,\mathbf{1}\big).$$

Using the identifications

$$\mathrm{Ext}^1\big(\mathcal{T}_N,\mathbf{1}\big)\cong T_N^\vee\otimes_{\underline{\mathbf{1}}}\mathrm{Ext}^1\big(\mathbf{1},\mathbf{1}\big)\cong T_N^\vee\otimes_{\underline{\mathbf{1}}}\mathrm{H}^1(\mathbf{1})$$

and $\mathrm{Ext}^2\big(\mathcal{E}_N,\mathbf{1}\big)\cong\mathrm{Ext}^2\big(\mathcal{T}_{N-1},\mathbf{1}\big)\cong T_{N-1}^\vee\otimes_{\underline{\mathbf{1}}}\mathrm{H}^2(\mathbf{1})$ by inductive hypothesis, the map β_N defines a morphism

$$\beta_N'\colon T_N^\vee\otimes_{\underline{\mathbf{1}}}\mathrm{H}^1(\mathbf{1})\longrightarrow T_{N-1}^\vee\otimes_{\underline{1}}\mathrm{H}^2(\mathbf{1})$$

and $T_{N+1}^\vee$ is the kernel of β_N'. Thanks to the identification $T_N^\vee=\mathrm{Ext}^1\big(\mathcal{E}_N,\mathbf{1}\big)$ we get $\mathrm{Ext}^1\big(\mathcal{T}_N,\mathbf{1}\big)\cong\mathrm{Ext}^1\big(\mathcal{E}_N,\mathbf{1}\big)\otimes_{\underline{\mathbf{1}}}\mathrm{Ext}^1\big(\mathbf{1},\mathbf{1}\big)$ providing a second description of β_N as a map

$$\beta_N''\colon\mathrm{Ext}^1\big(\mathcal{E}_N,\mathbf{1}\big)\otimes_{\underline{\mathbf{1}}}\mathrm{Ext}^1\big(\mathbf{1},\mathbf{1}\big)\longrightarrow\mathrm{Ext}^2\big(\mathcal{E}_N,\mathbf{1}\big)$$

as follows. Given $\mathcal{G}\in\mathrm{Ext}^1\big(\mathcal{E}_N,\mathbf{1}\big)$ corresponding to a unique morphism $f_{\mathcal{G}}\colon\mathcal{T}_N\to\mathbf{1}$ and a class $\mathcal{F}\in\mathrm{Ext}^1\big(\mathbf{1},\mathbf{1}\big)$ we take the unique extension $\mathcal{F}'\in\mathrm{Ext}^1\big(\mathcal{T}_N,\mathbf{1}\big)$ obtained by pulling–back the extension $\mathcal{F}$ via $f_{\mathcal{G}}$ and then $\beta_N''\big(\mathcal{G}\otimes\mathcal{F}\big)$ is the Yoneda two extension of $\mathcal{E}_N$ by $\mathbf{1}$ given by the composite complex

$$\mathcal{F}'*\mathcal{E}_{N+1}:=0\longrightarrow\mathbf{1}\longrightarrow\mathcal{F}'\longrightarrow\mathcal{T}_N\longrightarrow\mathcal{E}_{N+1}\longrightarrow\mathcal{E}_N\longrightarrow 0$$

(see [Ve, §3.2.4]). In particular we have a natural morphism of complexes

$$\begin{array}{ccccccccccccccc}
\mathcal{F}' * \mathcal{E}_{N+1} & = & 0 & \longrightarrow & \mathbf{1} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{T}_N & \longrightarrow & \mathcal{E}_{N+1} & \longrightarrow & \mathcal{E}_N & \longrightarrow & 0 \\
\downarrow & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
\mathcal{F} * \mathcal{G} & = & 0 & \longrightarrow & \mathbf{1} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathbf{1} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E}_N & \longrightarrow & 0.
\end{array}$$

Thus $\beta''_N\big(\mathcal{G} \otimes \mathcal{F}\big) = \mathcal{G} * \mathcal{F}$ by [Ve, Prop. 3.2.2]. Due to [Ve, Prop. 3.2.5] the map $\big(\mathcal{G}, \mathcal{F}\big) \mapsto \mathcal{G} * \mathcal{F}$ is minus the cup product of Ext-groups in the sense of derived functors. Consider the diagram:

$$\begin{array}{ccc}
\mathrm{Ext}^1\big(\mathcal{E}_N, \mathbf{1}\big) \otimes_{\underline{\mathbf{1}}} \mathrm{Ext}^1\big(\mathbf{1}, \mathbf{1}\big) & \xrightarrow{\beta''_N} & \mathrm{Ext}^2\big(\mathcal{E}_N, \mathbf{1}\big) \\
\alpha_{N-1} \otimes \mathrm{id}\downarrow & & \downarrow\wr \\
\mathrm{Ext}^1\big(\mathcal{T}_{N-1}, \mathbf{1}\big) \otimes_{\underline{\mathbf{1}}} \mathrm{Ext}^1\big(\mathbf{1}, \mathbf{1}\big) & \xrightarrow{\delta_N} & \mathrm{Ext}^2\big(\mathcal{T}_{N-1}, \mathbf{1}\big) \\
\downarrow\wr & & \downarrow\wr \\
T_{N-1}^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{H}^1\big(\mathbf{1}\big) \otimes_{\underline{\mathbf{1}}} \mathrm{H}^1\big(\mathbf{1}\big) & \xrightarrow{\mathrm{Id} \otimes \cup} & T_{N-1}^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{H}^2\big(\mathbf{1}\big).
\end{array} \tag{2}$$

Here the map δ_N is the cup product (with a minus sign). The top square is defined by the inclusion $\mathcal{T}_{N-1} \subset \mathcal{E}_N$ so that it commutes as we have proven that β''_N is the cup product (with a minus sign). The right top vertical map is an isomorphism by inductive hypothesis. The lower square is defined identifying $\mathrm{Ext}^1\big(\mathcal{T}_{N-1}, \mathbf{1}\big) \cong T_{N-1}^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{Ext}^1\big(\mathbf{1}, \mathbf{1}\big)$ and $\mathrm{Ext}^2\big(\mathcal{T}_{N-1}, \mathbf{1}\big) \cong T_{N-1}^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{Ext}^2\big(\mathbf{1}, \mathbf{1}\big)$ so that we can can re-write the middle row as

$$T_{N-1}^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{Ext}^1\big(\mathbf{1}, \mathbf{1}\big) \otimes_{\underline{\mathbf{1}}} \mathrm{Ext}^1\big(\mathbf{1}, \mathbf{1}\big) \longrightarrow T_{N-1}^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{Ext}^2\big(\mathbf{1}, \mathbf{1}\big),$$

which is the identity on $T_{N-1}^\vee$ and it is the cup product $\mathrm{Ext}^1\big(\mathbf{1}, \mathbf{1}\big) \otimes_{\underline{\mathbf{1}}} \mathrm{Ext}^1\big(\mathbf{1}, \mathbf{1}\big) \longrightarrow \mathrm{Ext}^2\big(\mathbf{1}, \mathbf{1}\big)$. We deduce that identifying $\mathrm{Ext}^1\big(\mathbf{1}, \mathbf{1}\big) \cong \mathrm{H}^1\big(\mathbf{1}\big)$ and $\mathrm{Ext}^2\big(\mathbf{1}, \mathbf{1}\big) \cong \mathrm{H}^2\big(\mathbf{1}\big)$, the map δ_N is obtained, up to sign, via the cup product

$$\cup \colon \mathrm{H}^1\big(\mathbf{1}\big) \otimes_{\underline{\mathbf{1}}} \mathrm{H}^1\big(\mathbf{1}\big) \longrightarrow \mathrm{H}^2\big(\mathbf{1}\big).$$

Hence, also the lower square commutes. As α_{N-1} is injective and coincides with ι_{N-1} by inductive hypothesis, we conclude that also $\alpha_{N-1} \otimes \mathrm{id}$ is injective, $\alpha_N = \iota_N$ and $T_{N+1}^\vee = \mathrm{Ker}\big(\beta''_N\big)$ coincides with R^{N+1}. This concludes the proof of the inductive step in (ii) and proves the inductive step of (vi).

Using the $\mathrm{Ext}^j = 0$ for $j \geq 3$ as $X \to S$ is of relative dimension 1, to prove (iv) and (v) for $n = N$ it suffices to show that the map

$$\beta_N \colon \mathrm{Ext}^1\big(\mathcal{T}_N, \mathbf{1}\big) \longrightarrow \mathrm{Ext}^2\big(\mathcal{E}_N, \mathbf{1}\big)$$

is surjective. Using the identifications of β_N with β''_N and the commutativity of the diagram (2), the map β_N is the map γ_N of Lemma 7.1. As the later is surjective by loc. cit. also β_N is surjective. In particular, $\mathrm{Ext}^2\big(\mathcal{E}_{N+1}, \mathbf{1}\big) \cong T_N^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{H}^2(\mathbf{1})$ is a finite and free $\underline{\mathbf{1}}$-module. As $\mathrm{Ext}^1\big(\mathcal{T}_N, \mathbf{1}\big) \cong T_N^\vee \otimes_{\underline{\mathbf{1}}} \mathrm{H}^1\big(\mathbf{1}\big)$ it also follows that $T_{N+1} = \mathrm{Ext}^1\big(\mathcal{E}_{n+1}, \mathbf{1}\big) = \mathrm{Ker}(\beta_N)$ is a finite and projective $\underline{\mathbf{1}}$-module of rank equal to

$$\mathrm{rk} T_{N+1} = \mathrm{rk} T_N \cdot \mathrm{rk} \mathrm{H}^1\big(\mathbf{1}\big) - \mathrm{rk} T_{N-1} \cdot \mathrm{rk} \mathrm{H}^2\big(\mathbf{1}\big).$$

As $\mathrm{rk} T_N \geq \mathrm{rk} T_{N-1}$ by inductive hypothesis, $\mathrm{rk}\mathrm{H}^1\big(\mathbf{1}\big) \geq 2$ and $\mathrm{rk}\mathrm{H}^2\big(\mathbf{1}\big) \leq 1$ as remarked above, it follows that $\mathrm{rk} T_{N+1} \geq \mathrm{rk} T_N$ and in particular $T_{N+1} \neq 0$. ☐

Now that we have proved the existence of the universal projective systems in the three categories of unipotent sheaves on X we will list some of their specific properties.

Corollary 3.6. *There is a unique action of G_K on the pointed étale system $\{(\mathcal{E}_n^{\mathrm{et}}, e_n^{\mathrm{et}})\}_n$ lifting the action on $X_{\overline{K}}$. Furthermore, each e_n^{et} is G_K-invariant.*

Proof. For every $\sigma \in G_K$ we have a unique morphism $f_\sigma\colon (\mathcal{E}_n^{\mathrm{et}}, e_n^{\mathrm{et}}) \to (\sigma^*(\mathcal{E}_n^{\mathrm{et}}), \sigma^*(e_n^{\mathrm{et}}))$ by universality. The map $\sigma \mapsto f_\sigma$ defines an action by uniqueness. In particular each f_σ is an isomorphism with inverse $f_{\sigma^{-1}}$. As b is a $\mathcal{O}_K$-valued point, it is G_K-invariant so that $\mathcal{E}_{n,b}^{\mathrm{et}} = \sigma^*(\mathcal{E}_n^{\mathrm{et}})_b$ and via this identification $\sigma^*(e_n^{\mathrm{et}}) = e_n^{\mathrm{et}}$. □

Let us recall the sequence of object $\{R^n\}_{n\geq 1}$ in $\mathrm{Uni}_{\mathbb{Q}_p}(X_{\overline{K}}^{\mathrm{ket}})^*$ and the fact that we have denoted

$$T_n = \mathrm{Hom}_{\mathbb{Q}_p}\big(R^n, \mathbb{Q}_p\big).$$

We deduce that T_n is naturally a $\mathbb{Q}_p$-representation of G_K, quotient of $\big(\mathrm{H}^1\big(X_{\overline{K}}^{\mathrm{ket}}\big)\big)^{\otimes n}$. The group $\mathrm{H}^1\big(X_{\overline{K}}^{\mathrm{ket}}\big)$ is the étale cohomology of the complement in $X_{\overline{K}}$ of the sections $\amalg_{i=1}^n s_i \otimes \overline{K}$ defining the log structure $N_{\overline{K}}$ (see [Il, Cor. 7.5]). Since the sections s_i are unramified over $\mathcal{O}_K$ by assumption, the latter $\mathrm{H}^1\big(X_{\overline{K}}^{\mathrm{ket}}\big)$ is a semistable representations of G_K in the sense of Fontaine. The same holds for its n-th tensor power and any of its quotients. We deduce from proposition 2.1:

Corollary 3.7. *For every n the G_K-representation T_n is $B_{\log}$-admissible.*

In the de Rham case, we will need a compatibility result under base change. Let R be a complete noetherian local domain of characteristic 0 and let $\iota_R\colon \mathcal{O} \to R$ be a continuous morphism of $\mathbb{W}(k)$-algebras. Consider the base change $\widetilde{X}_R := \widetilde{X}\widehat{\otimes}_{\mathcal{O}}\mathrm{Spf}(R)$ as a p-adic formal scheme. It inherits a log structure $\widetilde{N}_R$ from the base change of the log structure on $\widetilde{X}$. Also the base change of $\widetilde{b}$ via ι defines a section $\widetilde{b}_R\colon \mathrm{Spf}(R) \to \widetilde{X}_R$. Consider the category $\mathrm{Uni}_{\mathrm{dR}}\big(\widetilde{X}_R, \widetilde{N}_R\big)$ of unipotent objects in the category of sheaves $\mathcal{O}_{\widetilde{X}_R}$-modules endowed with integrable log connection relative to $\mathrm{Spf}(R)$ (up to p-isogenies) and the category $\mathrm{Uni}_{\mathrm{dR}}^*\big(\widetilde{X}_R, \widetilde{N}_R\big)$ where we further consider a section of the pull-back via $\widetilde{b}_R$. For example, we will consider the following two cases which will be important later:

(i) The unique map of $\mathbb{W}(k)$-algebras $\mathcal{O}_{\mathrm{cris}} \to \mathcal{O}_K$ mapping $Z \to \pi$;

(ii) The unique map of $\mathbb{W}(k)$-algebras $\mathcal{O}_{\mathrm{cris}} \to \mathbb{W}(k)\llbracket x\rrbracket$ mapping $Z \to px$.

On the other hand, as R is a complete local ring, $\widetilde{X}$ is a projective formal scheme over $\mathrm{Spf}(R)$ and we can algebraize it to a projective algebraic curve X_R with log structure N_R and with a section b_R by GAGA. Consider the category $\mathrm{Uni}_{\mathrm{dR}}^*\big(X_R, N_R\big)$ of unipotent objects in the category of sheaves of $\mathcal{O}_{X_R}$-modules with integrable log connections relative to $\mathrm{Spec}\big(R\big)$ (up to p-isogenies) and sections of the pull-back via b_R.

