Constructing the Cayley Graphs of $F_d/D_{n+1}(F_d)$

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1 Introduction

For any group $G$ and any unitary ring $R$, there is a normal sequence of finite index subgroups $D_n(G)$ associated with $R$, called the dimension subgroups of $G$. The finite quotients $G/D_n(G)$ are important objects of study, arising in a variety of mathematical contexts, but the structure of the corresponding Cayley graphs is not well understood. Here, we outline a method with which to compute these graphs, and apply it to calculate the spectra of their Laplacian operators, in the case for which $G$ is a finitely generated free group, and $R$ is the field $F_p$, of order $p$.

2 Dimension Subgroups and the Algebra of Augmentation Ideals

Definition 1. For $G$ a (finitely presented) group, and $p$ prime, the augmentation ideal $I$ of the group ring $F_pG$ is the kernel of the augmentation mapping $\phi: F_pG \rightarrow F_p$ given by $\phi(\sum_{g \in G} r_g \cdot g) = \sum_{g \in G} r_g$, where $r_g \in F_p$ for each $g \in G$. The $n$th mod $p$ dimension subgroup $D_{n}^p(G)$ of $G$ is the group $\{ g \in G | g - e \in I^n \} \trianglelefteq G$.

It is immediate from the definition that $I$ is freely generated over $F_p$ by $\{ g - e \}_{g \in G\setminus \{e\}}$. Further, $|F_p G/I^{n+1}|$ is finite, for each $n$. This is because, if $w = x_1 \cdots x_{n+1}$ is a word of length $n+1$ in generators $x_i$ of $G$, then $w - (x_1 - e) \cdots (x_{n+1} - e)$ is a linear combination of words of length at most $n$ in the $x_i$, and $\equiv w \pmod{I^{n+1}}$. The natural (group) homomorphism $\psi: D_{n}^p(G) \rightarrow I^n/I^{n+1}$, given by $\psi(g) = g - e$, induces a monomorphism $\hat{\psi}: D_{n}^p(G)/D_{n+1}^p(G) \rightarrow I^n/I^{n+1}$, since, for $g, h \in D_{n}^p(G)$,

$g \equiv h \pmod{D_{n+1}^p(G)}$ iff $g \cdot h^{-1} \in D_{n+1}^p(G)$ iff $g \cdot h^{-1} - e \in I^{n+1}$ iff $\overline{g - e} = \overline{h - e}$ iff $\psi(g) = \psi(h)$

so that $\hat{\psi}$ is well-defined and injective. In particular, $D_{n}^p(G)/D_{n+1}^p(G)$ is a finite $p$-group. Indeed, Jennings [1] has established the following result, relating the ranks of the $D_{n}^p(G)/D_{n+1}^p(G)$ and $I^n/I^{n+1}$.

Theorem 1 (Jennings). Where $|I^n/I^{n+1}| = p^{a_n}$ and $|D_{n}^p(G)/D_{n+1}^p(G)| = p^{d_n}$,

$$\sum_{r=0}^{\infty} a_r \cdot x^r = \prod_{s=1}^{p-1} \left( \sum x^{st} \right)^{d_s}.$$
Since the only contribution made to the coefficient of $x^n$ from the right-hand side comes from the terms of the product for which $s \leq n$, we have, for each $n$,

$$a_n = \text{Coefficient}_{x^n}(\prod_{s=1}^{n} \left( \sum_{t=0}^{p-1} x^{st} \right)^{d_s}) = d_n + \text{Coefficient}_{x^n}(\prod_{s=1}^{n-1} \left( \sum_{t=0}^{p-1} x^{st} \right)^{d_s}),$$

where $\text{Coefficient}_{x^n}(\prod_{s=1}^{n} \left( \sum_{t=0}^{p-1} x^{st} \right)^{d_s})$ is a polynomial in $d_1, \ldots, d_{n-1}$ alone, so we may calculate $d_n$, and hence $|G/D_{n+1}^p(G)| = p^{\sum_{i=1}^{n} d_i}$, by recursion, provided we can also calculate the $a_i$.

