IUTT Workshop, Oxford, December, 2015: Brief Summary

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Here is the key idea behind Mochizuki's proposed proof of Szpiro's conjecture.

> Let E/F be an elliptic curve over a number field F, $K = F(E[l])$, and $\underline{V}$ a set of places of K that restrict bijectively to the places of $F_{mod} = \mathbb{Q}(j(E))$ (satisfying some other conditions omitted here). Let q_v be the Tate parameters at the primes of bad reduction in $\underline{V}$. Then the difference in degrees of the arithmetic divisors
>
> $$\frac{1}{4l(l-1)} \sum_{j=1}^{(l-1)/2} \sum_{v \in \underline{V}^{bad}} (q_v^{j^2})$$
>
> and
>
> $$(1/2l) \sum_{v \in \underline{V}^{bad}} (q_v)$$
>
> can be bounded by a constant depending only on F and $\epsilon > 0$. This becomes possible by interpreting the various $q_v^{j^2/2l}$ as special values of a carefully chosen theta function. The function has a category-theoretic version, the *Frobenioid*[1]*-theoretic theta function.* The categorical theta function is extremely canonical and rigid, which allows a comparison of degrees of special values.

Many will object that this summary says very little. There are likely to be a few mistakes as well (my own, that is). However, I feel it is a statement that would be acceptable as a key idea *if the proof were already verified to be correct.* In that sense, it seems a reasonable guide, and an answer to many preliminary queries.

Unfortunately, after around the middle of the week, the lectures were difficult to follow, in spite of valiant effort on the part of the lecturers. Therefore, as to how the statement above is concretely realised, most people were still mystified at the end of the workshop. Employing Mochizuki's terminology, people had become somewhat familiar with the 'theta link', but were still quite confused by the 'log link' in the log-theta lattice. (To get a *very* rough idea of what these words mean, see my Mathoverflow article on Frobenioids.)

My own feeling is that all participants contributed in some important way. There were some tense moments amid heated dicussions, but they all seemed to add up in the end to constructive dialogue. How many people are willing to put further effort into understanding IUTT is obviously impossible to know. On the positive side, I was reminded of an aphorism told to me by Steve Zucker: 'Learning doesn't occur only at the moment of understanding[2].'

Finally, an analogy I rather like is with the *motivic polylogarithm.* When proving the Bloch-Kato conjecture for the Riemann zeta function, one needs to evaluate the regulator on the motivic cohomology of a cyclotomic field in terms of values of polylogarithms. All one is interested in the end are numbers. However, to calculate them, elements in motivic cohomology are expressed as specialisation of a motivic polylogarithmic sheaf living on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The rigidity of the sheaf-theoretic polylogarithm allows us to identify Hodge and étale realisations quite easily, which then can be specialised to the original regulator values of interest. In this sense, the basic philosophy behind Mochizuki's categorical theta function appears to belong to a canonical tradition.

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[1]Essentially a Galois category enhanced by divisors

[2]He meant this as advice for calculus students. I believe the formulation is original to him