Constructing the Cayley Graphs of $F_d/D_n^p(F_d)$

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1 Introduction

For any group G and any unitary ring R, there is a normal sequence of finite index subgroups $D_n(G)$ associated with R, called the *dimension subgroups* of G. The finite quotients $G/D_n(G)$ are important objects of study, arising in a variety of mathematical contexts, but the structure of the corresponding Cayley graphs is not well understood. Here, we outline a method with which to compute these graphs, and apply it to calculate the spectra of their Laplacian operators, in the case for which G is a finitely generated free group, and R is the field $\mathbb{F}_p$, of order p.

2 Dimension Subgroups and the Algebra of Augmentation Ideals

Definition 1. For G a (finitely presented) group, and p prime, the *augmentation ideal* I of the group ring $\mathbb{F}_pG$ is the kernel of the *augmentation mapping* $\phi\colon \mathbb{F}_pG \to \mathbb{F}_p$ given by $\phi(\sum_{g\in G} r_g \cdot g) = \sum_{g\in G} r_g$, where $r_g \in \mathbb{F}_p$ for each $g \in G$. The *nth mod p dimension subgroup* $D_n^p(G)$ of G is the group $\{g \in G \mid g - e \in I^n\} \trianglelefteq G$.

It is immediate from the definition that I is freely generated over $\mathbb{F}_p$ by $\{g-e\}_{g\in G\setminus\{e\}}$. Further, $\left|\mathbb{F}_pG/I^{n+1}\right|$ is finite, for each n. This is because, if $w = x_1 \cdots x_{n+1}$ is a word of length $n+1$ in generators x_i of G, then $w - (x_1 - e)\cdots(x_{n+1} - e)$ is a linear combination of words of length at most n in the x_i, and $\equiv w$ (mod I^{n+1}). The natural (group) homomorphism $\psi\colon D_n^p(G) \to I^n/I^{n+1}$, given by $\psi(g) = \overline{g-e}$, induces a monomorphism $\hat{\psi}\colon D_n^p(G)/D_{n+1}^p(G) \to I^n/I^{n+1}$, since, for $g, h \in D_n^p(G)$,

$$g \equiv h (\mathrm{mod}\ D_{n+1}^p(G)) \text{ iff } g\cdot h^{-1} \in D_{n+1}^p(G) \text{ iff } g\cdot h^{-1} - e \in I^{n+1} \text{ iff } \overline{g-e} = \overline{h-e} \text{ iff } \psi(g) = \psi(h)$$

so that $\hat{\psi}$ is well-defined and injective. In particular, $D_n^p(G)/D_{n+1}^p(G)$ is a finite p-group. Indeed, Jennings [1] has established the following result, relating the ranks of the $D_n^p(G)/D_{n+1}^p(G)$ and I^n/I^{n+1}.

Theorem 1 (Jennings)**.** *Where* $\left|I^n/I^{n+1}\right| = p^{a_n}$ *and* $\left|D_n^p(G)/D_{n+1}^p(G)\right| = p^{d_n}$,

$$\sum_{r=0}^{\infty} a_r \cdot x^r = \prod_{s=1}^{\infty} (\sum_{t=0}^{p-1} x^{st})^{d_s}.$$

Since the only contribution made to the coefficient of x^n from the right-hand side comes from the terms of the product for which $s \leq n$, we have, for each n,

$$a_n = \mathrm{Coefficient}_{x^n}(\prod_{s=1}^{n}(\sum_{t=0}^{p-1} x^{st})^{d_s}) = d_n + \mathrm{Coefficient}_{x^n}(\prod_{s=1}^{n-1}(\sum_{t=0}^{p-1} x^{st})^{d_s}),$$

where $\mathrm{Coefficient}_{x^n}(\prod_{s=1}^{n-1}(\sum_{t=0}^{p-1} x^{st})^{d_s})$ is a polynomial in $d_1, \ldots, d_{n-1}$ alone, so we may calculate d_n, and hence $\left|G/D_{n+1}^{p}(G)\right| = p^{\sum_{i=1}^{n} d_i}$, by recursion, provided we can also calculate the a_i. For this purpose, we shall require the following results, adapted from [2]:

Lemma 2. *If G has a presentation with d generators and r defining relations, then there is an exact sequence of G-modules:*

$$\mathbb{F}_p G^{\bigoplus r} \xrightarrow{\alpha} \mathbb{F}_p G^{\bigoplus d} \xrightarrow{\beta} I \to 0$$

Lemma 3. *Under the hypothesis of the previous lemma, where for $l \in \mathbb{N}$, $A_l = \alpha^{-1}((I^l)^{\bigoplus d})$, there is an exact sequence of G-modules:*

$$A_l/A_{l+1} \xrightarrow{\alpha_l} (I^l/I^{l+1})^{\bigoplus d} \xrightarrow{\beta_l} I^{l+1}/I^{l+2} \to 0,$$

where α_l and β_l are induced by α and β, respectively.

In the case for which G is free on a finite set $X = \{x_i\}_{i=1}^{d}$, we may take $r = 0$, so that $A_l = 0$ for each l, and $\beta_l \colon (I^l/I^{l+1})^{\bigoplus d} \cong I^{l+1}/I^{l+2}$, with the isomorphism given explicitly by $\beta_l(\gamma_1, \ldots, \gamma_d) = \sum_{i=1}^{d} \gamma_i \cdot (x_i - e)$. Now, by Jennings' Theorem, $a_0 = 1$, and by Lemma 3, $a_{l+1} = d \cdot a_l$ for each l, hence $a_n = d^n$.

Furthermore, since $\mathbb{F}_p G/I = \{\lambda \cdot e + I \mid \lambda \in \mathbb{F}_p\}$, it follows by induction from Lemma 3 that $\{(x_{i_1} - e) \cdots (x_{i_n} - e) + I^{n+1} \mid i_j \in \{1, \ldots, d\}\}$ is a basis for I^n/I^{n+1}, and $B_n := \{e + I^{n+1}\} \cup \{(x_{i_1} - e) \cdots (x_{i_k} - e) + I^{n+1} \mid i_j \in \{1, \ldots, d\}, k \in \{1, \ldots, n\}\}$ is a basis for $\mathbb{F}_p G/I^{n+1}$. This leads us to:

Lemma 4. *For $k \in \{1, \ldots, d\}$, let $\pi_k \in End(\mathbb{F}_p G/I^{n+1})$ be given by $\pi_k(\gamma) = \gamma \cdot x_k$ (where $End(\mathbb{F}_p G/I^{n+1})$ is the set of $\mathbb{F}_p$-vector space endomorphisms of $\mathbb{F}_p G/I^{n+1}$). Then the matrix $M_k^{(n)} \in \mathbb{M}_{\sum_{i=0}^{n} d^i}(\mathbb{F}_p)$ of π_k wrt basis B_n (the elements of which being ordered first by product length, then lexicographically in the $x_i - e$), is given by:*

$$(M_k^{(n)})_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i = d \cdot (j-1) + k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

so that

$$M_k^{(n+1)} = \left(\begin{array}{c|c} M_k^{(n)} & 0 \\ \hline A_k^{(n)} & I_{d^{n+1}} \end{array} \right), \text{ where } (A_k^{(n)})_{i,j} = \begin{cases} 1 & \text{if } i = d \cdot (j - \sum_{i=0}^{n-1} d^i - 1) + k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. $\pi_k(\overline{e}) = \overline{x_k} = \overline{(x_k - e)} + \overline{e}$.

$$\pi_k(\overline{(x_{i_1} - e) \cdots (x_{i_m} - e)}) = \overline{(x_{i_1} - e) \cdots (x_{i_m} - e)} \cdot x_k$$

$$= \begin{cases} \overline{(x_{i_1} - e) \cdots (x_{i_m} - e) \cdot (x_k - e)} + \overline{(x_{i_1} - e) \cdots (x_{i_m} - e)} & \text{if } 1 \leq m < n, \\ \overline{(x_{i_1} - e) \cdots (x_{i_m} - e)} & \text{if } m = n \end{cases}$$

as $(x_{i_1} - e) \cdots (x_{i_n} - e) \cdot (x_k - e) \in I^{n+1}$.