Lemma 3.8. *(i) If ι_R factors via $\iota_R^{\mathrm{cris}}\colon \mathcal{O}_{\mathrm{cris}} \to R$ For every n the universal object in $\mathrm{Uni}_{\mathrm{dR}}^*\big(\widetilde{X}_R, \widetilde{N}_R\big)$ of index $\leq n$ exists and its base change to $R[p^{-1}]$ is uniquely isomorphic to the base change of $\big(\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris}}, e_n^{\mathrm{cris}}\big)$ via ι_R^{cris}.*

(ii) For every n the universal object in $\mathrm{Uni}_{\mathrm{dR}}^*\big(X_R, N_R\big)$ of index $\leq n$ exists and its base via the canonical map of ringed toposes $\rho\colon X_R \to \widetilde{X}_R$ is uniquely isomorphic to the universal object in $\mathrm{Uni}_{\mathrm{dR}}^*\big(\widetilde{X}_R, \widetilde{N}_R\big)$ of index $\leq n$.

Proof. The existence statements in (i) and in (ii) are proven arguing as in Proposition 3.4. It also follows from loc. cit. that the cohomology groups $\mathrm{Ext}^j\big(\_,\mathbf{1}\big)$ of the universal objects (for $j=0$, 1 and 2) are free $R[p^{-1}]$-modules of finite rank both in the case of $\widetilde{X}_R$ and of X_R, non-zero for $j=1$. The claimed canonical isomorphisms are constructed using the analogue of Proposition 3.3.

(ii) The claim is proven by induction on n using the analogue of Proposition 3.3 and proving that the image via ρ of universal system $\big(\mathcal{E}_n,e_n\big)_n$ in $\mathrm{Uni}^*_{\mathrm{dR}}\big(X_R,N_R\big)$ satisfies Properties (i)–(iii) of loc. cit. Property (i) is clear and Property (ii) is preserved by ρ. Property (iii) follows if we show that $\mathrm{Ext}^j\big(\mathcal{E}_n,\mathbf{1}\big)\cong\mathrm{Ext}^j\big(\rho\big(\mathcal{E}_n\big),\mathbf{1}\big)$ for $j=0$, 1 and 2. These are de Rham cohomology groups of $\mathcal{E}_n^\vee$ and $\rho\big(\mathcal{E}_n\big)^\vee=\rho\big(\mathcal{E}_n^\vee\big)$ respectively. Using the de Rham-Hodge spectral sequences they can be expressed in terms of coherent cohomology groups which are isomorphic by GAGA.

(i) It suffices to show that both $\big(\mathcal{E}^{\mathrm{cris}}_{n,\widetilde{X}},e^{\mathrm{cris}}_n\big)_n$ and the universal system in $\mathrm{Uni}^*_{\mathrm{dR}}\big(\widetilde{X}_R,\widetilde{N}_R\big)$ are canonically isomorphic to the base change of the universal system $\big(\mathcal{E}_n,e_n\big)_n$ in $\mathrm{Uni}^*_{\mathrm{dR}}\big(\widetilde{X}_{\mathcal{O}},\widetilde{N}_{\mathcal{O}}\big)$ via the maps $\mathcal{O}\to\mathcal{O}_{\mathrm{cris}}$ and $\iota_R\colon\mathcal{O}\to R$ respectively. We rely on the analogue of Proposition 3.3 proving that such base change satisfies properties (i), (ii) and (iii) of loc. cit. In both cases Property (i) holds trivially true and Property (ii) also holds true as the base change of a short exact sequence of locally free sheaves is still a short exact sequence of locally free sheaves. For Property (iii) it suffices to show that the formation of $\mathrm{Ext}^j\big(\_,\mathbf{1}\big)$ commutes with base change of $\mathcal{E}_n$. More precisely, these groups coincide with de Rham cohomology groups of $\mathcal{E}_n^\vee$ and we are left to show that their base change via $\mathcal{O}\to\mathcal{O}_{\mathrm{cris}}$, resp. via $\iota_R\colon\mathcal{O}\to R$, map isomorphically onto the de Rham cohomology groups of the base change of $\mathcal{E}_n^\vee$ to $\widetilde{X}\widehat{\otimes}\mathcal{O}_{\mathrm{cris}}$, resp. to $\widetilde{X}_R$. We can calculate such groups using the hypercohomology of the logarithmic de Rham complex of $\mathcal{E}_n^\vee$ with respect to an affine covering of X_k. As $\mathcal{E}_n^\vee$ is a locally free sheaf, the formation of the logarithmic de Rham complex commutes with base change. As $\mathrm{Ext}^j\big(\mathcal{E}_n,\mathbf{1}\big)$ are finite and locally free $R[p^{-1}]$-modules, as remarked above, one then deduces the claimed commutation with base change arguing, for example, as in the proof of [Be, Prop. V.3.5.2]. □

3.7 Fundamental groups

Following the discussion in [Ha, §2] we show how the existence of an object as in Proposition 3.3 allows us to construct a fundamental group scheme. It follows from 3.3 that $\mathcal{E}_{n,b}\cong\mathrm{End}(\mathcal{E}_n)$ via the map taking $w\in\mathcal{E}_{n,b}$ to the unique endomorphism $g\colon\mathcal{E}_n\to\mathcal{E}_n$ such that $g_b(e_n)=w$. Hence, $A_n:=\mathcal{E}_{n,b}$ has a (non necessarily commutative) ring structure having e_n as identity element. Set $A_\infty:=\lim\limits_{\infty\leftarrow n}A_n$. For every n and m there is a unique morphism $c_{n,m}\colon\mathcal{E}_{n+m}\to\mathcal{E}_n\otimes\mathcal{E}_m$ sending $e_{n+m}\to e_n\otimes e_m$. Let $c_{n,m,b}\colon\mathcal{E}_{n+m,b}\to\mathcal{E}_{n,b}\otimes_{\underline{1}}\mathcal{E}_{m,b}$ be the induced map and let

$$c\colon A_\infty\longrightarrow A_\infty\otimes_{\underline{1}}A_\infty$$

be the limits of the morphisms $c_{n,m,b}$ over all n and m. Let $\varepsilon_\infty\colon A_\infty\to\underline{1}$ be the map induced by the projection $\varepsilon_n\colon\mathcal{E}_{n,b}\to\mathcal{E}_{1,b}=\underline{1}$. Then, A_∞ has a natural structure of co-commutative and co-associative Hopf algebra with comultiplication c_∞ and co-unit ε. Its dual $A_\infty^\vee:=\mathrm{Hom}_{\mathcal{C}}\big(A_\infty,\underline{1}\big)$ is then a commutative, associative, unitary ring with Hopf algebra structure. Let $G^{\mathrm{univ}}:=\mathrm{Spec}\big(A_\infty^\vee\big)$ be the associated group scheme over $\mathrm{Spec}(\underline{1})$, called the *fundamental group scheme* of Uni. It is flat over $\mathrm{Spec}(\underline{1})$.

Depending on the category we are working in we write $G^{\mathrm{et}}\big((X_{\overline{K}},N_{\overline{K}}),b_{\overline{K}}\big)$, $G^{\mathrm{cris}}\big((X,N),\widetilde{b}\big)$ or $G^{\mathrm{dR}}\big((X_K,N_K),b_K\big)$ for G^{univ}.

Proposition 3.9. *In the étale and in the de Rham case the category* Uni *together with the fibre functor $F\colon \mathrm{Uni}\to\mathcal{C}$ is a neutral Tannakian category, equivalent to the category of representations of G^{univ} on finite dimensional $\underline{\mathbf{1}}$-vector spaces.*

Proof. See [Ha, Thm. 2.9]. □

4 Geometrically semi-stable sheaves

4.1 Faltings' site and Fontaine's period sheaves

We provide the analogue of the constructions in §2.1 in the relative setting. In [AI, §2.2.3] we have introduced a site $\mathfrak{X}_{\overline{K}}$ called Faltings' site as follows:

i) the objects of the underlying category consist of pairs $\big(U,W\big)$ such that $U\in X^{\mathrm{ket}}$ and $W\in U_{\overline{K}}^{\mathrm{fket}}$ is Kummer finite étale over $U_{\overline{K}}$;

ii) a morphism $(U',W')\longrightarrow(U,W)$ consists of a pair (α,β), where $\alpha\colon U'\longrightarrow U$ is a morphism in X^{ket} and $\beta\colon W'\longrightarrow W\times_{U_{\overline{K}}}U'_{\overline{K}}$ is a morphism in $U_{\overline{K}}^{\prime,\mathrm{fket}}$;

iii) the topology is generated by the following families $\{(U_i,W_i)\longrightarrow(U,W)\}_{i\in I}$:

α) $\{U_i\longrightarrow U\}_{i\in I}$ is a covering in X^{fket} and $W_i\cong W\times_{U_{\overline{K}}}U_{i,\overline{K}}$ for every $i\in I$.

or

β) $U_i\cong U$ for all $i\in I$ and $\{W_i\longrightarrow W\}_{i\in I}$ is a covering in $U_{\overline{K}}^{\mathrm{fket}}$.

We have morphisms of sites

$$v\colon X^{\mathrm{ket}}\longrightarrow\mathfrak{X}_{\overline{K}},U\mapsto\big(U,U_{\overline{K}}\big)$$

and

$$z\colon\mathfrak{X}_{\overline{K}}\longrightarrow X_{\overline{K}}^{\mathrm{ket}},\qquad (U,W)\mapsto W,$$

inducing a morphism of associated toposes of sheaves

$$v_*\colon\mathrm{Sh}\big(\mathfrak{X}_{\overline{K}}\big)\longrightarrow\mathrm{Sh}\big(X^{\mathrm{ket}}\big),\quad z_*\colon\mathrm{Sh}\big(X_{\overline{K}}^{\mathrm{ket}}\big)\longrightarrow\mathrm{Sh}\big(\mathfrak{X}_{\overline{K}}\big).$$

In [AI, §2.3] we have also defined an ind-continuous sheaf of periods $\mathbb{B}_{\log}$ i.e., this sheaf is an inductive limit of inverse systems of sheaves. We summarize its key properties:

(1) it is a sheaf of $v^*\big(\mathcal{O}_{\widetilde{X}}\big)\widehat{\otimes}_{\mathcal{O}}B_{\log}$-modules. Here $v^*\big(\mathcal{O}_{\widetilde{X}}\big)\widehat{\otimes}_{\mathcal{O}}B_{\log}$ is viewed as the inductive limit with respect to the multiplication by t on the inverse system $v^*\big(\mathcal{O}_{\widetilde{X}}\big)\widehat{\otimes}_{\mathcal{O}}A_{\log}/(p,Z)^n$ for $n\in\mathbb{N}$;

(2) there is an integrable connection $\nabla_{\mathbb{W}(k)}\colon\mathbb{B}_{\log}\longrightarrow\mathbb{B}_{\log}\otimes_{\mathcal{O}_{\widetilde{X}}}\omega_{\widetilde{X}/\mathbb{W}(k)}$ (here we write $\omega_{\widetilde{X}/\mathbb{W}(k)}$ for the module of log differentials and we set $\omega_{\widetilde{X}/\mathbb{W}(k)}$ for $v^*\big(\omega_{\widetilde{X}/\mathbb{W}(k)}\big)$ by abuse of notation).

(3) thanks to [AI, §2.3.3& §2.3.4] $\mathbb{B}_{\log}$ it is endowed with a decreasing, exhaustive filtration $\mathrm{Fil}^n\mathbb{B}_{\log}$ by ind-continuous sheaves. The connection $\nabla_{\mathbb{W}(k)}$ satisfies Griffiths' transversality with respect to the filtration;

(4) For every small object $U=\mathrm{Spec}(R_U)$ of $X^{\rm ket}$ and for every choice of Frobenius on the open $\widetilde{U}$ of $\widetilde{X}$ defined by the special fiber U_k of U, the sheaf $\mathbb{B}_{\log}$ restricted to objects over $\bigl(U,U_{\overline{K}}\bigr)$ is endowed with a Frobenius morphism compatible with Frobenius on $\mathbb{B}_{\rm cris}^\nabla$ and $\mathbb{B}_{\log}^\nabla$.

4.2 Localizations

Fix (U,M_U) with $U=\mathrm{Spec}(R_U)$ a small object of $X^{\rm ket}$, mapping surjectively onto $\mathrm{Spec}(\mathcal{O}_K)$, an algebraic closure $\mathbb{C}_U$ of $\mathrm{Frac}(R_U)$ and $\mathbb{C}_U^{\log}=(\mathbb{C}_U,N_{\mathbb{C}})$ a log geometric point of $\bigl(\mathrm{Spec}(R_U),N_U\bigr)$ over $\mathbb{C}_U$. Let $\widetilde{U}=\mathrm{Spf}\bigl(\widetilde{R}_U\bigr)$ be the formal open subscheme of $\widetilde{X}$ associated to the special fiber U_k of U and $\widetilde{N}_U$ the induced log structure on $\widetilde{U}$.

Let $\mathcal{G}_{U_K}$ be the Kummer étale Galois group $\pi_1^{\log}\bigl(\mathrm{Spec}\bigl(R_U[p^{-1}]\bigr),\mathbb{C}_U^{\log}\bigr)$, see [Il, §4.5], classifying Kummer étale covers of $\mathrm{Spec}\bigl(R_U[p^{-1}]\bigr)$. It sits in an exact sequence

$$0\longrightarrow G_{U_K}\longrightarrow \mathcal{G}_{U_K}\longrightarrow G_K\longrightarrow 0$$

where G_{U_K} is the geometric Kummer étale Galois group $\pi_1^{\log}\bigl(\mathrm{Spec}\bigl(R_U\otimes_{\mathcal{O}_K}\overline{K}\bigr),\mathbb{C}_U^{\log}\bigr)$.

We write $\bigl(\overline{R}_U,\overline{N}_U\bigr)$ for the direct limit of all the normal extensions $R_U\otimes_{\mathcal{O}_K}\mathcal{O}_{\overline{K}}\to S$, all log structures N_S on $\mathrm{Spec}(S[1/p])$ and all maps $\bigl(R_{U,\overline{K}},N_{U,\overline{K}}\bigr)\to\bigl(S[1/p],N_S\bigr)\to\bigl(\mathbb{C}_U,N_{\mathbb{C}}\bigr)$ such that $(R_{U,\overline{K}},N_{U,\overline{K}})\to(S[1/p],N_S)$ is finite Kummer étale. In [AI, §2.2.6] we have explained how to associate to a (ind-continuous) sheaf $\mathcal{F}$ on $\mathfrak{X}_{\overline{K}}$ a continuous representation

$$\mathcal{F}(\overline{R}_U):=\lim_{W=\mathrm{Spec}(S[1/p])}\mathcal{F}\bigl(U,W\bigr)$$

of G_{U_K}, where the limit is taken over all (S,N_S). Next we will describe the localizations of the sheaves $\mathbb{B}_{\rm cris}^\nabla$, $\mathbb{B}_{\log}^\nabla$ and $\mathbb{B}_{\log}$.