For this purpose, we shall require the following results, adapted from [2]:

**Lemma 2.** If $G$ has a presentation with $d$ generators and $r$ defining relations, then there is an exact sequence of $G$-modules:

$$\mathbb{F}_p G^\oplus r \xrightarrow{\alpha} \mathbb{F}_p G^\oplus d \xrightarrow{\beta} I \to 0$$

**Lemma 3.** Under the hypothesis of the previous lemma, where for $l \in \mathbb{N}$, $A_l = \alpha^{-1}((I^l)^{\oplus d})$, there is an exact sequence of $G$-modules:

$$A_l/A_{l+1} \xrightarrow{\alpha_l} (I^l/I^{l+1})^{\oplus d} \xrightarrow{\beta_l} I^{l+1}/I^{l+2} \to 0,$$

where $\alpha_l$ and $\beta_l$ are induced by $\alpha$ and $\beta$, respectively.

In the case for which $G$ is free on a finite set $X = \{x_i\}_{i=1}^d$, we may take $r = 0$, so that $A_l = 0$ for each $l$, and $\beta_l: (I^l/I^{l+1})^{\oplus d} \cong I^{l+1}/I^{l+2}$, with the isomorphism given explicitly by $\beta_l(\gamma_1, \ldots, \gamma_d) = \sum_{i=1}^{d} \gamma_i \cdot (x_i - e)$. Now, by Jennings’ Theorem, $a_0 = 1$, and by Lemma 3, $a_{i+1} = d \cdot a_i$ for each $i$, hence $a_n = d^n$.

Furthermore, since $\mathbb{F}_p G/I = \{\lambda \cdot e + I \mid \lambda \in \mathbb{F}_p\}$, it follows by induction from Lemma 3 that $\{(x_{i_1} - e) \cdots (x_{i_j} - e) + I^{n+1} \mid i_j \in \{1, \ldots, d\}\}$ is a basis for $I^n/I^{n+1}$, and $B_n := \{e + I^{n+1}\} \cup \{(x_{i_1} - e) \cdots (x_{i_k} - e) + I^{n+1} \mid i_j \in \{1, \ldots, d\}, k \in \{1, \ldots, n\}\}$ is a basis for $\mathbb{F}_p G/I^{n+1}$. This leads us to:

**Lemma 4.** For $k \in \{1, \ldots, d\}$, let $\pi_k \in \text{End}(\mathbb{F}_p G/I^{n+1})$ be given by $\pi_k(\gamma) = \gamma \cdot x_k$ (where $\text{End}(\mathbb{F}_p G/I^{n+1})$ is the set of $\mathbb{F}_p$-vector space endomorphisms of $\mathbb{F}_p G/I^{n+1}$). Then the matrix $M_k^{(n)} \in \mathbb{M}_{\sum_{i=0}^{d} (\mathbb{F}_p)}$ of $\pi_k$ wrt basis $B_n$ (the elements of which being ordered first by product length, then lexicographically in the $x_i - e$), is given by:

$$(M_k^{(n)})_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i = d \cdot (j - 1) + k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

so that

$$M_k^{(n+1)} = \left( \begin{array}{c|c} M_k^{(n)} & 0 \\ \hline A_k^{(n)} & I_{d^{n+1}} \end{array} \right),$$

where $(A_k^{(n)})_{i,j} = \begin{cases} 1 & \text{if } i = d \cdot (j - \sum_{i=0}^{n-1} d^i - 1) + k, \\ 0 & \text{otherwise.} \end{cases}$
Proof. \( \pi_k(\overline{e}) = \overline{x_k} = (x_k - e) + \overline{e} \).

\[
\pi_k((x_{i_1} - e) \cdots (x_{i_m} - e)) = (x_{i_1} - e) \cdots (x_{i_m} - e) \cdot x_k
\]

\[
= \begin{cases} 
(x_{i_1} - e) \cdots (x_{i_m} - e) \cdot (x_k - e) + (x_{i_1} - e) \cdots (x_{i_m} - e) & \text{if } 1 \leq m < n, \\
(x_{i_1} - e) \cdots (x_{i_m} - e) & \text{if } m = n
\end{cases}
\]

as \((x_{i_1} - e) \cdots (x_{i_m} - e) \cdot (x_k - e) \in I^{n+1}\).

