Chapter 2

Compactness etc.

2.4 Compactifications

An application of the Stone-Čech Compactification

The material in this handout is taken from *Algebra in the Stone-Čech Compactification* by Hindman and Strauss.

I'm going to prove a theorem about arithmetic on $\mathbb{N}$, using the Stone-Čech compactification. The application is, possibly, rather surprising.

Theorem 2.4.23 *(The Finite Sums Theorem) Let $\mathbb{N}$ be divided into finitely many disjoint sets $A_1, \ldots, A_n$. Then there is an infinite subset B of $\mathbb{N}$ such that, for some i, not only is $B \subseteq A_i$, but, whenever $n_1, \ldots, n_k$ are distinct elements of B, then their sum $n_1 + \cdots + n_k \in A_i$ also.*

The first thing we do is embed $\mathbb{N}$ (with the usual discrete topology) into its Stone-Čech compactification $\langle h, \beta\mathbb{N}\rangle$, and then extend the operation of addition to $\beta\mathbb{N}$, using the Stone-Čech property. (Remember that, at the level of intuition, we like to identify $\mathbb{N}$ with its image under h, so pretend that h is the identity. But I will keep on writing h, for the sake of formal correctness.)

Actually, $+$ is a function of two variables, and the Stone-Čech property only talks about functions of one variable. So we split $+$ into two parts: the operation $\rho_n : m \mapsto m + n$ of adding a given constant value on the *right*, and the corresponding operation of adding a given constant value on the *left*, and extend them one at a time:

Definition 2.4.24 *For each $n \in \mathbb{N}$, define ρ_n as a function from $\mathbb{N}$ to $\mathbb{N}$ by $\rho_n(m) = m + n$. Noticing that $h \circ \rho_n : \mathbb{N} \to \beta\mathbb{N}$ is continuous, we use the Stone-Čech property to define a function $\beta\rho_n : \beta\mathbb{N} \to \beta\mathbb{N}$ so that for all $m \in \mathbb{N}$, $\beta\rho_n(h(m)) = h(\rho_n(m))$.*

(Intuitively, at this stage $\beta\rho_n(p)$, for $p \in \beta\mathbb{N}$, is a sum $p + h(n)$: in other words, we now know how to add a natural number on the right to any element of $\beta\mathbb{N}$.)

Now, for $p \in \beta\mathbb{N}$, and $n \in \mathbb{N}$, define $\lambda_p(n) = \beta\rho_n(p)$: so $\lambda_p : \mathbb{N} \to \beta\mathbb{N}$. Use the Stone-Čech property to define $\beta\lambda_p : \beta\mathbb{N} \to \beta\mathbb{N}$ so that if $n \in \mathbb{N}$, then $\beta\lambda_p(h(n)) = \lambda_p(n)$.

If $p, q \in \beta\mathbb{N}$, we define $p + q = \lambda_p(q)$.

It turns out, rather surprisingly, that $+$ on the Stone-Čech compactification is very unlike $+$ on the natural numbers. For instance, it is not commutative; worse than that, even though $p + q$ is certainly continuous as a function of q, it turns out, bafflingly, not to be continuous as a function of p. (Exercise: why can you not prove that $p+q$ is continuous as a function of p?) One can, though, show that $+$ is associative on $\beta\mathbb{N}$, and I will assume this without comment.

Another odd property of $+$ on $\beta\mathbb{N}$ is that there are *idempotents*: elements $p \neq h(0)$ such that $p + p = p$. We will now set out to show that this is so.

Definition 2.4.25 *Let $\mathcal{Z} = \{A \subseteq \beta\mathbb{N} \setminus \{h(0)\} : A + A \subseteq A$ and A is compact and non-empty$\}$.*

(By $A + A$ I mean, of course, $\{p + q : p, q \in A\}$.)

We notice, first, that $\mathcal{Z}$ is non-empty, since $\beta\mathbb{N} \setminus \{h(0)\}$ is in $\mathcal{Z}$, for $\{h(0)\}$ is open in $\beta\mathbb{N}$ (why?) and so its complement is closed.

Next, we can prove that any chain in $\mathcal{Z}$ has a *lower* bound (this is the point where we use compactness, in the form that any family of closed sets with the finite intersection property has a non-empty intersection); and Zorn's Lemma (suitably adapted) allows us to show that $\mathcal{Z}$ then has a *minimal* element.

Definition 2.4.26 *Let A be some minimal element of $\mathcal{Z}$.*

Now in fact, A contains exactly one element, which is an idempotent. But we only need the rather weaker:

Lemma 2.4.27 *A contains an idempotent.*

Proof. Let $p \in A$. We will show that $p + p = p$.

First, we show that $A + p = A$, which is clearly a step in the right direction. (By $A + p$, of course, we mean $\{q + p : q \in A\}$.)

For, let $B = A + p$.

Then $B + B = A + p + A + p \subseteq A + A + A + p$ (since $p \in A$) and, since $A \in \mathcal{Z}$, we know that $A + A \subseteq A$, so $B + B \subseteq A + p = B$.

Hence $B \in \mathcal{Z}$.

Also $p \in A$, and $A + A \subseteq A$, so $B = A + p \subseteq A + A \subseteq A$. But A is a *minimal* element of $\mathcal{Z}$, so $B = A$.