Put

$$\widetilde{\mathbf{E}}^+_{\overline{R}_U}:=\lim_{\leftarrow}\overline{R}_U/p\overline{R}_U$$

where the projective limits are taken with respect to Frobenius $x\mapsto x^p$, with log structure provided by the the inverse image of the log structure on $\overline{R}_U/p\overline{R}_U$ defined by $\overline{N}_U$. We get an induced log structure on $\mathbb{W}\bigl(\widetilde{\mathbf{E}}^+_{\overline{R}_U}\bigr)$ applying the Teichmüller lift of the log structure on $\widetilde{\mathbf{E}}^+_{\overline{R}_U}$. There is a natural map Θ

$$\Theta\colon \widetilde{\mathbf{E}}^+_{\overline{R}_U}\longrightarrow \widehat{\overline{R}}_U,$$

strict with respect to the log structures. Extending the morphism Θ $\widetilde{R}_U$-linearly we obtain a homomorphism of $\widetilde{R}_U$–algebras

$$\Theta_{\widetilde{R}_U,\log}\colon \mathbb{W}\bigl(\widetilde{\mathbf{E}}^+_{\overline{R}_U}\bigr)\otimes_{\mathbb{W}(k)}\widetilde{R}_U\longrightarrow \widehat{\overline{R}}_U.$$

We consider on $\mathbb{W}\bigl(\widetilde{\mathbf{E}}^+_{\overline{R}_U}\bigr)\otimes_{\mathbb{W}(k)}\widetilde{R}_U$ the log structure defined as the product of the log structures on $\mathbb{W}\bigl(\widetilde{\mathbf{E}}^+_{\overline{R}_U}\bigr)$ and on $\widetilde{R}_U$. Then, $\Theta_{\widetilde{R}_U,\log}$ respects the log structures.

Let $\mathrm{A}_{\log}(\widetilde{R}_U)$ be the p–adic completion of the log divided power envelope $\Bigl(\mathbb{W}\bigl(\widetilde{\mathbf{E}}^+_{\overline{R}_U}\bigr)\otimes_{\mathbb{W}(k)}\widetilde{R}_U\Bigr)^{\rm logDP}$ of $\mathbb{W}\bigl(\widetilde{\mathbf{E}}^+_{\overline{R}_U}\bigr)\otimes_{\mathbb{W}(k)}\widetilde{R}_U$ with respect to $\mathrm{Ker}\bigl(\Theta_{\widetilde{R}_U,\log}\bigr)$ (compatible with the canonical divided power structure on $p\mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\overline{R}_U}\big)\otimes_{\mathbb{W}(k)}\widetilde{R}_U)$ in the sense of [K2, Def. 5.4]. It is endowed with a filtration coming from the DP filtration. For every choice of a lift of Frobenius on $\widetilde{R}_U$, compatible with the given Frobenius on $\mathcal{O}$, we get an induced Frobenius morphism on $\mathrm{A}_{\log}\big(\widetilde{R}_U\big)$. Define

$$\mathrm{B}_{\log}\big(\widetilde{R}_U\big) := \mathrm{A}_{\log}\big(\widetilde{R}_U\big)\big[t^{-1}\big],$$

with induced filtration and Frobenius, once chosen a lift of Frobenius on $\widetilde{R}_U$. It follows from [AI, §2.3.6]:

Proposition 4.1. *We have an isomorphism of algebras, compatible with filtrations, actions of G_{U_K} and Frobenius:* $\mathbb{B}_{\log}\big(\overline{R}_U\big) \cong \mathrm{B}_{\log}\big(\widetilde{R}_U\big)$

Recall that we have chosen a $\mathcal{O}_K$-valued point b of X and a $\mathcal{O}$-valued point $\widetilde{b}$ of $\widetilde{X}$ lifting $\widehat{b}$, the $\mathcal{O}_K$-valued point associated to $\widehat{X}$. Choose a small open subscheme U of X such that b factors through U. Thus b defines a ring homomorphism $R_U \to \mathcal{O}_K$. Choose an extension to a morphism $\overline{b}\colon \overline{R}_U \to \mathcal{O}_{\overline{K}}$. Then $\overline{b}$ defines a morphism $\widetilde{\mathbf{E}}^+_{\overline{R}_U} \to \widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}$ and hence a morphism on Witt vectors $w(\overline{b})$. We get a commutative diagram

$$\begin{array}{ccc}
\mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\overline{R}_U}\big)\otimes_{\mathbb{W}}\widetilde{R}_U & \stackrel{\Theta_{\log}}{\longrightarrow} & \widehat{\overline{R}}_U \\
\big\downarrow w(\overline{b})\otimes\widetilde{b} & & \big\downarrow\widehat{\overline{b}} \\
\mathbb{W}\big(\widetilde{\mathbf{E}}^+_{\mathcal{O}_{\overline{K}}}\big)\otimes_{\mathbb{W}}\mathcal{O} & \stackrel{\theta\otimes\theta_{\mathcal{O}}}{\longrightarrow} & \widehat{\mathcal{O}}_{\overline{K}}
\end{array}$$

This induces a morphism $\mathrm{A}_{\log}\big(\widetilde{R}_U\big) \longrightarrow A_{\log}$ and, inverting t, a morphism of $B_{\log}$-algebras

$$b_{\log}\colon \mathrm{B}_{\log}\big(\widetilde{R}_U\big) \longrightarrow B_{\log} \tag{3}$$

4.3 Geometrically and arithmetically semistable sheaves

$\mathbb{Q}_p$-adic étale sheaves. By a p–adic sheaf $\mathbb{L}$ on $X_{\overline{K}}$ we mean a continuous system $\{\mathbb{L}_n\} \in \mathrm{Sh}(X_{\overline{K}})^{\mathbb{N}}$ such that $\mathbb{L}_n$ is a locally constant sheaf of $\mathbb{Z}/p^n\mathbb{Z}$–modules, free of finite rank, and $\mathbb{L}_n = \mathbb{L}_{n+1}/p^n\mathbb{L}_{n+1}$ for every $n \in \mathbb{N}$. The category of p-adic sheaves on $X_{\overline{K}}$ is an abelian tensor category. Define $\mathrm{Sh}(X_{\overline{K}})_{\mathbb{Q}_p}$ to be the full subcategory of $\mathrm{Ind}\big(\mathrm{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}\big)$ consisting of inductive systems of the form $(\mathbb{L})_{i\in\mathbb{Z}}$ where $\mathbb{L}$ is a p-adic étale sheaf and the transition maps $\mathbb{L} \to \mathbb{L}$ are given by multiplication by p. The functor z_* is a fully faithful functor of abelian tensor categories from $\mathrm{Sh}(X_{\overline{K}})_{\mathbb{Q}_p}$ to the category of ind-continuous sheaves on $\mathfrak{X}_{\overline{K}}$. Abusing notations we still write $\mathbb{L}$ instead of $z_*(\mathbb{L})$.

4.3.1 The functor $\mathbb{D}^{\mathrm{geo}}_{\mathrm{cris}}$

Given a $\mathbb{Q}_p$–adic sheaf $\mathbb{L}$ on $X^{\mathrm{ket}}_{\overline{K}}$ define

$$\mathbb{D}^{\mathrm{geo}}_{\mathrm{cris}}(\mathbb{L}) := v_{\overline{K},*}\Big(\mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{B}_{\log}\Big).$$

It is a sheaf of $\mathcal{O}_{\widetilde{X}}\widehat{\otimes}_{\mathcal{O}}B_{\log}$-modules in $\mathrm{Sh}\big(X^{\mathrm{ket}}\big)$. We get a functor

$$\mathbb{D}^{\mathrm{geo}}_{\mathrm{cris}}\colon \mathrm{Sh}\big(\mathfrak{X}_{\overline{K}}\big)_{\mathbb{Q}_p} \longrightarrow \mathrm{Mod}\left(\mathcal{O}_{\widetilde{X}}\widehat{\otimes}B_{\log}\right).$$

We have the following explicit description given in [AI, §2.4.3]. For every small object (U, N_U) of $X^{\rm ket}$, let $V := \mathbb{L}\big(\overline{R}_U\big)$ be the localization of $\mathbb{L}$. It is a representation of G_{U_K}. Set $\mathrm{D}_{\log}^{\rm geo}\big(V\big) := \Big(V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\log}\big(\widetilde{R}_U\big)\Big)^{G_{U_K}}$. Then, $\mathbb{D}_{\rm cris}^{\rm geo}\big(\mathbb{L}|_U\big) \cong \mathrm{D}_{\log}^{\rm geo}\big(V\big)$.

Definition 4.2. A $\mathbb{Q}_p$-adic sheaf $\mathbb{L} = \{\mathbb{L}_n\}_n$ on $X_{\overline{K}}^{\rm ket}$ is called *geometrically semistable* if

i. there exists a coherent $\mathcal{O}_{\widetilde{X}}\widehat{\otimes}_{\mathcal{O}}A_{\log}$-submodule $D(\mathbb{L})$ of $\mathbb{D}_{\rm cris}^{\rm geo}\big(\mathbb{L}\big)$ such that:

   (a) it is stable with respect to the connection $\nabla_{\mathbb{L},\mathbb{W}(k)}$ and $\nabla_{\mathbb{L},\mathbb{W}(k)}|_{D(\mathbb{L})}$ is integrable and topologically nilpotent on $D(\mathbb{L})$;

   (b) $\mathbb{D}_{\rm cris}^{\rm geo}\big(\mathbb{L}\big) \cong D(\mathbb{L})\otimes_{A_{\log}}B_{\log}$;

   (c) there exist integers h and $n \in \mathbb{N}$ such that for every small object U of $X^{\rm ket}$ the map $t^h\varphi_{\mathbb{L},U}$ sends $D(\mathbb{L})|_U$ to $D(\mathbb{L})|_U$ and multiplication by t^n on $D(\mathbb{L})|_U$ factors via $t^h\varphi_{\mathbb{L},U}$.

ii. $\mathbb{D}_{\rm cris}^{\rm geo}\big(\mathbb{L}\big)$ is locally free of finite rank on $X^{\rm ket}$ as $\mathcal{O}_{\widetilde{X}}\widehat{\otimes}_{\mathcal{O}}B_{\log}$-module.

iii. the natural map $\alpha_{\log,\mathbb{L}}\colon \mathbb{D}_{\rm cris}^{\rm geo}\big(\mathbb{L}\big)\otimes_{\big(\mathcal{O}^{\rm geo}_{\widetilde{X},\log}\big)}\mathbb{B}_{\log,\overline{\mathrm{K}}} \longrightarrow \mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{B}_{\log,\overline{\mathrm{K}}}$ is an isomorphism in the category $\mathrm{Ind}\big(\mathrm{Sh}(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}\big)$.

We let $\mathrm{Sh}\big(X_{\overline{K}}^{\rm ket}\big)_{\rm gs}$ be the full subcategory of $\mathbb{Q}_p$-adic étale sheaves on $X_{\overline{K}}^{\rm ket}$ consisting of geometrically semistable sheaves. We have the following fundamental result [AI, Prop. 2.26 & Prop. 3.68]:

Proposition 4.3. *(i) The category of geometrically semistable representations is closed under duals, tensor products and extensions. In particular, the category* $\mathrm{Uni}_{\mathbb{Q}_p}\big(X_{\overline{K}}^{\rm ket}\big)$ *of unipotent* $\mathbb{Q}_p$*-adic étale sheaves is a full subcategory of* $Sh\big(X_{\overline{K}}^{\rm ket}\big)_{\rm gs}$.

(ii) The functor $\mathbb{D}_{\rm cris}^{\rm geo}$*, from the category of geometrically semistable representations to the category of* $\mathcal{O}_{\widetilde{X}}\widehat{\otimes}_{\mathcal{O}}B_{\log}$*-modules, commutes with duals and tensor products and moreover it is exact.*

Fix (U, M_U) with $U = \mathrm{Spec}(R_U)$ a small object of $X^{\rm ket}$ as in §4.2. It follows from [AI, Prop. 2.26& Prop. 3.65] that if $\mathbb{L}$ is a geometrically semistable $\mathbb{Q}_p$-adic étale sheaves on $X_{\overline{K}}$ and if $V := \mathbb{L}(\overline{R}_U)$ is the associated representation of G_{U_K}, then setting $\mathrm{D}_{\log}^{\rm geo}\big(V\big) := \big(V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\log}\big(R_U\big)\big)^{G_{U_K}}$ we deduce from 4.1:

$$\mathbb{D}_{\rm cris}^{\rm geo}\big(\mathbb{L}\big)(U) = \mathrm{D}_{\log}^{\rm geo}(V) \tag{4}$$

and

$$\mathrm{D}_{\log}^{\rm geo}(V) \otimes_{\mathrm{B}_{\log}\left(\widetilde{R}_U\right)^{G_{U_K}}} \mathrm{B}_{\log}\big(\widetilde{R}_U\big) \longrightarrow V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\log}\big(\widetilde{R}_U\big)$$

is an isomorphism, compatible with Galois actions, filtrations and Frobenius. In particular, pulling-back this isomorphism via the section $b_{\log}\colon \mathrm{B}_{\log}\big(\widetilde{R}_U\big) \longrightarrow B_{\log}$ defined in (3), we get a G_K-equivariant isomorphism of $B_{\log}$-modules:

$$b_{\log}^*\big(\mathrm{D}_{\log}^{\rm geo}\big(V\big)\big) \cong b_{\overline{K}}^*\big(V\big) \otimes_{\mathbb{Q}_p} B_{\log} \tag{5}$$

The connection $\nabla_{\mathbb{L},\mathbb{W}(k)}$ induces a connection on $b_{\log}^*\big(\mathrm{D}_{\log}^{\mathrm{geo}}(V)\big)$ compatible with the connection on the RHS of equation (5) inducing the connection $B_{\log}$ described in §2.1 and trivial on $b_{\overline{K}}^*(V)$. The filtration on $\mathrm{D}_{\log}^{\mathrm{geo}}(V)$ defines a filtration on $b_{\log}^*\big(\mathrm{D}_{\log}^{\mathrm{geo}}(V)\big)$ (a priori not strictly) compatible with the filtration on $b_{\overline{K}}^*(V)\otimes_{\mathbb{Q}_p} B_{\log}$ defined by requiring that $b_{\overline{K}}^*(V)$ are in Fil^0 and the filtration on $B_{\log}$ is as in §2.1.