□

Example 1. Letting $d = 2$, in the case for which $n = 1$ we have an ordered $\mathbb{F}_p$-basis $B_1 = \{e + I^2, x_1 - e + I^2, x_2 - e + I^2\}$ for $\mathbb{F}_pG/I^2$, with respect to which the matrices of π_1 and π_2 (as defined above) are:

$$M_1^{(1)} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \; M_2^{(1)} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right)$$

respectively. Likewise, in the case for which $n = 2$, we have an $\mathbb{F}_p$-basis $B_2 = \{e + I^3,$ $x_1 - e + I^3, x_2 - e + I^3, (x_1 - e)^2 + I^3, (x_1 - e)(x_2 - e) + I^3, (x_2 - e)(x_1 - e) + I^3, (x_2 - e)^2 + I^3\}$ for $\mathbb{F}_pG/I^3$, and the corresponding matrices for π_1 and π_2 are now:

$$M_1^{(2)} = \left(\begin{array}{c|cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \; M_2^{(2)} = \left(\begin{array}{c|cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Note in particular that the 3×3 upper-left submatrices of $M_1^{(2)}$ and $M_2^{(2)}$ are precisely $M_1^{(1)}$ and $M_2^{(1)}$, respectively.

3 A Construction of $Cay(F_d/D_n^p(F_d), X)$

Proposition 5. *With G and X as above, let Γ be the graph with vertex set $\mathbb{F}_pG/I^n$ and edge set $\{(r, r \cdot x) \mid r \in \mathbb{F}_pG/I^n, x \in X\}$. Then $Cay(G/D_n^p(G), X)$ is graph-isomorphic to Γ_0, the path-component of Γ containing the vertex $\overline{e}$.*

Proof. Let $f \colon V(Cay(G/D_n^p(G), X)) \to V(\Gamma_0)$ be given by $f(g) = \overline{g}$. For $g, h \in G$,

$$g \equiv h (\text{mod } D_n^p(G)) \text{ iff } g \cdot h^{-1} \in D_n^p(G) \text{ iff } g \cdot h^{-1} - e \in I^n \text{ iff } \overline{g} = \overline{h} \text{ iff } f(g) = f(h)$$

so f is well-defined and injective. Clearly, any vertex of Γ_0 is represented by some $\overline{w}$, for w a word in X, so f is surjective. Finally,

$$\begin{aligned}(g,h) \in E(Cay(G/D_n^p(G),X)) \quad &\text{iff } \exists k \in \{1,\ldots,d\} \text{ s.t. } h \equiv g \cdot x_k (\text{mod } D_n^p(G)) \\ &\text{iff } \exists k \in \{1,\ldots,d\} \text{ s.t. } h \cdot x_k^{-1} \cdot g^{-1} \in D_n^p(G) \\ &\text{iff } \exists k \in \{1,\ldots,d\} \text{ s.t. } h \cdot x_k^{-1} \cdot g^{-1} - e \in I^n \\ &\text{iff } \exists k \in \{1,\ldots,d\} \text{ s.t. } \overline{h} = \overline{g} \cdot x_k \\ &\text{iff } (f(g), f(h)) \in E(\Gamma_0)\end{aligned}$$

□

Indeed, we may alternatively regard Γ as the graph with vertex set V comprising the coordinate vectors (drawn from $\mathbb{F}_p{}^{\sum_{i=0}^{n-1} d^i}$) of the elements of $\mathbb{F}_pG/I^n$ wrt B_{n-1}, and edge set $\{(v, M_k^{(n)} \cdot v) \mid v \in V, k \in \{1,\ldots,d\}\}$, so that Γ_0 is the path-component of Γ containing the vertex $(1,0,\ldots,0)$.

This provides us with an explicit method for computing $Cay(G/D_n^p(G),X)$:

1. Set $V = \{(1,0,\ldots,0)^T\}$, $E = \emptyset$
2. Repeatedly replace V by $V \cup \{M_k^{(n)} \cdot v \mid v \in V, k \in \{1,\ldots,d\}\}$, until $|V| = p^{\sum_{i=1}^{n-1} d_i}$, as in Jennings' Theorem.
3. For each $v \in V$ in turn, replace E by $E \cup \{(v, M_k^{(n)} \cdot v) \mid k \in \{1,\ldots,d\}\}$.
4. Then $Cay(G/D_n^p(G),X) \cong (V,E)$.