So, indeed, $A + p = A$.

Our next step is to investigate those $y \in A$ such that $y + p = p$, in hopes that p will turn out to be one of them.

So, let $C = \{y \in A : y + p = p\}$; this is non-empty, since $A + p = A$. We will show that $C = A$, so that, indeed, $p \in C$.

Now, if y, $z \in C$, then y, $z \in A$, so, since $A + A \subseteq A$, it follows that $y + z \in A$. Also, $(y + z) + p = y + (z + p) = y + p = p$, using the fact that z, and then y, is in C. Therefore $y + z \in C$.

So $C + C \subseteq C$. Now also, $C = A \cap (\beta\lambda_y)^{-1}\{p\}$ is closed in the compact space $\beta\mathbb{N}$, hence it is compact. So C is an element of $\mathcal{Z}$. But $C \subseteq A$, so, since A is minimal, $C = A$.

Hence, for all $y \in A$, $y \in C$, so $y + p = p$.

Hence in particular $p \in A$ implies $p + p = p$, as required. $\square$

We now turn to the proof of the Finite Sums Theorem.

Suppose $\mathbb{N}$ is the disjoint union of finitely many sets $A_1, \ldots, A_n$.

By a question on the problem sheets, $\beta\mathbb{N}$ is the disjoint union of the sets $\overline{h(A_1)}, \ldots, \overline{h(A_n)}$, which we can then prove are all also open, since each is the complement of a finite union of closed sets.

Let p be an idempotent. Suppose $p \in \overline{h(A_i)}$.

Well, $+$ is continuous on the right, by which I mean that the function $\beta\lambda_r$ is continuous for all r.

So, applying continuity of $\beta\lambda_p$, we note that whenever $U \ni p$ is open, there exists $V \ni p$ which is also open such that $V \subseteq (\beta\lambda_p)^{-1}(U)$; in other words, for all $q \in V$, $\beta\lambda_p(q) = p + q \in U$. (Actually, of course, I could have taken $V = (\beta\lambda_p)^{-1}(U)$; but for my purposes I don't care about the reverse inclusion.)

Also, while $+$ is not always continuous on the left, we do know that if $n \in \mathbb{N}$, the function $\beta\rho_n$ is continuous. It then follows that if $h(n) \in V \cap h(\mathbb{N})$, by continuity of $\beta\rho_n$ at p, there is $W \ni p$ open, depending on n, such that for all $r \in W$, $\beta\rho_n(r) = \lambda_r(n) = \beta\lambda_r(n) = r + h(n) \in U$.

Now we proceed to the construction of B. We construct it as a sequence $n_0, n_1, \ldots$.

Let $U_0 = \overline{A_i}$: recall that this was open. Find V_0 open such that $p \in V_0$, and for all $q \in V_0$, $p + q \in U_0$.

Now pick n_0 such that $h(n_0) \in U_0 \cap V_0$. (Remember that this is possible because $h(\mathbb{N})$ is dense.)

Now find open $W_0 \ni p$ such that for all $r \in W_0$, $r + h(n_0) \in U_0$.

Next, let $U_1 = U_0 \cap W_0$, noting that U_1 is open and $p \in U_1$. Find $V_1 \ni p$ open such that for all $q \in V_1$, $p + q \in U_1$.

Now pick $n_1 > n_0$ such that $h(n_1) \in U_1 \cap V_1$. (It is possible to choose $n_1 > n_0$, because p is a limit point of $h(\mathbb{N})$, and so each open neighbourhood of it contains infinitely many values of $h(m)$.)

Now find $W_1 \ni p$ such that for all $r \in W_1$, $r + h(n_1) \in U_1$.

Let $U_2 = U_1 \cap W_1 \cap W_0$, and continue.

Let $B = \{n_k : k \in \mathbb{N}\}$. We show that any finite sum of distinct elements of B, is in A_i.

Suppose $k_1 < k_2 < \cdots k_m$. We show that $n_{k_m} + \cdots + n_{k_1} + n_{k_0} \in A_i$.

Well, n_{k_m} was chosen so that $h(n_{k_m})$ belongs to $U_{k_m} \subseteq W_{k_{m-1}}$. This means that, by definition of $W_{k_{m-1}}$, $h(n_{k_m}) + h(n_{k_{m-1}}) \in U_{k_{m-1}}$. Now $U_{k_{m-1}}$ is a subset of $W_{k_{m-2}}$. Hence, since $(h(n_{k_m}) + h(n_{k_{m-1}})) \in W_{k_{m-2}}$, $(h(n_{k_m}) + h(n_{k_{m-1}})) + h(n_{k_{m-2}}) \in U_{k_{m-2}}$.

Continuing in this way, we find that $h(n_{k_m}) + \cdots + h(n_{k_1}) + h(n_{k_0}) \in U_{k_0}$. Now $U_{k_0} \subseteq U_{k_0-1} \subseteq \cdots \subseteq U_1 \subseteq U_0 = \overline{h(A_i)}$. So $h(n_{k_m}) + \cdots + h(n_{k_1}) + h(n_{k_0}) \in \overline{h(A_i)}$.

We can now drop all the h's, pulling back from $\beta\mathbb{N}$ to $\mathbb{N}$, and see that $n_{k_m} + \cdots + n_{k_1} + n_{k_0} \in A_i$, as required.