We come to the main result of this section. Due to Corollary 3.6 there is an action of G_K on the pointed étale system $\big\{\big(\mathcal{E}_n^{\mathrm{et}}, e_n^{\mathrm{et}}\big)\big\}_n$ lifting the action on $X_{\overline{K}}$ and such that e_n^{et} is G_K-invariant for every n. Arguing as in [AI, Lemma 3.3], or using directly equation (4), we deduce that for every $n\in\mathbb{N}$ the sheaf of $\mathcal{O}_{\widetilde{X}}\widehat{\otimes}_{\mathcal{O}}B_{\log}$-modules $\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{E}_n^{\mathrm{et}}\big)$ is endowed with an action of G_K such that $e_n^{\mathrm{et}}\otimes 1\in b_{\log}^*\big(\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{E}_n^{\mathrm{et}}\big)\big)$ (via the identification in (5)) is G_K-invariant.

On the other hand the universal pointed system $\big\{\big(\mathcal{E}_n^{\mathrm{crys}}, e_n^{\mathrm{crys}}\big)\big\}_{n\in\mathbb{N}}$ on the crystalline site $\big(X_0/\mathcal{O}_{\mathrm{cris}}\big)_{\log}^{\mathrm{cris}}$ provides by evaluation at $\widetilde{X}\widehat{\times}_{\mathcal{O}}\mathrm{Spf}(\mathcal{O}_{\mathrm{crys}})$ a system of sheaves of $\widehat{\mathcal{O}}_{\widetilde{X}}^{\mathrm{DP}}\big[p^{-1}\big]$-modules with integrable logarithmic connection $\big(\mathcal{E}_{n,\widetilde{X}}^{\mathrm{crys}},\nabla_n\big)$ relative to $\mathcal{O}_{\mathrm{cris}}$ on X_0^{ket} (see §3.4) and compatible sections e_n^{crys} of $\widetilde{b}^*\big(\mathcal{E}_{n,\widetilde{X}}^{\mathrm{crys}}\big)$.

Theorem 4.4. *There exist unique isomorphisms*

$$\alpha_n\colon \mathcal{E}_{n,\widetilde{X}}^{\mathrm{crys}}\widehat{\otimes}_{\mathcal{O}_{\mathrm{crys}}}B_{\log}\longrightarrow \mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{E}_n^{\mathrm{et}}\big)$$

of $\mathcal{O}_{\widetilde{X}}\widehat{\otimes}_{\mathcal{O}}B_{\log}$-modules on X_0^{ket} with logarithmic connection with respect to $B_{\log}$, which are compatible for varying n and such that $b_{\log}^*(\alpha_n)$ sends e_n^{crys} to $e_n^{\mathrm{et}}\otimes 1$ for every $n\in\mathbb{N}$. Moreover, for every $n\in\mathbb{N}$ the isomorphism α_n is G_K-equivariant.

Proof. We proceed by induction on n. For $n=1$ we know that $\mathcal{E}_{1,\widetilde{X}}^{\mathrm{crys}}=\widehat{\mathcal{O}}_{\widetilde{X}}^{\mathrm{DP}}\big[p^{-1}\big]$ with connection given by the usual derivation and $e_n^{\mathrm{crys}}=1$. On the other hand $\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{E}_1^{\mathrm{et}}\big)=\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathbf{1}\big)=\mathcal{O}_{\widetilde{X}}\widehat{\otimes}_{\mathcal{O}}B_{\log}$ with connection given by the usual derivation and $e_n^{\mathrm{et}}\otimes 1=1$. Thus the claim follows for $n=1$.

Assume that the statement is proven for n. Let us prove if for $n+1$. Note that $\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{E}_{n+1}^{\mathrm{et}}\big)$ is, as a module with connection, an extension of $\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{E}_n^{\mathrm{et}}\big)$ by $\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{T}_n^{\mathrm{et}}\big)=T_n^{\mathrm{et}}\otimes_{\mathbb{Q}_p}\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathbf{1}\big)$ where T_n^{et} is a $B_{\log}$-admissible representation of G_K (see corollary 3.7). Hence, such an extension is defined by a class

$$c_{n+1}\in\mathbb{H}_{\mathrm{dR}}^1\left(X_0^{\mathrm{ket}},T_n^{\mathrm{et}}\otimes_{\mathbb{Q}_p}\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{E}_n^{\mathrm{et}}\big)^\vee\right)\cong\mathbb{H}_{\mathrm{dR}}^1\left(X_0^{\mathrm{ket}},\mathcal{E}_{n,\widetilde{X}}^{\mathrm{crys},\vee}\right)\widehat{\otimes}_{\mathcal{O}_{\mathrm{crys}}}B_{\log}\otimes_{\mathbb{Q}_p}T_n^{\mathrm{et}}.$$

The last isomorphism is a G_K-equivariant isomorphism of $B_{\log}$-modules obtained using the inductive hypothesis. We have also used the inductive hypothesis to identify $\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{E}_n^{\mathrm{et}}\big)$ with $\mathcal{E}_{n,\widetilde{X}}^{\mathrm{crys}}\widehat{\otimes}_{\mathcal{O}_{\mathrm{crys}}}B_{\log}$ and the fact that $\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}$ commutes with duals (Proposition 4.3). As T_n^{et} is $B_{\log}$-admissible, setting $D_{\log}\big(T_n^{\mathrm{et}}\big):=\big(B_{\log}\otimes_{\mathbb{Q}_p}T_n^{\mathrm{et}}\big)^{G_K}$, the natural G_K-equivariant map $D_{\log}\big(T_n^{\mathrm{et}}\big)\otimes_{B_{\log}^{G_K}}B_{\log}\longrightarrow T_n^{\mathrm{et}}\otimes B_{\log}$ of $B_{\log}$-modules is an isomorphism. The existence of a G_K-action on $\mathbb{D}_{\mathrm{cris}}^{\mathrm{geo}}\big(\mathcal{E}_{n+1}^{\mathrm{et}}\big)$ translates into the fact that c_{n+1} is G_K-invariant. Thus, c_{n+1} defines a class in

$$\begin{aligned} c_{n+1}&\in\left(\mathbb{H}_{\mathrm{dR}}^1\left(X_0^{\mathrm{ket}},\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris},\vee}\right)\widehat{\otimes}_{\mathcal{O}_{\mathrm{cris}}}D_{\log}\big(T_n^{\mathrm{et}}\big)\otimes_{B_{\log}^{G_K}}B_{\log}\right)^{G_K}\\ &=\mathbb{H}_{\mathrm{dR}}^1\left(X_0^{\mathrm{ket}},\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris},\vee}\right)\widehat{\otimes}_{\mathcal{O}_{\mathrm{cris}}}D_{\log}\big(T_n^{\mathrm{et}}\big)=\mathbb{H}_{\mathrm{dR}}^1\left(X_0^{\mathrm{ket}},\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris},\vee}\widehat{\otimes}_{\mathcal{O}_{\mathrm{cris}}}D_{\log}\big(T_n^{\mathrm{et}}\big)\right).\end{aligned}$$

By Proposition 3.4 we have that $T_n^{\rm et}=\mathrm{Ext}^1\big(\mathcal{E}_n^{\rm et},\mathbf{1}\big)\cong \mathrm{H}^1\big(X_{\overline{K}}^{\rm ket},\mathcal{E}_n^{\rm et,\vee}\big)$. Hence, $D_{\log}\big(T_n^{\rm et}\big)\cong T_n^{\rm cris}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}^{G_K}$ with $T_n^{\rm crys}:=\mathrm{H}^1\big(X_0/\mathcal{O}_{\rm cris},\mathcal{E}_n^{\rm cris,\vee}\big)$ by [AI, Thm. 1.1]. As the latter group is $\mathrm{Ext}^1\big(\mathcal{E}_n^{\rm cris},\mathbf{1}\big)\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}^{G_K}$, again in virtue of Proposition 3.4, also $\mathcal{E}^{\rm cris}_{n+1,\widetilde{X}}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}^{G_K}$ is an extension of $\mathcal{E}^{\rm cris}_{n,\widetilde{X}}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}^{G_K}$ by $D_{\log}\big(T_n^{\rm et}\big)$.

We conclude that $\mathcal{E}^{\rm cris}_{n+1,\widetilde{X}}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}$ and $\mathbb{D}^{\rm geo}_{\rm cris}\big(\mathcal{E}_{n+1}^{\rm et}\big)$ are *both* extensions of $\mathcal{E}^{\rm cris}_{n,\widetilde{X}}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}$ (using the inductive hypothesis for the existence of α_n) by $T_n^{\rm et}\otimes_{\mathbb{Q}_p}B_{\log}\cong T_n^{\rm cris}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}$.

Arguing as in Proposition 3.3 one proves that there is a unique isomorphism α_{n+1} of such extensions such that $b_{\log}^*(\alpha_{n+1})$ sends $e_{n+1}^{\rm cris}$ to $e_{n+1}^{\rm et}\otimes 1$. The existence follows from the fact that the map

$$\mathrm{Ext}^1\big(\mathcal{E}^{\rm cris}_{n,\widetilde{X}}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log},\mathbf{1}\big)\to \mathrm{Ext}^1\big(\mathcal{E}^{\rm cris}_{n+1,\widetilde{X}}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log},\mathbf{1}\big)$$

is zero as it is the base change $\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}$ of $\mathrm{Ext}^1\big(\mathcal{E}^{\rm cris}_{n,\widetilde{X}},\mathbf{1}\big)\to\mathrm{Ext}^1\big(\mathcal{E}^{\rm cris}_{n+1,\widetilde{X}},\mathbf{1}\big)$ which is zero by loc. cit. The uniqueness follows from the fact that the projection $\mathcal{E}^{\rm cris}_{n,\widetilde{X}}\to\mathcal{E}^{\rm cris}_{1,\widetilde{X}}=\mathbf{1}$ induces an isomorphism $\mathrm{Hom}\big(\mathcal{E}^{\rm cris}_{n,\widetilde{X}},\mathbf{1}\big)=\mathrm{Hom}\big(\mathbf{1},\mathbf{1}\big)$ and, hence, it provides an isomorphism $\mathrm{Hom}\big(\mathcal{E}^{\rm cris}_{n,\widetilde{X}}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log},\mathbf{1}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}\big)=B_{\log}$ after base change $\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}$.

We are only left to prove that α_{n+1} is G_K-equivariant. This follows from its uniqueness as both $e_{n+1}^{\rm cris}$ and $e_{n+1}^{\rm et}\otimes 1$ are G_K-invariant (see Corollary 3.6). □

4.3.2 The functor $\mathbb{D}^{\rm ar}_{\rm cris}$

Consider a lisse $\mathbb{Q}_p$-adic sheaf $\mathbb{L}$ on $X_K^{\rm ket}$. We view it as a sheaf on $X_{\overline{K}}^{\rm ket}$ endowed with an auxiliary action of G_K lifting the action on $X_{\overline{K}}$. As in [AI, Lemma 3.3], or using directly equation (4), one can prove that the sheaf $v_*\big(\mathbb{B}_{\log}\big)$ and more generally $\mathbb{D}^{\rm geo}_{\rm cris}(\mathbb{L})$ is endowed with an action of G_K.

We wish to study $\mathbb{D}^{\rm geo}_{\rm cris}(\mathbb{L})^{G_K}$. For $\mathbb{L}=\mathbb{Q}_p$ the sheaf $\mathbb{D}^{\rm geo}_{\rm cris}(\mathbb{L})^{G_K}=v_*\big(\mathbb{B}_{\log}\big)^{G_K}$ contains $\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{X}}[p^{-1}]$ but is not known to be equal to it. But is is very close. Namely, given a small object U of $X^{\rm ket}$ and a choice of Frobenius on the formal open subscheme $\widetilde{U}$ of $\widetilde{X}$ associated to U, it is proven in [AI, Lemma 2.25] that the second power of Frobenius φ^2 on $v_*\big(\mathbb{B}_{\log}\big)|_U$ factors via $\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{U}}[p^{-1}]$. One defines

$$\mathbb{D}^{\rm ar}_{\rm cris}(\mathbb{L})|_U=\big(\mathbb{D}^{\rm geo}_{\rm cris}(\mathbb{L})\big)^{G_K}|_U\otimes^{\varphi^2}_{v_*\left(\mathbb{B}_{\log}|_U\right)}\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{U}}[p^{-1}].$$

Following [AI, §2.4.4] we say that $\mathbb{L}|_{U_{\overline{K}}}$ is semi-stable if

i. $\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathbb{L}\big)|_U$ is in $\mathrm{Coh}\big(\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{U}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\big)$ (the full subcategory of sheaves of $\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{U}}$-modules isomorphic to $F\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ for some coherent sheaf F of $\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{U}}$-modules on $U_0^{\rm ket}$);

ii. the natural map $\alpha_{\log,\mathbb{L}}\colon\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathbb{L}\big)|_U\otimes_{\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{U}}}\mathbb{B}_{\rm log,K}\longrightarrow\mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{B}_{\log}$ is an isomorphism in the category $\mathrm{Ind}\left(\mathrm{Sh}(\mathfrak{U}_{\overline{K}})^{\mathbb{N}}\right)$ of inductive systems of continuous sheaves.

We say that $\mathbb{L}$ is semi-stable if there exists a covering of X by open small subschemes $\{U_i\}$ and for every i there exists a lift of Frobenius on $\widetilde{U}_i$ such that $\mathbb{L}|_{U_{i,\overline{K}}}$ is semistable for every i. We let $\mathrm{Sh}\big(X_K^{\mathrm{ket}}\big)_{\mathrm{ss}}$ be the full sub-category of $\mathbb{Q}_p$-adic étale sheaves on $\mathfrak{X}_{\overline{K}}$ consisting of semi-stable sheaves.

For a semi-stable sheaf $\mathbb{L}$ the elements $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}\big(\mathbb{L}\big)|_{U_i}$ glue to an element $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}\big(\mathbb{L}\big)$ of $\mathrm{Coh}\big(\widehat{\mathcal{O}}^{\mathrm{DP}}_{\widetilde{X}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\big)$. We obtain a functor

$$\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}\colon \mathrm{Sh}\big(X_K^{\mathrm{ket}}\big)_{\mathrm{ss}} \longrightarrow \mathrm{Coh}\big(\widehat{\mathcal{O}}^{\mathrm{DP}}_{\widetilde{X}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\big)$$

and we have a natural $B_{\log}$-linear and G_K-equivariant map

$$\beta_{\mathbb{L}}\colon \mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}\big(\mathbb{L}\big)\widehat{\otimes}_{\mathcal{O}_{\mathrm{crys}}} B_{\log} \longrightarrow \mathbb{D}^{\mathrm{geo}}_{\mathrm{cris}}(\mathbb{L}),$$

functorial in $\mathbb{L}$.