\[\square\]

Example 1. Letting \( d = 2 \), in the case for which \( n = 1 \) we have an ordered \( \mathbb{F}_p \)-basis \( B_1 = \{ e + I^2, x_1 - e + I^2, x_2 - e + I^2 \} \) for \( \mathbb{F}_p G / I^2 \), with respect to which the matrices of \( \pi_1 \) and \( \pi_2 \) (as defined above) are:

\[
M_1^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
M_2^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\]

respectively. Likewise, in the case for which \( n = 2 \), we have an \( \mathbb{F}_p \)-basis \( B_2 = \{ e + I^3, x_1 - e + I^3, (x_1 - e)^2 + I^3, (x_2 - e)(x_1 - e) + I^3, (x_2 - e)^2 + I^3 \} \) for \( \mathbb{F}_p G / I^3 \), and the corresponding matrices for \( \pi_1 \) and \( \pi_2 \) are now:

\[
M_1^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},
M_2^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.
\]

Note in particular that the \( 3 \times 3 \) upper-left submatrices of \( M_1^{(2)} \) and \( M_2^{(2)} \) are precisely \( M_1^{(1)} \) and \( M_2^{(1)} \), respectively.

3 A Construction of \( \text{Cay}(F_d / D_n^p(F_d), X) \)

Proposition 5. With \( G \) and \( X \) as above, let \( \Gamma \) be the graph with vertex set \( \mathbb{F}_p G / I^n \) and edge set \( \{(r, r \cdot x) \mid r \in \mathbb{F}_p G / I^n, x \in X\} \). Then \( \text{Cay}(G / D_n^p(G), X) \) is graph-isomorphic to \( \Gamma_0 \), the path-component of \( \Gamma \) containing the vertex \( \overline{e} \).

Proof. Let \( f : V(\text{Cay}(G / D_n^p(G), X)) \rightarrow V(\Gamma_0) \) be given by \( f(g) = \overline{g} \). For \( g, h \in G \),

\[
g \equiv h \mod D_n^p(G) \iff g \cdot h^{-1} \in D_n^p(G) \iff g \cdot h^{-1} - e \in I^n \iff \overline{g} = \overline{h} \iff f(g) = f(h)
\]

so \( f \) is well-defined and injective. Clearly, any vertex of \( \Gamma_0 \) is represented by some \( \overline{w} \), for \( w \) a word in \( X \), so \( f \) is surjective. Finally,
\[(g, h) \in E(Cay(G/D_p^n(G), X)) \iff \exists k \in \{1, \ldots, d\} \text{ s.t. } h \equiv g \cdot x_k (\text{mod } D_p^n(G))\]
\[(g, h) \in E(Cay(G/D_p^n(G), X)) \iff \exists k \in \{1, \ldots, d\} \text{ s.t. } h \equiv g \cdot x_k^{-1} \cdot g^{-1} \in D_p^n(G)\]
\[(g, h) \in E(Cay(G/D_p^n(G), X)) \iff \exists k \in \{1, \ldots, d\} \text{ s.t. } h \equiv g \cdot x_k^{-1} \cdot g^{-1} - e \in I^n\]
\[(g, h) \in E(Cay(G/D_p^n(G), X)) \iff \exists k \in \{1, \ldots, d\} \text{ s.t. } \bar{h} = \bar{g} \cdot x_k\]
\[(g, h) \in E(Cay(G/D_p^n(G), X)) \iff (f(g), f(h)) \in E(\Gamma_0)\]

Indeed, we may alternatively regard \(\Gamma\) as the graph with vertex set \(V\) comprising the coordinate vectors (drawn from \(\mathbb{F}_p^\sum_{i=0}^{n-1} d_i\)) of the elements of \(\mathbb{F}_p G/I^n\) wrt \(B_{n-1}\), and edge set \(\{(v, M_k^{(n)} \cdot v) \mid v \in V, k \in \{1, \ldots, d\}\}\), so that \(\Gamma_0\) is the path-component of \(\Gamma\) containing the vertex \((1,0,\ldots,0)\).