4 An Application to Spectral Theory

Clearly, we could equally regard the above construction as a method for computing the adjacency matrix A of $Cay(G/D_n^p(G),X)$. For A is the $|V| \times |V|$ matrix with (i,j)th entry equal to the number of edges running from v_i to v_j, for $v_1,\ldots,v_{|V|}$ an enumeration of V, and depends only on the choice of enumeration. We would like therefore to be able to apply our method to calculations in spectral graph theory; for instance by computing the spectrum of the Laplacian on $Cay(G/D_n^p(G),X)$. However, for this purpose, we must instead work with $A + A^T$, which is the adjacency matrix of $Cay(G/D_n^p(G),X^{\pm})$, since the Laplacian operator is defined only on undirected or symmetric graphs. The Laplacian matrix of (V,E) is then $L := 2dI_{|V|} - (A + A^T)$.

We conclude by exhibiting $Cay(F_2/D_n^2(F_2),X)$, together with the spectrum of its Laplacian, for $n = 2$, 3 and 4. For the sake of clarity, we also include colour-coded illustrations of the Cayley graphs. These graphs and spectra were calculated using Wolfram Mathematica 7.0, following the above method. The program used (which takes values for n, p and d and returns $Cay(F_d/D_n^p(F_d),X)$ and the corresponding spectrum) can be found at [3].

4.1 $F_2/D_2^2(F_2)$:

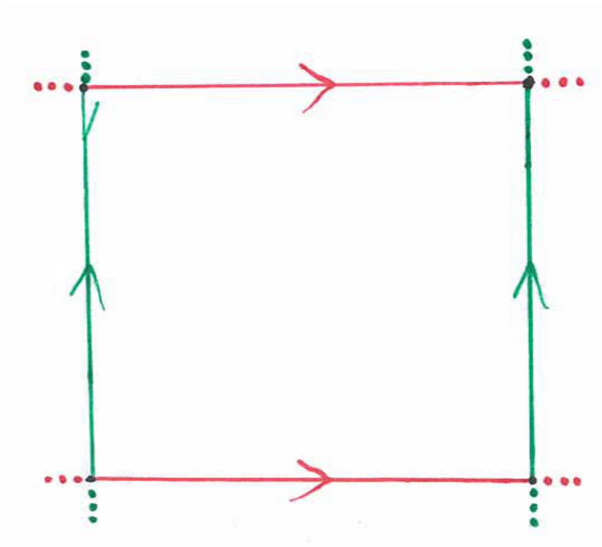


Figure 1: Author's illustration of $Cay(F_2/D_2^2(F_2), X)$.

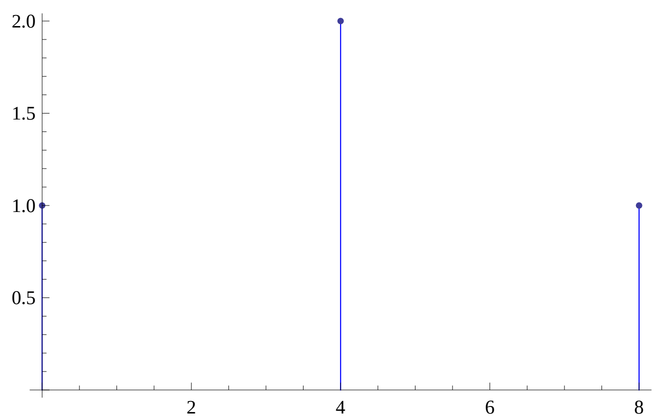
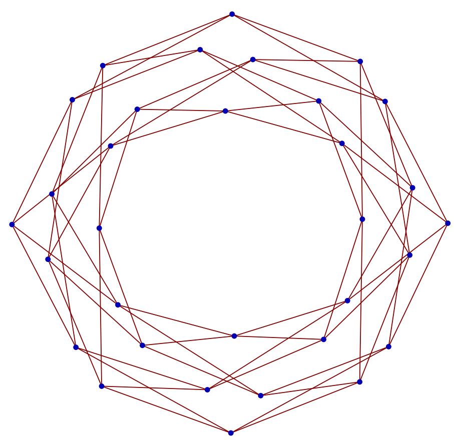


Figure 2: Spectrum of the Laplacian operator on $Cay(F_2/D_2^2(F_2), X)$, as outputted by [3]. The eigenvalues are 0, 4 and 8.