Proposition 4.5. *The functor $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}$ has the following extra properties:*

(1) the map $\beta_{\mathbb{L}}\colon \mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}\big(\mathbb{L}\big)\widehat{\otimes}_{\mathcal{O}_{\mathrm{cris}}} B_{\log} \longrightarrow \mathbb{D}^{\mathrm{geo}}_{\mathrm{cris}}(\mathbb{L})$ is an isomorphism for every $\mathbb{L}$. It commutes with connections relative to $\mathbb{W}(k)$ and is strict with respect to the filtrations on $\mathbb{D}^{\mathrm{geo}}_{\mathrm{cris}}(\mathbb{L})$ and the filtration on $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}\big(\mathbb{L}\big)\widehat{\otimes}_{\mathcal{O}_{\mathrm{cris}}} B_{\log}$ composite of the filtrations on $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}\big(\mathbb{L}\big)$ and on $B_{\log}$;

(2) $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})$ is a finite and projective $\widehat{\mathcal{O}}^{\mathrm{DP}}_{\widetilde{X}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$-module;

(3) $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})$ is endowed with a decreasing, exhaustive filtration $\mathrm{Fil}^n\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})$, for $n\in\mathbb{Z}$, strictly compatible with the filtration on $\mathbb{D}^{\mathrm{geo}}_{\mathrm{cris}}(\mathbb{L})$ via $\beta_{\mathbb{L}}$ and with finite and projective $\mathcal{O}_{\widehat{X}_K}$-modules as graded pieces;

(4) $\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})$ is endowed with an integrable and topologically nilpotent connection

$$\nabla_{\mathbb{L},\mathbb{W}(k)}\colon \mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L}) \longrightarrow \mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})\otimes_{\mathcal{O}_{\widetilde{X}}} \omega^1_{\widetilde{X}/\mathbb{W}(k)}$$

compatible with the connection on $\mathbb{D}^{\mathrm{geo}}_{\mathrm{cris}}(\mathbb{L})$ via $\beta_{\mathbb{L}}$ and such that the filtration satisfies Griffiths' transversality;

(5) given a small U object of X^{ket} and a choice of Frobenius on the formal open subscheme $\widetilde{U}$ of $\widetilde{X}$ associated to U, we have a Frobenius operator $\varphi_{\mathbb{L}}\colon \mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})|_U \longrightarrow \mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})|_U$ compatible with the Frobenius on $\mathbb{D}^{\mathrm{geo}}_{\mathrm{cris}}(\mathbb{L})|_U$ via $\beta_{\mathbb{L}}$ and horizontal with respect to $\nabla_{\mathbb{L},\mathbb{W}(k)}$;

(6) if write

$$\nabla_{\mathbb{L},\mathcal{O}}\colon \mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L}) \longrightarrow \mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L})\otimes_{\mathcal{O}_{\widetilde{X}}} \omega^1_{\widetilde{X}/\mathcal{O}}$$

for the connection induced by $\nabla_{\mathbb{L},\mathbb{W}(k)}$, then $\big(\mathbb{D}^{\mathrm{ar}}_{\mathrm{cris}}(\mathbb{L}),\nabla_{\mathbb{L},\mathcal{O}}\big)$ uniquely defines an isocrystal on $\big(X_0/\mathcal{O}_{\mathrm{cris}}\big)^{\mathrm{cris}}_{\log}$ in the sense of §3.4 and the local Frobenii define the structure of an F-isocrystal.

Proof. The first claim is proven in [AI, Prop. 2.26]. Claims (2)–(5) follow from [AI, Prop. 2.28]. Claim (6) follows from [AI, Cor. 2.29]. □

Concerning statement (6) of Proposition 4.5 we recall that absolute Frobenius on X_0 and the given Frobenius $\varphi_{\mathcal{O}}$ on $\mathcal{O}$ define a morphism of sites

$$F\colon \big(X_0/\mathcal{O}_{\mathrm{cris}}\big)^{\mathrm{cris}}_{\log} \longrightarrow \big(X_0/\mathcal{O}_{\mathrm{cris}}\big)^{\mathrm{cris}}_{\log}.$$

The category of *F-isocrystals* consist of pairs $(\mathcal{E},\varphi)$ where $\mathcal{E}$ is an isocrystal and $\varphi\colon F^*(\mathcal{E})\to\mathcal{E}$ is an isomorphism of isocrystals.

Cohomology of semistable sheaves: By construction we have an isomorphism

$$\alpha_{\log,\mathbb{L}}\colon \mathbb{D}^{\rm ar}_{\rm cris}\big(\mathbb{L}\big)\otimes_{\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{X}}}\mathbb{B}_{\rm log,K}\longrightarrow \mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{B}_{\log},$$

compatible with all extra structures (connections, local Frobenii, filtrations). It follows from [AI, §2.4.9] that there are isomorphsims

$$\mathrm{H}^i\left(X^{\rm ket}_{\overline{K}},\mathbb{L}\right)\otimes_{\mathbb{Q}_p} B_{\log}\cong \mathrm{H}^i\left(\mathfrak{X}_{\overline{K}},\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathbb{L}\big)\otimes_{\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{X}}}\mathbb{B}_{\rm log,K}\right)$$

and

$$\mathrm{H}^i\left(\mathfrak{X}_{\overline{K}},\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathbb{L}\big)\otimes_{\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{X}}}\mathbb{B}_{\rm log,K}\right)\cong \mathrm{H}^i\left(X,\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathbb{L}\big)\right)\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}.$$

Therefore one obtains the comparison isomorphism

$$\mathrm{H}^i\left(X^{\rm ket}_{\overline{K}},\mathbb{L}\right)\otimes_{\mathbb{Q}_p} B_{\log}\cong \mathrm{H}^i\left(X,\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathbb{L}\big)\right)\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}$$

as filtered $B_{\log}$-modules, compatible with derivations, Frobenius, and G_K-action.

5 Comparison of universal objects

As a consequence of Theorem 4.4 we immediately have the following

Corollary 5.1. *For every $n\in\mathbb{N}$ the universal étale object $\mathcal{E}^{\rm et}_n$ on $X^{\rm ket}_{\overline{K}}$, with its natural action of G_K, is semistable and*

$$\mathcal{E}^{\rm cris}_{n,\widetilde{X}}\cong \mathbb{D}^{\rm ar}_{\rm cris}\big(\mathcal{E}^{\rm et}_n\big)$$

as isocrystals on $\big(X_0/\mathcal{O}_{\rm cris}\big)^{\rm cris}_{\log}$, compatibly for varying n.

In particular, $e^{\rm crys}_n$ defines an element of $\widetilde{b}^*\big(\mathcal{E}^{\rm cris}_{n,\widetilde{X}}\big)$. Due to 4.5 we may complete the equation (5) to an isomorphism

$$\rho_n\colon \mathcal{E}^{\rm crys}_{n,b}\widehat{\otimes}_{\mathcal{O}_{\rm cris}}B_{\log}\cong b^*_{\log}\big(\mathbb{D}^{\rm geo}_{\rm cris}\big(\mathcal{E}^{\rm et}_n\big)\big)\cong \mathcal{E}^{\rm et}_{n,b}\otimes_{\mathbb{Q}_p}B_{\log}. \tag{6}$$

These are G_K-equivariant isomorphisms of $B_{\log}$-modules, compatible for varying n, commuting with Frobenius φ and by 4.4 the image of $e^{\rm cris}_n\otimes 1$ is $e^{\rm et}_n\otimes 1$. Here we write $\mathcal{E}^{\rm cris}_{n,b}:=\widetilde{b}^*\big(\mathcal{E}^{\rm cris}_{n,\widetilde{X}}\big)$ and $\mathcal{E}^{\rm et}_{n,b}:=b^*_{\overline{K}}\big(\mathcal{E}^{\rm et}_n\big)$.

Using 5.1 we get that $\big(\mathcal{E}^{\rm cris}_{n,\widetilde{X}},e^{\rm cris}_n\big)$ has further structure:

Theorem 5.2. *(i) The universal crystalline system $\{\mathcal{E}^{\rm cris}_n\}_n$ is endowed with a Frobenius morphism $\{\varphi_n\}_n$ making it an F-isocrystal and moreover $e^{\rm cris}_n$ is fixed by Frobenius. In particular, $\mathcal{E}^{\rm cris}_{n,b}$ is a free $\mathcal{O}_{\rm cris}\big[p^{-1}\big]$-module and Frobenius is étale;*

(ii.a) the connection on $\left\{\mathcal{E}^{\rm cris}_{n,\widetilde{X}}\right\}_n$ relative to $\mathcal{O}_{\rm cris}\big[p^{-1}\big]$ can be extended to an integrable, topologically nilpotent, log connection $\{\nabla_{n,\mathbb{W}}\}_n$ relative to $\mathbb{W}(k)\big[p^{-1}\big]$;

(ii.b) the connection $\nabla_{n,\mathbb{W}}$ induces a connection $\nabla_{n,b}$ on $\mathcal{E}^{\rm cris}_{n,b}$ such that Frobenius is horizontal and in (6) it is compatible with the connection on $\mathcal{E}^{\rm et}_{n,b}\otimes_{\mathbb{Q}_p} B_{\log}$ which is trivial on $\mathcal{E}^{\rm et}_{n,b}$ and is the connection on $B_{\log}$ defined in §2.1;

(iii.a) the $\mathcal{O}_{\widehat{X}_K}$-modules $\mathcal{E}^{\rm cris}_{n,\widetilde{X}}$ are endowed with decreasing, exhaustive filtrations $\mathrm{Fil}^\bullet\mathcal{E}^{\rm cris}_{n,\widetilde{X}}$, strictly compatible with the filtrations for varying n, $\nabla_{n,\mathbb{W}}$ satisfies Griffith's transversality with respect to the filtration. Moreover the graded quotients of the filtration are finite and projective $\mathcal{O}_{\widehat{X}_K}$-modules;

(iii.b) for each $n\geq 1$ the filtration at (iii.a) induces by pull-back a filtration on $\mathcal{E}^{\rm cris}_{n,b}$ such that via the isomorphism ρ_n in (6) is strictly compatible with the filtration on $\mathcal{E}^{\rm et}_{n,b}\otimes_{\mathbb{Q}_p} B_{\log}$ defined by: $\mathcal{E}^{\rm et}_{n,b}$ is endowed with the trivial filtration and the filtration of $B_{\log}$ is the one defined in §2.1.

Proof. (i) the first part has already been proven. The element $e^{\rm cris}_n$ is fixed by Frobenius as $e^{\rm et}_n\otimes 1$ is. The last statement follows as $\mathcal{E}^{\rm cris}_{n,\widetilde{X}}$ is a Frobenius isocrystal so that its pull back via $\widetilde{b}$ is also a Frobenius isocrystal.

(ii.a) follows directly from Proposition 4.5.

(ii.b) the horizontality of Frobenius follows from 4.5. The first isomorphism in (6) is compatible with the connections induced from $\nabla_{n,\mathbb{W}}$ and the given connection on $\mathbb{D}^{\rm geo}_{\rm cris}\big(\mathcal{E}^{\rm et}_n\big)$ by Proposition 4.5(i). The second isomorphism is compatible with the given connection on $\mathcal{E}^{\rm et}_{n,b}\otimes_{\mathbb{Q}_p} B_{\log}$ by the discussion following (6). In particular $e^{\rm crys}_n$ is horizontal as $e^{\rm et}_n\otimes 1$ is.

(iii.a) the claim, except the strict compatibility of the filtrations via the surjection $\mathcal{E}^{\rm cris}_{n+1,\widetilde{X}}\to\mathcal{E}^{\rm cris}_{n,\widetilde{X}}$, follows from Proposition 4.5. The compatibility of the filtrations via $\mathcal{E}^{\rm cris}_{n+1,\widetilde{X}}\to\mathcal{E}^{\rm cris}_{n,\widetilde{X}}$ follows from the functoriality of $\mathbb{D}^{\rm ar}_{\rm cris}$.

(iii.b) The compatibility of the filtrations follow immediately. In particular $e^{\rm cris}_n$ is in Fil^0 as $e^{\rm et}_n\otimes 1$ is.

We are left to prove the strict compatibility in (ii.a) and (ii.b). Fix (U,M_U) with $U=\mathrm{Spec}(R_U)$ a small object of $X^{\rm ket}$ as in §4.2. Write $V_i:=\mathcal{E}^{\rm et}_i(\overline{R}_U)$ for $i=n$ or $n+1$. The natural isomorphism

$$\alpha_U\colon \mathbb{D}^{\rm ar}_{\rm cris}\big(\mathcal{E}^{\rm et}_i\big)(U)\widehat{\otimes}_{\widetilde{R}_U^{DP}}\mathrm{B}_{\log}\big(\widetilde{R}_U\big)\longrightarrow V_i\otimes_{\mathbb{Q}_p}\mathrm{B}_{\log}\big(\widetilde{R}_U\big)$$

is strictly compatible with the filtrations due to [AI, Prop. 2.28(5)], considering on the RHS the composite of the trivial filtration $\mathrm{Fil}^0V=V$ and the given filtration on $B_{\log}\big(\widetilde{R}_U\big)$. If b factors via U, then ρ_n is obtained by pull-back of via $b^*_{\log}$ of α_U and (iii.b) follows.