This provides us with an explicit method for computing \(Cay(G/D_p^n(G), X)\):

1. Set \(V = \{(1,0,\ldots,0)^T\}, E = \emptyset\).
2. Repeatedly replace \(V\) by \(V \cup \{M_k^{(n)} \cdot v \mid v \in V, k \in \{1, \ldots, d\}\}\), until \(|V| = p^\sum_{i=0}^{n-1} d_i\), as in Jennings’ Theorem.
3. For each \(v \in V\) in turn, replace \(E\) by \(E \cup \{(v, M_k^{(n)} \cdot v) \mid k \in \{1, \ldots, d\}\}\).
4. Then \(Cay(G/D_p^n(G), X) \cong (V, E)\).

4 An Application to Spectral Theory

Clearly, we could equally regard the above construction as a method for computing the adjacency matrix \(A\) of \(Cay(G/D_p^n(G), X)\). For \(A\) is the \(|V| \times |V|\) matrix with \((i, j)\)th entry equal to the number of edges running from \(v_i\) to \(v_j\), for \(v_1, \ldots, v_{|V|}\) an enumeration of \(V\), and depends only on the choice of enumeration. We would like therefore to be able to apply our method to calculations in spectral graph theory; for instance by computing the spectrum of the Laplacian on \(Cay(G/D_p^n(G), X)\). However, for this purpose, we must instead work with \(A + A^T\), which is the adjacency matrix of \(Cay(G/D_p^n(G), X^\pm)\), since the Laplacian operator is defined only on undirected or symmetric graphs. The Laplacian matrix of \((V, E)\) is then \(L := 2dI_{|V|} - (A + A^T)\).

We conclude by exhibiting \(Cay(F_2/D_2^n(F_2), X)\), together with the spectrum of its Laplacian, for \(n = 2, 3\) and \(4\). For the sake of clarity, we also include colour-coded illustrations of the Cayley graphs. These graphs and spectra were calculated using Wolfram Mathematica 7.0, following the above method. The program used (which takes values for \(n\), \(p\) and \(d\) and returns \(Cay(F_d/D_p^n(F_d), X)\) and the corresponding spectrum) can be found at [3].
4.1 $F_2/D_2^2(F_2)$:

Figure 1: Author’s illustration of $Cay(F_2/D_2^2(F_2), X)$.

Figure 2: Spectrum of the Laplacian operator on $Cay(F_2/D_2^2(F_2), X)$, as outputted by [3]. The eigenvalues are 0, 4 and 8.

This is just $\mathbb{Z}_2$. Indeed, since $\hat{\psi}: F_d/D_2^p(F_d) \to I/I^2$, as defined above, is injective, and since $a_1 = d_1 = d$ (by Jennings' Theorem and Lemma 3), $F_d/D_2^p(F_d) \cong I/I^2 \cong \mathbb{Z}_d^p$, for any $p$ and $d$. 
4.2 $F_2/D_3^2(F_2)$:

Figure 3: Diagram of $Cay(F_2/D_3^2(F_2), X)$, as outputted by [3], and corresponding author’s illustration.

Figure 4: Spectrum of the Laplacian operator on $Cay(F_2/D_3^2(F_2), X)$. The eigenvalues are 0, 2, 4, 6, 8 and $4 \pm 2\sqrt{2}$. 
4.3 $F_2/D_4^2(F_2)$:

Figure 5: Diagram of $Cay(F_2/D_4^2(F_2), X)$, as outputted by [3].
Figure 6: Author’s illustration of $\text{Cay}(F_2/D_3^2(F_2), X)$. 
Figure 7: Spectrum of the Laplacian operator on $Cay(F_2/D_4^2(F_2), X)$. The eigenvalues are 0, 2, 4, 6, 8, $4 \pm 2\sqrt{2}$, $3 \pm \sqrt{5}$ and $5 \pm \sqrt{5}$.

References