This is just $\mathbb{Z}_2^2$. Indeed, since $\hat{\psi}\colon F_d/D_2^p(F_d) \to I/I^2$, as defined above, is injective, and since $a_1 = d_1 = d$ (by Jennings' Theorem and Lemma 3), $F_d/D_2^p(F_d) \cong I/I^2 \cong \mathbb{Z}_p^d$, for any p and d.

4.2 $F_2/D_3^2(F_2)$:



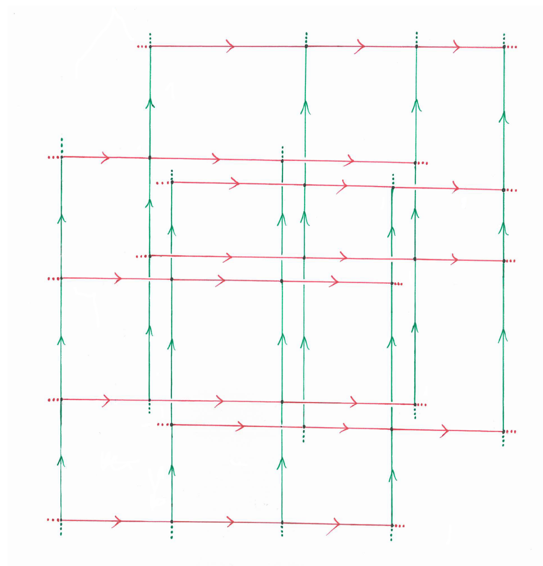


Figure 3: Diagram of $Cay(F_2/D_3^2(F_2), X)$, as outputted by [3], and corresponding author's illustration.

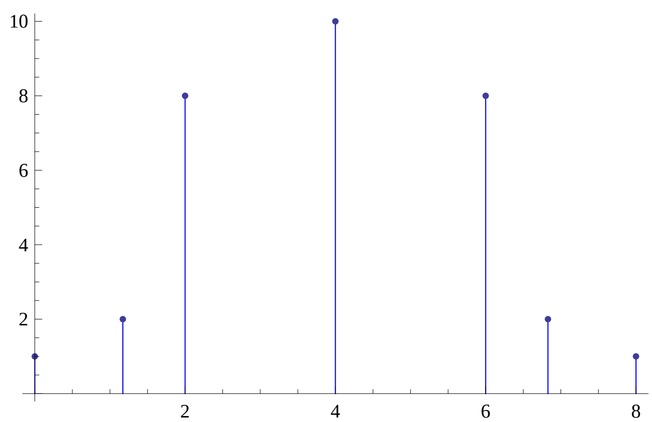


Figure 4: Spectrum of the Laplacian operator on $Cay(F_2/D_3^2(F_2), X)$. The eigenvalues are 0, 2, 4, 6, 8 and $4 \pm 2\sqrt{2}$.

4.3 $F_2/D_4^2(F_2)$:

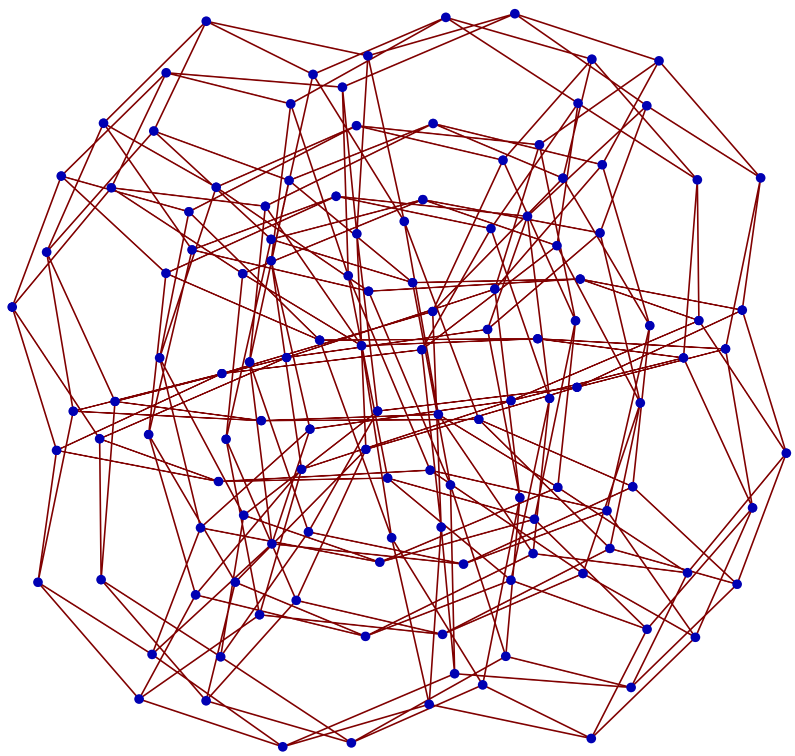


Figure 5: Diagram of $Cay(F_2/D_4^2(F_2), X)$, as outputted by [3].