We also deduce that the map

$$\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathcal{E}^{\rm et}_{n+1}\big)(U)\widehat{\otimes}_{\widetilde{R}_U^{DP}}B_{\log}\big(\widetilde{R}_U\big)\longrightarrow\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathcal{E}^{\rm et}_n\big)(U)\widehat{\otimes}_{\widetilde{R}_U^{DP}}B_{\log}\big(\widetilde{R}_U\big)$$

is strictly compatible with the filtrations, namely it induces a surjective map on the graded quotients. As

$$\mathrm{Gr}^h\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathcal{E}^{\rm et}_i\big)(U)\widehat{\otimes}_{\widetilde{R}_U^{DP}}\mathrm{B}_{\log}\big(\widetilde{R}_U\big)=\bigoplus_{a+b=h}\mathrm{Gr}^a\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathcal{E}^{\rm et}_i\big)(U)\widehat{\otimes}_{R_U\otimes K}\mathrm{Gr}^b\mathrm{B}_{\log}\big(\widetilde{R}_U\big)$$

by [AI, Prop. 3.29(4)] and $\mathrm{Gr}^b\mathrm{B}_{\log}\big(\widetilde{R}_U\big)$ is free $\widehat{\overline{R}}_U[p^{-1}]$-module by [AI, Prop. 3.15], we deduce that the surjection $\mathcal{E}^{\rm cris}_{n+1,\widetilde{X}}\to\mathcal{E}^{\rm cris}_{n,\widetilde{X}}$ induces a surjective map on the graded quotients, i.e., it is strictly compatible with the filtrations. This proves (iii.a). $\square$

Let $\mathcal{T}_n^{\mathrm{et}} := \mathrm{Ker}\big(\mathcal{E}_{n+1}^{\mathrm{et}} \to \mathcal{E}_n^{\mathrm{et}}\big)$ and $\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris}} := \mathrm{Ker}\big(\mathcal{E}_{n+1,\widetilde{X}}^{\mathrm{cris}} \to \mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris}}\big)$. Write $T_{n,b}^{\mathrm{et}} := b_{\overline{K}}^*\big(\mathcal{T}_n^{\mathrm{et}}\big)$. It is a finite dimensional representation of G_K. Set $\mathcal{T}_{n,b}^{\mathrm{cris}} := \widetilde{b}^*\big(\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris}}\big)$. It is a filtered $\mathcal{O}_{\mathrm{cris}}\big[p^{-1}\big]$-module, endowed with a filtration, a Frobenius linear operator φ and a logarithmic connection ∇ obtained by pull-back from $\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris}}$. Composing it with the derivation $Z\frac{\partial}{\partial Z}$ we get a derivation N. Also $B_{\log}^{G_K}$ is a filtered $\mathcal{O}_{\mathrm{cris}}\big[p^{-1}\big]$-module, endowed with a filtration, a Frobenius linear operator and a derivation. In particular, using the conventions of §2.1:

Corollary 5.3. *(i) The modules $\mathcal{E}_{n,b}^{\mathrm{cris}} \otimes_{\mathcal{O}_{\mathrm{cris}}} B_{\log}^{G_K}$ and $\mathcal{T}_{n,b}^{\mathrm{cris}} \otimes_{\mathcal{O}_{\mathrm{crys}}} B_{\log}^{G_K}$ endowed with a composite filtration, a composite Frobenius linear operator and a composite derivation define objects in the category $\mathcal{MF}_{B_{\log}^{G_K}}(\varphi, N)$.*

(iii) The G_K-representations $\mathcal{E}_{n,b}^{\mathrm{et}}$ and $\mathcal{T}_{n,b}^{\mathrm{et}}$ are semi-stable in the sense of Fontaine, and in particular $B_{\log}$-admissible and

$$D_{\log}\big(\mathcal{E}_{n,b}^{\mathrm{et}}\big) \cong \mathcal{E}_{n,b}^{\mathrm{cris}} \otimes_{\mathcal{O}_{\mathrm{cris}}} B_{\log}^{G_K}, \quad D_{\log}\big(\mathcal{T}_{n,b}^{\mathrm{et}}\big) \cong \mathcal{T}_{n,b}^{\mathrm{cris}} \otimes_{\mathcal{O}_{\mathrm{cris}}} B_{\log}^{G_K}$$

in $\mathcal{MF}_{B_{\log}^{G_K}}(\varphi, N)$.

Proof. (i) As $\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris}}$ is the kernel of a morphism of isocrystals, it is an isocrystal and the same holds for its pull-back $\mathcal{T}_{n,b}^{\mathrm{cris}}$ via $\widetilde{b}^*$. In particular, it is a free $\mathcal{O}_{\mathrm{cris}}\big[p^{-1}\big]$-module, φ is horizontal with respect to N and étale. As the connection of $\mathcal{E}_{n+1,\widetilde{X}}^{\mathrm{cris}}$ satisfies Griffiths transversality, the induced connection on $\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris}}$ does as well with respect to the induced filtration and hence also the pull-back connection on $\mathcal{T}_{n,b}^{\mathrm{cris}}$ satisfies Griffith transversality with respect to the pull-back filtration. Hence, the axioms for $\mathcal{T}_{n,b}^{\mathrm{cris}} \otimes_{\mathcal{O}_{\mathrm{crys}}} B_{\log}^{G_K}$ to be in $\mathcal{MF}_{B_{\log}^{G_K}}(\varphi, N)$ hold. For $\mathcal{E}_{n,b}^{\mathrm{cris}} \otimes_{\mathcal{O}_{\mathrm{cris}}} B_{\log}^{G_K}$ this follows from 5.2.

(ii) As $\mathcal{T}_n^{\mathrm{et}}$ is constant, we have that $\mathcal{T}_{n,b}^{\mathrm{et}} \cong T_n^{\mathrm{et}}$ as representations of G_K. The latter is semistable thanks to Propositions 2.1 and 3.7. It follows from Theorem 5.2 that the natural map

$$\big(\mathcal{T}_{n,b}^{\mathrm{cris}} \otimes_{\mathcal{O}_{\mathrm{cris}}} B_{\log}^{G_K}\big) \otimes_{B_{\log}^{G_K}} B_{\log} \longrightarrow \mathcal{T}_{n,b}^{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\log},$$

defined in (6), is an isomorphism of filtered $B_{\log}$-modules and it is G_K-equivariant and compatible with connections on the two sides. Hence, $D_{\log}\big(\mathcal{T}_{n,b}^{\mathrm{et}}\big) := \big(\mathcal{T}_{n,b}^{\mathrm{et}} \otimes_{\mathbb{Q}_p} B_{\log}\big)^{G_K}$ coincides with $\mathcal{T}_{n,b}^{\mathrm{cris}} \otimes_{\mathcal{O}_{\mathrm{cris}}} B_{\log}^{G_K}$ (as filtered $B_{\log}^{G_K}$-modules, strictly compatible with filtrations and compatible with connections). The second claim follows. The statement for $\mathcal{E}_{n,b}^{\mathrm{et}}$ follows directly from (6) and Theorem 5.2.

□

We know from Lemma 3.8 that the base-change of the universal pointed de Rham object $\big(\mathcal{E}_n^{\mathrm{dR}}, e_n^{\mathrm{dR}}\big)$ of index $\leq n$ on X_K is the algebrization of the base-change of the universal pointed crystalline object $\big(\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris}}, e_n^{\mathrm{cris}}\big)$ via the map $\mathcal{O}_{\mathrm{cris}}\big[p^{-1}\big] \to K$, $Z \mapsto \pi$, as module with connection on $\widehat{X}_K$, with a section of the pull-back via $\widehat{b}_K$. Let $\mathrm{Fil}^\bullet \mathcal{E}_n^{\mathrm{dR}}$ be the image of the filtration on $\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris}}$. It is called the *Hodge filtration.* We write $\mathcal{E}_{n,b}^{\mathrm{dR}}$ for the K-vector space $b_K^*\big(\mathcal{E}_n^{\mathrm{dR}}\big)$ with induced pull-back filtration.

Corollary 5.4. *(1) For every $n \geq 1$ the filtration $\mathrm{Fil}^\bullet \mathcal{E}_n^{\mathrm{dR}}$ is decreasing and exhaustive with quotients which are locally free $\mathcal{O}_{X_K}$-modules;*

(2) the connection satisfies Griffiths' transversality with respect to the filtration;

(3) the maps $\mathcal{E}_{n+1}^{\mathrm{dR}} \to \mathcal{E}_n^{\mathrm{dR}}$ are strictly compatible with respect to the filtrations;

(4) the filtration on $\mathcal{E}_{n,b}^{\mathrm{dR}}$ coincides with the image of the filtration on $\mathcal{E}_{n,b}^{\mathrm{crys}}$ via the isomorphism

$$t_n \colon \mathcal{E}_{n,b}^{\mathrm{cris}} \otimes_{\mathcal{O}_{\mathrm{cris}}} K \longrightarrow \mathcal{E}_{n,b}^{\mathrm{dR}}.$$

In particular $e_n^{\mathrm{dR}} = t_n\big(e_n^{\mathrm{cris}}\big)$ is in Fil^0.

Proof. Claims (1), (2) and (3) follow from Theorem 5.2 and [AI, Cor. 2.29] where the behaviour of the filtration on $\mathbb{D}_{\mathrm{cris}}^{\mathrm{ar}}(\mathbb{L})$ under base change via $\mathcal{O}_{\mathrm{crys}} \to K$ is studied. The map t_n is compatible and surjective on filtrations by construction. As e_n^{dR} corresponds via t_n to e_n^{cris} by 3.8 it follows from theorem 5.2 that it lies in Fil^0. □

As $\mathcal{T}_n^{\mathrm{dR}} := \mathrm{Ker}\big(\mathcal{E}_{n+1}^{\mathrm{dR}} \to \mathcal{E}_n^{\mathrm{dR}}\big)$ is constant, then $\mathcal{T}_n^{\mathrm{dR}} \cong \mathcal{T}_{n,b}^{\mathrm{dR}} \otimes_K \mathbf{1}$ with $\mathcal{T}_{n,b}^{\mathrm{dR}} := b_K^*\big(\mathcal{T}_n^{\mathrm{dR}}\big)$. By definition the Hodge filtration on $\mathcal{T}_n^{\mathrm{dR}}$ is the image of the filtration on $\mathcal{T}_{n,b}^{\mathrm{crys}}$ and it is uniquely determined by the filtration on $\mathcal{T}_{n,b}^{\mathrm{dR}}$. By Corollary 5.4(4) the latter coincides with the image of the filtration on $\mathcal{T}_{n,b}^{\mathrm{cris}}$ via t_n. On the other hand, as the G_K-representation $\mathcal{T}_{n,b}^{\mathrm{et}}$ is semistable by 5.3, following Fontaine we can associate to it a filtered (φ, N)-module $D_{\mathrm{st}}\big(T_{n,b}^{\mathrm{et}}\big)$: it is a K_0-vector space with Frobenius and monodromy operator and a filtration on $D_{\mathrm{st}}\big(T_{n,b}^{\mathrm{et}}\big) \otimes_{K_0} K$. It follows from 2.1 and 5.3 that $D_{\mathrm{st}}\big(\mathcal{E}_{n,b}^{\mathrm{et}}\big) \otimes_{K_0} K = \mathcal{T}_{n,b}^{\mathrm{dR}}$ and $D_{\mathrm{st}}\big(\mathcal{E}_{n,b}^{\mathrm{et}}\big) \otimes_{K_0} K = \mathcal{T}_{n,b}^{\mathrm{dR}}$ as K-vector space. Then,

Lemma 5.5. *(i) We have $\mathcal{E}_{n,b}^{\mathrm{dR}} = D_{\mathrm{st}}\big(\mathcal{E}_{n,b}^{\mathrm{et}}\big) \otimes_{K_0} K$ and $\mathcal{T}_{n,b}^{\mathrm{dR}} = D_{\mathrm{st}}\big(\mathcal{T}_{n,b}^{\mathrm{et}}\big) \otimes_{K_0} K$ as filtered K-vector spaces;*

(ii) There is a unique filtration on the system $\big(\mathcal{E}_n^{\mathrm{dR}}\big)_n$ inducing the filtration on $\mathcal{T}_{n,b}^{\mathrm{dR}}$ provided by the identification with obtained $D_{\mathrm{st}}\big(\mathcal{T}_{n,b}^{\mathrm{et}}\big) \otimes_{K_0} K$ and such that properties (1)–(3) of Corollary 5.4 hold.

Proof. Claim (i) follows from 2.1, Corollary 5.3 and Corollary 5.4. For claim (ii) one argues as in [Ha, Prop. 3.3& Lemma 3.6]. □

6 Proofs of the Theorems in section §1

6.1 The proof of Theorems 1.7 and 1.8

Write $G^{\mathrm{et}}\big((X_{\overline{K}}, N_{\overline{K}}), b_{\overline{K}}\big) := \mathrm{Spec}\big(A_\infty^{\mathrm{et},\vee}\big)$, set $G^{\mathrm{crys}}\big((X, N), \widetilde{b}\big) := \mathrm{Spec}\big(A_\infty^{\mathrm{crys},\vee}\big)$ and finally denote $G^{\mathrm{dR}}\big((X_K, N_K), b_K\big) := \mathrm{Spec}\big(A_\infty^{\mathrm{dR},\vee}\big)$ as in §3.7 using the universal systems $\big(\mathcal{E}_n^{\mathrm{et}}, e_n^{\mathrm{et}}\big)_n$, $\big(\mathcal{E}_n^{\mathrm{crys}}, e_n^{\mathrm{crys}}\big)_n$ and $\big(\mathcal{E}_n^{\mathrm{dR}}, e_n^{\mathrm{dR}}\big)_n$ respectively. Then, Corollary 5.3 and Lemma 5.5 imply that

(i) $A_\infty^{\mathrm{et},\vee} = \lim\limits_{n\to\infty} \mathcal{E}_{n,b}^{\mathrm{et},\vee}$ is endowed with an action of G_K such that each $\mathcal{E}_{n,b}^{\mathrm{et}}$ is a semistable or equivalently a $B_{\log}$-admissible representation of G_K (in the sense of §2.1);

(ii) $A_\infty^{\mathrm{cris},\vee}=\lim_{n\to\infty}\mathcal{E}_{n,b}^{\mathrm{cris},\vee}$ and $\mathcal{E}_{n,b}^{\mathrm{crys},\vee}\widehat{\otimes}_{\mathcal{O}_{\mathrm{cris}}}B_{\log}^{G_K}$ is an object in $\mathcal{MF}_{B_{\log}^{G_K}}(\varphi,N)$ (in the sense of §2.1) isomorphic to $D_{\log}\big(\mathcal{E}_{n,b}^{\mathrm{et},\vee}\big)$ for every n;

(iii) $A_\infty^{\mathrm{dR},\vee}=\lim_{n\to\infty}\mathcal{E}_{n,b}^{\mathrm{dR},\vee}$, each $\mathcal{E}_{n,b}^{\mathrm{dR},\vee}$ is a filtered K-vector space, identified with $\mathcal{E}_{n,b}^{\mathrm{cris},\vee}\otimes_{\mathcal{O}_{\mathrm{cris}}}K=D_{\mathrm{st}}\big(\mathcal{E}_{n,b}^{\mathrm{et},\vee}\big)\otimes_{K_0}K$ (as filtered K-vector spaces).

We obtain identifications

$$A_\infty^{\mathrm{cris},\vee}\otimes_{\mathcal{O}_{\mathrm{cris}}}B_{\log}^{G_K}\cong D_{\log}\big(\mathcal{E}_{n,b}^{\mathrm{et},\vee}\big)$$

of $B_{\log}^{G_K}$-modules, compatibly with Frobenius, monodromy operators N and strictly compatibly with the filtrations and

$$A_\infty^{\mathrm{dR},\vee}\cong A_\infty^{\mathrm{cris},\vee}\otimes_{\mathcal{O}_{\mathrm{cris}}}K\cong D_{\mathrm{st}}\big(A_\infty^{\mathrm{et},\vee}\big)\otimes_{K_0}K$$

as filtered K-vector spaces.

The correspondence as universal objects between $\mathcal{E}_{n,b}^{\mathrm{et}}$, $\mathcal{E}_{n,b}^{\mathrm{cris}}$ and $\mathcal{E}_{n,b}^{\mathrm{dR}}$, of the sections e_n^{et}, e_n^{cris} and e_n^{dR} proven in Theorem 5.2 and in Corollary 5.4 and the compatibilities for varying n imply that the isomorphisms displayed above respect the structures as Hopf algebras over $B_{\log}^{G_K}$ (resp. Hopf K-algebras for the second one). This proves Theorem 1.7 and, using **??**, also Theorem 1.8 except for Theorem 1.7(ii) and Theorem 1.8(iv). These claims follow from (5.1) and the discussion after Proposition 4.5.

6.2 The proof of Theorem 1.9

First of all we characterize the integrable log connection $\{\nabla_{n,\mathbb{W}}\}_n$ on $\Big\{\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris}}\Big\}_n$ relative to $\mathbb{W}(k)\big[p^{-1}\big]$ extending the universal one relative to $\mathcal{O}_{\mathrm{cris}}\big[p^{-1}\big]$ provided by Theorem 5.2.

Consider the exact sequence

$$0\longrightarrow\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris}}\longrightarrow\mathcal{E}_{n+1,\widetilde{X}}^{\mathrm{cris}}\longrightarrow\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris}}\longrightarrow 0.$$

The compatibility of $\nabla_{n+1,\mathbb{W}}$ and $\nabla_{n,\mathbb{W}}$ provide an integrable connection on $\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris}}$ relative to $\mathbb{W}(k)$. Then,

Proposition 6.1. *(1) The logarithmic connection on $\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris}}$ relative to $\mathbb{W}(k)$ described above is the unique one for which*

a) $\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris}}$ is constant, namely the tensor product of an $\mathcal{O}_{\mathrm{cris}}[p^{-1}]$-module $T_{n,\widetilde{X}}^{\mathrm{cris}}$ with log connection with $\mathbf{1}$ with the standard derivation;

b) the induced map

$$T_{n,\widetilde{X}}^{\mathrm{cris}}=\mathbb{H}_{\mathrm{dR}}^0\big(X_0/\mathcal{O}_{\mathrm{cris}},\mathcal{T}_{n,\widetilde{X}}^{\mathrm{cris},\vee}\big)\longrightarrow\mathbb{H}_{\mathrm{dR}}^1\big(X_0/\mathcal{O}_{\mathrm{cris}},\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris},\vee}\big),$$

which we know to be an isomorphism by the discussion in §3.5, is compatible with respect to the induced Gauss–Manin connections considering $\nabla_{n,\mathbb{W}}$ on $\mathcal{E}_{n,\widetilde{X}}^{\mathrm{cris}}$.

(2) Given the connection $\nabla_{n,\mathbb{W}}$ on $\mathcal{E}^{\rm crys}_{n,\widetilde{X}}$ and the connection on $\mathcal{T}^{\rm cris}_{n,\widetilde{X}}$ relative to $\mathbb{W}(k)$ described in (1), then $\nabla_{n+1,\mathbb{W}}$ is the unique connection on $\mathcal{E}^{\rm cris}_{n+1,\widetilde{X}}$ which is compatible with the two and with the universal one relative to $\mathcal{O}_{\rm cris}\big[p^{-1}\big]$ and such that $e^{\rm cris}_{n+1}$ is horizontal for the induced connection on $\mathcal{E}^{\rm cris}_{n+1,b}$.

Proof. (1) The uniqueness is clear. The object $\mathcal{T}^{\rm cris}_{n,\widetilde{X}}$ is constant for the connection relative to $\mathbb{W}(k)$ as it coincides with $\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathcal{T}^{\rm et}_n\big)$ by 5.2, $\mathcal{T}^{\rm et}_n$ is constant and $\mathbb{D}^{\rm ar}_{\rm cris}\big(\mathcal{T}^{\rm et}_n\big)$ commutes with tensor product.

The given connection on $\mathcal{T}^{\rm cris}_{n,\widetilde{X}}$ satisfies also property (b) taking the long exact sequence of de Rham cohomology groups associated to the dual of the short exact sequence displayed before the lemma.

(2) Suppose we have two connections $\nabla_{n+1,\mathbb{W}}$ and $\nabla'_{n+1,\mathbb{W}}$ with the properties in (2). Their difference provides a homomorphism

$$\mathcal{E}^{\rm cris}_{n,\widetilde{X}} \longrightarrow \mathcal{T}^{\rm cris}_{n,\widetilde{X}} \otimes_{\mathcal{O}_{\rm cris}} \omega^1_{\mathcal{O}_{\rm cris}/\mathbb{W}(k)}.$$

Such morphism must factor via $\mathcal{E}^{\rm cris}_{1,\widetilde{X}} = \mathbf{1}$ by the discussion in §3.5 and it is determined by the image of $1 = e^{\rm cris}_1$ in the pull back via $\widetilde{b}$. As $e^{\rm cris}_1$ is the image of $e^{\rm cris}_{n+1}$ it must be zero. □

Let $\iota_0\colon \mathcal{O}_{\rm crys} \to R := \mathbb{W}(k)\llbracket x\rrbracket$ be the unique morphism of $\mathbb{W}(k)$-algebras sending $Z \to px$. Thanks to Lemma 3.8 the base change of $\big(\mathcal{E}^{\rm cris}_{n,\widetilde{X}}, e^{\rm cris}_n\big)$, as $\widehat{\mathcal{O}}^{\rm DP}_{\widetilde{X}}$-module with connection and with section $e^{\rm cris}_n$, via ι_R is isomorphic, as a pointed module with connection on $\widehat{X}_K$, to the module with connection associated to the universal pointed de Rham object $\big(\mathcal{E}^{\rm dR}_{n,R}, e^{\rm dR}_{n,R}\big)$ of index $\leq n$ on X_R (with log poles along N_R).

As the base change $R\big[p^{-1}\big] \to R' := K_0\llbracket x\rrbracket$ is flat, the base change of $\big(\mathcal{E}^{\rm dR}_{n,R}, e^{\rm dR}_{n,R}\big)$ via the map $R \to R'$ is the universal object of $\big(\mathcal{E}^{\rm dR}_{n,R'}, e^{\rm dR}_{n,R'}\big)$ index $\leq n$ on $X_{R'} := X_R \otimes_R R'$ (with log poles along $N_{R'}$). The section $\widetilde{b}$ defines a section $b_R \in X_R(R)$ and by further base change a section $b_{R'} \in X_{R'}(R')$. Write $\mathcal{E}^{\rm dR}_{n,R',b}$ for $b^*_{R'}\big(\mathcal{E}^{\rm dR}_{n,R'}\big)$.

Lemma 6.2. *(i) The connections on $\{\mathcal{E}^{\rm dR}_{n,R'}\}_n$ uniquely extend to a compatible, log, integrable connection relative to K_0 satisfying the requirements of Proposition 6.1(2). In particular, $\mathcal{E}^{\rm dR}_{n,R',b}$ is endowed with an integrable logarithmic connection $\nabla_{n,R',b}$ relative to R', considering on R' the logarithmic structure defined by x.*

(ii) We have isomorphisms $D_{\rm st}\big(\mathcal{E}^{\rm et}_{n,b}\big) \cong \mathcal{E}^{\rm dR}_{n,R',b}/x\mathcal{E}^{\rm dR}_{n,R'}$, compatible for varying n, as K_0-vector spaces endowed with nilpotent operators, where on the LHS we consider the monodromy operator and on the RHS we consider the residue of $\nabla_{n,R',b}$ at $x = 0$.

Proof. (i) The claimed extension follows from 5.2 by base change to R'. The uniqueness follows arguing as in the proof of Proposition 6.1(2).

(ii) By construction $\mathcal{E}^{\rm dR}_{n,R',b}/x\mathcal{E}^{\rm dR}_{n,R'}$ is equal to the base change of $\mathcal{E}^{\rm crys}_{n,b}$ via the morphism $\mathcal{O}_{\rm crys} \to K_0$ sending $Z \to 0$ and the residues of the connections on these two K_0-vector spaces coincide. The second claim follows then from Corollary 5.3 and the description of the monodromy operator on $D_{\rm st}\big(\mathcal{E}^{\rm et}_{n,b}\big)$ starting from the connection on $D_{\rm log}\big(\mathcal{E}^{\rm et}_{n,b}\big)$ provided by Proposition 2.1. □

As the special fiber of $X_{R'}$ at $x=0$ has the same dual graph as the special fiber of X in order to prove Theorem 1.9 we are left to show that:

Proposition 6.3. *The special fiber $\overline{X}'_R$ of $X_{R'}$ at $x=0$ has good reduction if and only if the residue of the connection on $\mathcal{E}^{\mathrm{dR}}_{n,R',b}$ is 0 for every $n \in \mathbb{N}$.*

Proof. To check this we may replace R' with the base change $K_0\llbracket x\rrbracket \to \mathbb{C}\llbracket x\rrbracket$ obtained by any field homomorphism $K_0 \to \mathbb{C}$. Notice that as such morphism is faithfully flat, the formation of universal de Rham pointed object $\big(\mathcal{E}^{\mathrm{dR}}_{n,R'}, e^{\mathrm{dR}}_n\big)$ commutes with base change. We may then assume that $R' = \mathbb{C}\llbracket x\rrbracket$. As the choice of $\widetilde{X}$ is auxiliary we may proceed as in [O, §2.3] and choose $\widetilde{X} \to \mathrm{Spf}(\mathcal{O})$ as arising from the completion of a 1-dimensional quotient of the henselization of the moduli stack of genus g curves at the k-valued point defined by the class of the curve $[X_k]$. In particular, we may assume that there exists a smooth irreducible affine curve $U = \mathrm{Spec}(A)$ over $\mathbb{C}$, a semistable genus g curve X_A over U, a section $b_A \in X_U(U)$ and a point κ of U such that R' is the completion of A at κ, X_A is smooth over $U\backslash\{\kappa\}$ and $\big(X_{R'}, b_{R'}\big)$ is isomorphic to the base-change of $\big(X_A, b_A\big)$ via $A \to R'$.

We denote by $X_A^{\mathrm{an}} \to U^{\mathrm{an}}$ the associated morphism of complex analytic spaces and by $\iota\colon X_A^{\mathrm{an}} \to X_A$ the associated map of ringed spaces. Then, arguing as in Lemma 3.8 we have that

(i) The base change of the pointed universal de Rham system $\big(\mathcal{E}^{\mathrm{dR}}_{n,A}, e^{\mathrm{dR}}_{n,A}\big)_n$ on X_A via $A \to R'$ is the pointed universal de Rham system $\big(\mathcal{E}^{\mathrm{dR}}_{n,R'}, e^{\mathrm{dR}}_n\big)_n$;

(ii) The pull back of the pointed universal de Rham system $\big(\mathcal{E}^{\mathrm{dR}}_{n,A}, e^{\mathrm{dR}}_{n,A}\big)_n$ on X_A via ι is the universal system $\big(\mathcal{E}^{\mathrm{dR}}_{n,U^{\mathrm{an}}}, e^{\mathrm{dR}}_{n,U^{\mathrm{an}}}\big)_n$ on X_A^{an}.

The Betti realization: On the analytic side we also have a pointed universal Betti system $\big(\mathcal{E}^{\mathrm{Be}}_{n,U^{\mathrm{an}}}, e^{\mathrm{Be}}_{n,U^{\mathrm{an}}}\big)_n$ in the category of unipotent local systems in finite dimensional $\mathbb{C}$-vector spaces on $X_A^{\mathrm{an,o}} := X_A^{\mathrm{an}}\backslash X_\kappa^{\mathrm{an}}$. Then $\mathcal{E}^{\mathrm{Be}}_{n,U^{\mathrm{an}}} \otimes_{\mathbb{C}} \mathcal{O}_{X_A^{\mathrm{an,o}}}$ defines a locally free $\mathcal{O}_{X_A^{\mathrm{an,o}}}$-module with connection defined by the derivation on $\mathcal{O}_{X_A^{\mathrm{an,o}}}$ and by requiring that $\mathcal{E}^{\mathrm{Be}}_{n,U^{\mathrm{an}}}$ are horizontal elements. By universality we get unique morphisms

$$\mathcal{E}^{\mathrm{dR}}_{n,U^{\mathrm{an}}}|_{X_A^{\mathrm{an,o}}} \longrightarrow \mathcal{E}^{\mathrm{Be}}_{n,U^{\mathrm{an}}} \otimes_{\mathbb{C}} \mathcal{O}_{X_A^{\mathrm{an,o}}}, \quad e^{\mathrm{dR}}_{n,U^{\mathrm{an}}} \mapsto e^{\mathrm{Be}}_{n,U^{\mathrm{an}}} \otimes 1.$$

They are compatible for varying n and they are isomorphisms. This description provides $\mathcal{E}^{\mathrm{dR}}_{n,U^{\mathrm{an,o}}}$ with a unique integrable connection relative to $\mathbb{C}$, extending the universal one relative to $U^{\mathrm{an,o}}$, such that $\mathcal{E}^{\mathrm{Be}}_{n,U^{\mathrm{an}}}$ is the set of solutions. Setting $\mathcal{E}^{\mathrm{an,dR}}_{n,b} := b_A^*\big(\mathcal{E}^{\mathrm{dR}}_{n,U^{\mathrm{an}}}\big)$ and $\mathcal{E}^{\mathrm{Be}}_{n,b} = b_A^*\big(\mathcal{E}^{\mathrm{Be}}_{n,U^{\mathrm{an}}}\big)$, we conclude that $\big\{\mathcal{E}^{\mathrm{an,dR}}_{n,b}\big\}_n$ is endowed with a connection with log poles at k and that

$$\mathcal{E}^{\mathrm{an,dR}}_{n,b}|_{U^{\mathrm{an,o}}} \cong \mathcal{E}^{\mathrm{Be}}_{n,b} \otimes_{\mathbb{C}} \mathcal{O}_{U^{\mathrm{an,o}}}, \quad e^{\mathrm{Be}}_{n,U^{\mathrm{an}}} \otimes 1 \mapsto e^{\mathrm{dR}}_{n,U^{\mathrm{an}}},$$

as $\mathcal{O}_{U^{\mathrm{an}}\backslash\{\kappa\}}$-modules with connection, compatibly for varying n. Here, $U^{\mathrm{an,o}} := U^{\mathrm{an}}\backslash\{\kappa\}$ and the connection on $\mathcal{E}^{\mathrm{Be}}_{n,b} \otimes_{\mathbb{C}} \mathcal{O}_{U^{\mathrm{an,o}}}$ has $\mathcal{E}^{\mathrm{Be}}_{n,b}$ as horizontal sections and is the standard derivative on $\mathcal{O}_{U^{\mathrm{an,o}}}$. Note that we also get an action of $\pi_1\big(U^{\mathrm{an,o}}, b_A\big)$ on $\mathcal{E}^{\mathrm{Be}}_{n,b}$ and hence on $\mathcal{E}^{\mathrm{an,dR}}_{n,b}|_{U^{\mathrm{an,o}}}$.

Topological vs algebraic monodromy: Let $X_A^o := X_A\backslash X_\kappa$ and $U^o := U\backslash\{\kappa\}$. Arguing as in the proof of Proposition 6.1(2), one shows by induction on n that the problem of extending the universal connection on $\big(\mathcal{E}^{\mathrm{dR}}_{n,A}, e^{\mathrm{dR}}_{n,A}\big)_n$ on X_A^o relative to U^o to a connection relative to $\mathbb{C}$ lives in the $\mathrm{H}^1\big(X_A^o, \mathcal{E}^{\mathrm{dR}}_{n,A}\mathcal{T}^{\mathrm{dR},\vee}_{n,A} \otimes_A \omega^{1,\vee}_{U^o/\mathbb{C}}\big)$. As such an extensions exists over $X_A^{\mathrm{an,o}}$, the pull back of such an obstruction via $\iota\colon X_A^{\mathrm{an}} \to X_A$ vanishes and, hence, it is zero. It follows from Proposition 6.1 that its base change via $A \to R'$ is the extension of the universal connection on $X_{R'}^o := X_{R'} \backslash X_\kappa$ relative to R' to a connection relative to $\mathbb{C}$. Putting everything together and base-changing to $R' = \mathbb{C}\llbracket x \rrbracket$, identified with the completed local ring of U and $U^{\mathrm{an,o}}$ at κ, we obtain isomorphisms of $R'\left[x^{-1}\right]$-modules, compatible with connections relative to $\mathbb{C}$ and compatible for varying n:

$$\mathcal{E}_{n,b}^{\mathrm{dR}} \otimes_{R'} R'\left[x^{-1}\right] \cong \mathcal{E}_{n,b}^{\mathrm{an,dR}} \otimes_{\mathcal{O}_{U^{\mathrm{an,o}}}} R'\left[x^{-1}\right] \cong \mathcal{E}_{n,b}^{\mathrm{Be}} \otimes_{\mathbb{C}} R'\left[x^{-1}\right],$$

where on the RHS we consider the unique connection relative to $\mathbb{C}$ having $\mathcal{E}_{n,b}^{\mathrm{Be}}$ as set of solutions. Note that we also get an action of the fundamental group $I_\kappa \cong \mathbb{Z}$ of a punctured disk of $U^{\mathrm{an,o}}$ with center in κ on $\mathcal{E}_{n,b}^{\mathrm{Be}}$ and that by construction the connection on $\mathcal{E}_{n,b}^{\mathrm{dR}} \otimes_{R'} R'\left[x^{-1}\right]$ extends to a logarithmic connection $\nabla_n^{\log}$ on $\mathcal{E}_{n,b}^{\mathrm{dR}}$.

It follows from [D1] that $\nabla_n^{\log}$ is regular if and only if its residue at $x = 0$ is trivial if and only if the action of I_κ on $\mathcal{E}_{n,b}^{\mathrm{Be}}$ is trivial. We are then reduced to prove that $\overline{X}_k$ is smooth if and only if the action of I_κ on $\mathcal{E}_{n,b}^{\mathrm{Be}}$ is trivial for every $n \in \mathbb{N}$.

If $\overline{X}_k$ is smooth then $\mathcal{E}_{n,U^{\mathrm{an}}}^{\mathrm{Be}}$ is a local system on the whole of X_A^{an} and $\mathcal{E}_{n,b}^{\mathrm{Be}}$ extends to a representation of the fundamental group of U^{an}. It follows that the action of I_κ is trivial.

Viceversa, if X_κ is singular, it follows from [O, Prop. 1.10] that the action of I_t on $\mathcal{E}_{n,b}^{\mathrm{Be}}$ is non trivial for n large enough. The conclusion follows. $\square$

7 Appendix. A simplicial lemma

In this appendix we prove a technical lemma used in the proof of Proposition 3.4.

Let R be a $\mathbb{Q}$-algebra and H a free R-module of rank $2g \geq 4$. We write H^n for the n-fold tensor product $H \otimes_R \cdots \otimes_R H$. Suppose we have a perfect, alternating pairing $gamma_1\colon H \otimes_R H \to R$. For every $n \in \mathbb{N}$ define inductively R-submodules $\iota_n\colon R^n \hookrightarrow H^n$ as follows: $R^1 := H$, $R^2 = \mathrm{Ker}\big(\gamma_1\big)$. For $n \geq 2$ set

$$\gamma_n\colon R^n \otimes_R H \overset{\iota_{n-1}\otimes 1}{\longrightarrow} R^{n-1} \otimes_R H \otimes_R H \overset{1\otimes\gamma_1}{\longrightarrow} R^{n-1},$$

put $R^{n+1} = \mathrm{Ker}\left(\gamma_n\right)$ and define $\iota_n\colon R^{n+1} \hookrightarrow R^n \otimes_R H \hookrightarrow H^{n+1}$ to be the natural inclusion.

We use γ_1 to define an isomorphism $H \cong H^\vee$. In particular

$$H \otimes_R H \cong H^\vee \otimes_R H \cong \mathrm{Hom}_R\big(H, H\big).$$

Write $\Delta \in H \otimes_R H$ for the image of $\mathrm{Id}\colon H \to H$. For every integer $n \geq 2$ and every integer $1 \leq i \leq n-1$ we define the following R-linear operators:

$$r_i^{(n)}\colon H^n \to H^{n-2}, v_1 \otimes \cdots \otimes v_n \mapsto \big(v_i \cup v_{i+1}\big) \cdot \otimes v_1 \otimes \cdots v_{i-1} \otimes v_{i+2} \otimes \cdots \otimes v_n$$

and

$$s_i^{(n)}\colon H^{n-2} \to H^n, v_1 \otimes \cdots \otimes v_{n-2} \mapsto v_1 \otimes \cdots \otimes v_{i-1} \otimes \Delta \otimes v_i \otimes \cdots \otimes v_n.$$

We have that

i. $r_i^{(n)} \circ s_i^{(n)}\colon H^n \to H^n$ coincides with $2g \cdot \mathrm{Id}$ for every $1 \leq i \leq n-1$.

ii. $r_{i+1}^{(n)} \circ s_i^{(n)} \colon H^n \to H^n$ is the identity map for every $1 \leq i \leq n-2$.

iii. $r_{i-1}^{(n)} \circ s_i^{(n)} \colon H^n \to H^n$ is the identity map for every $2 \leq i \leq n-1$.

iv. $r_j^{(n)} \circ s_i^{(n)} = s_i^{(n-4)} \circ r_j^{(n-2)}$ for every $1 \leq j \leq i-2 \leq n-3$.

v. $r_j^{(n)} \circ s_i^{(n)} = s_i^{(n-4)} \circ r_{j-2}^{(n-2)}$ for every $1 \leq i+2 \leq j \leq n-1$.

The first property follows as $r_i^{(n)} \circ s_i^{(n)}$ is simply multiplication by $\gamma_1(\Delta)$. If $\{\alpha_i\}_i$ is a basis for H and $\{\beta_j\}$ is the dual basis with respect to γ_1, then $\Delta = \sum_i \alpha_i \otimes \beta_i$ and $\gamma_1(\Delta) = \sum_i \gamma_1(\alpha_i \otimes \beta_i) = \sum_i 1 = 2g$.

The second property follows if we show that for every $v \in H$ we have $\sum_i \alpha_i \cdot \gamma_1(\beta_i \otimes v) = v$. By linearity this can be verified for $v = \alpha_h$ an element of the basis in which case $\gamma_1(\beta_i \otimes v) = \gamma_1(\beta_i \otimes \alpha_h)$ is 0 for $i \neq h$ and is 1 for $i = h$. Then $\sum_i \alpha_i \cdot \gamma_1(\beta_i \otimes \alpha_h) = \alpha_h$. Similarly the third property follows by showing that for every $v \in H$ we have $v = \sum_i \gamma_1(v \otimes \alpha_i) \otimes \beta_i$. The fourth and fifth properties are readily verified.

Claim: $R^n = \cap_{i=1}^{n-1} \mathrm{Ker}(r_i^{(n)})$ for $n \geq 2$.

This is verified by induction on n. For $n = 2$ it is clear as $r_1^{(1)} = \gamma_1$. Using the inductive hypothesis for n, as H is a flat R-module then $R^n \otimes_R H$ is the intersection of the kernels $\cap_{i=1}^{n-1} \mathrm{Ker}(r_i^{(n)} \otimes 1) = \cap_{i=1}^{n-1} \mathrm{Ker}(r_i^{(n+1)})$. Here we use that $r_i^{(n)} \otimes 1 \colon H^{n+1} \cong H^n \otimes_R H \to H^{n-2} \otimes_R H \cong H^{n-1}$ is $r_i^{(n+1)}$. By definition $R^{n+1} \subset R^n \otimes_R H$ is the kernel of γ_n which is $r_n^{(n+1)}$ restricted to $R^n \otimes_R H$ and thus it coincides with $\cap_{i=1}^{n} \mathrm{Ker}(r_i^{(n+1)})$. This proves the claim.

We are now ready to prove the following

Lemma 7.1. *The map γ_n is surjective for every $n \in \mathbb{N}$.*

Proof. For $n = 1$ it is trivial. Fix $n \geq 2$ and $\gamma \in R^{n-1} \subset H^{n-1}$. Then, set $\xi_i := s_i^{(n+1)}(\gamma)$ for $i = 1, \ldots, n$. Write $\xi := \sum_{i=1}^n a_i \xi_i$ with a_i integers defined as follows: $a_1 = 1$, $a_2 + 2ga_1 = 0$ and $a_j + 2ga_{j-1} + a_{j-2} = 0$ for every $3 \leq j \leq n$. Since $\gamma \in R^{n-1}$ we have $r_s^{(n-1)}(\gamma) = 0$ for every $1 \leq s \leq n-2$. It then follows from properties (iv) and (v) above, setting $a_i = 0$ for $i < 0$ or $i > n$, that

$$r_h^{(n+1)}(\xi) = a_{h-1} r_h^{(n+1)}(\xi_{h-1}) + a_h r_h^{(n+1)}(\xi_h) + a_{h+1} r_h^{(n+1)}(\xi_{h+1})$$

for $1 \leq h \leq n$. This is equal to $(a_{h-1} + 2ga_h + a_{h+1})\gamma$ by Properties (i), (ii) and (iii). This is zero for every $h \leq n-1$ by definition of the a_h's so that $\xi \in R^n \otimes_{\underline{\mathbf{1}}} H$ and $\gamma_n(\xi) = r_n^{(n+1)}(\xi) = (a_{n-1} + 2ga_n)\gamma$. If we prove that $u_n := (a_{n-1} + 2ga_n)$ is a unit in $\mathbb{Q}$ and, hence in R, we conclude that γ_n is surjective.

Let $x_{1,2} := -g \pm \sqrt{g^2 - 1}$ be the roots of the polynomial $X^2 + 2gX + 1$. Then one proves by induction on h that $a_{h+1} = \left(2g(x_1^h - x_2^h) - (x_1^{h-1} - x_2^{h-1})\right)/(x_2 - x_1)$. It follows that $(x_2 - x_1)u_n = 4g^2(x_1^{n-1} - x_2^{n-1}) - (x_1^{n-3} - x_2^{n-3})$ and an elementary estimate shows that $u_n \neq 0$ in $\mathbb{Q}$. $\square$

References

[AI] F. Andreatta, A. Iovita: *Semistable sheaves and comparison isomorphisms in the semistable case* J. Inst. Math. Jussieu **12** (2013), 77–151.

[Be] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique* $p > 0$. LNM **407**, Springer, Berlin-Heidelberg-New York, 1974.

[Bre] C. Breuil: *Représentations p-adiques semi-stables et transversalité de Griffiths*. Math. Ann. **307** (1997), 191–224.

[CI] R. Coleman, A. Iovita: *The Frobenius and monodromy operators for curves and abelian varieties*. Duke Math. J. **97** (1999), 171 V-215.

[CF] P. Colmez, J.-M. Fontaine: *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), 1–43.

[D1] P. Deligne, *Équations diff'erentielles à points singuliers réguliers*. LNM **163** (1970), pp. iii+133.

[D2] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*. In " Galois groups over $\mathbb{Q}$" (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ. **16** (1989), 79- V297.

[Fo] J.-M. Fontaine, *Sur certains types de représentations p-adiques du group de Galois d'un corps local; construction d'un anneau de Barsotti-Tate*, Ann. of Math. **115** (1982), 529–577.

[Fo] J.-M. Fontaine: *Le corps des périodes p-adiques. With an appendix by Pierre Colmez*. In "Périodes p-adiques (Bures-sur-Yvette, 1988)". Astérisque **223** (1994), 59–111.

[Ha] M. Hadian: *Motivic Fundamental Groups and Integral Points*. Duke Math. J. **160** (2011), 503 V565

[HK] O. Hyodo, K. Kato: *Semi-stable reduction and crystalline cohomology with logarithmic poles*, Astérisque **223** (1994), 321–347.

[Il] L. Illusie: *An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology*. In " Cohomologies p-adiques et applications arithmétiques, II". Astérisque **279** (2002), 271–322.

[K1] K. Kato: *Semi-stable reduction and p-adic étale cohomology*. In Périodes p-adiques (Bures-sur-Yvette, 1988). Astérisque **223** (1994), 269–293.

[K2] K. Kato: *Logarithmic structures of Fontaine-Illusie*, in Proceedings of JAMI conference, (1988), 191-224.

[Ka] N. Katz: *Nilpotent connexions and the monodromy theorem: applications of a result of Turrittin*. Publ. Math. IHES **39** (1970), 176–232.

[O] T. Oda: *A Note on Ramification of the Galois Representation of the Fundamental Group of an Algebraic Curve, II*, J. N. T. **53** (1995), 342-355.

[Ol] M. Olsson, *Towards non-abelian p-adic Hodge theory in the good reduction case.* Mem. Amer. Math. Soc. **210** (2011), vi+157.

[Ve] J. L.-Verdier, *Des catégories dérivées des catégories abéliennes. With a preface by Luc Illusie.* Edited and with a note by Georges Maltsiniotis. Astérisque **239** (1996), xii+253 pp.

[Vo] V. Vologodsky, *Hodge structures on the fundamental group and its applications to p-adic integration.* Mosc. Math. J. **3** (2003), 205 V-247.