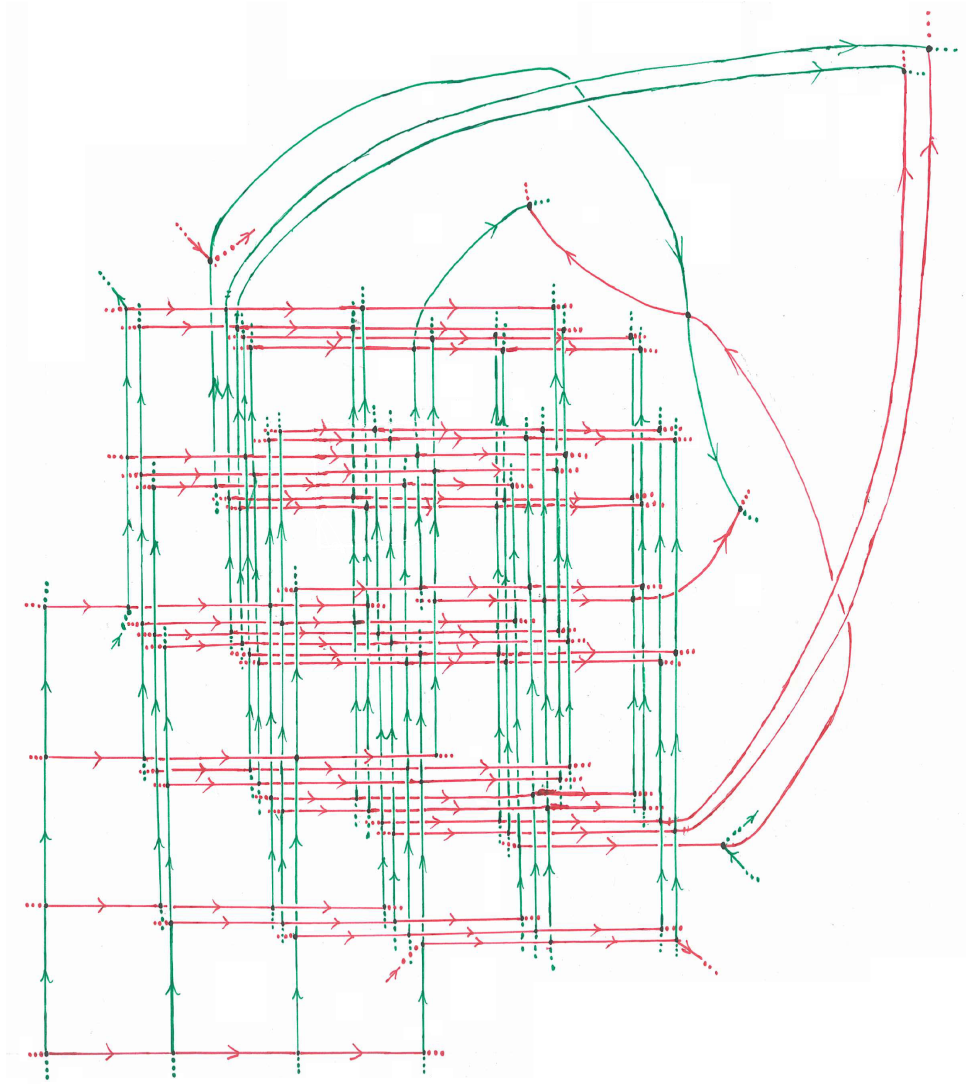


Figure 6: Author's illustration of $Cay(F_2/D_4^2(F_2), X)$.

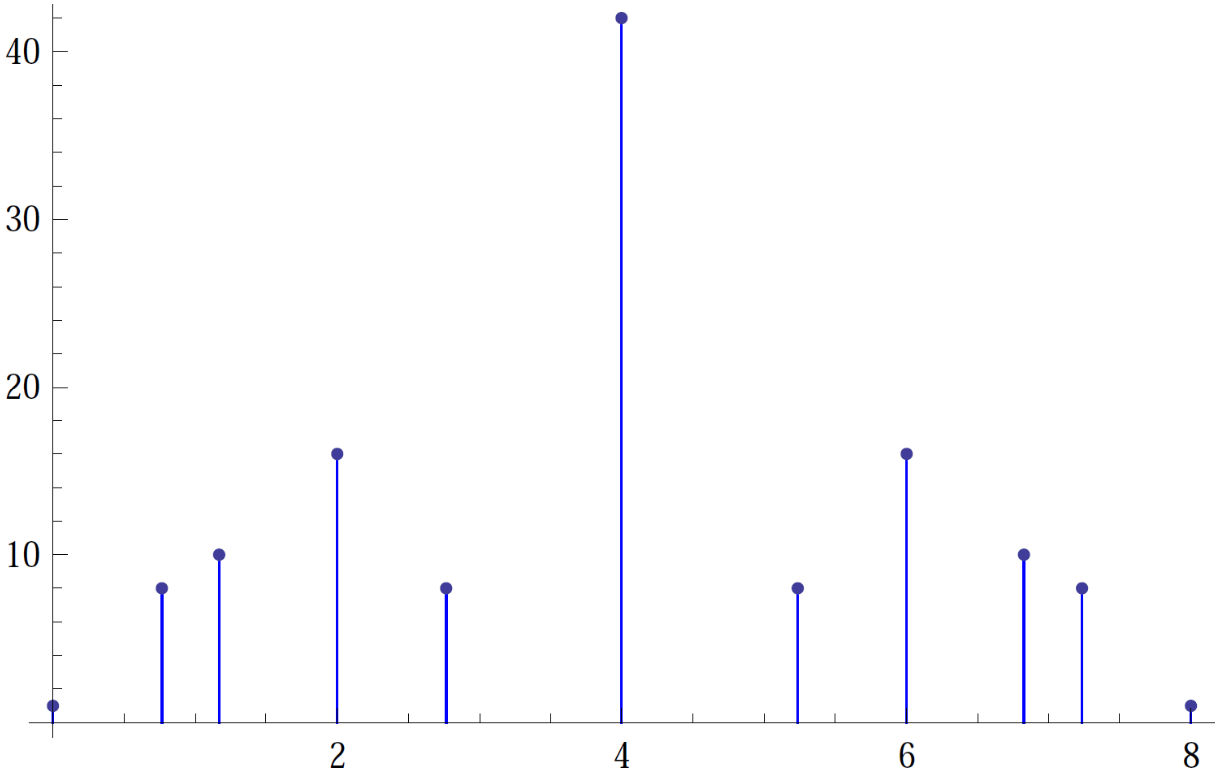


Figure 7: Spectrum of the Laplacian operator on $Cay(F_2/D_4^2(F_2), X)$. The eigenvalues are 0, 2, 4, 6, 8, $4 \pm 2\sqrt{2}$, $3 \pm \sqrt{5}$ and $5 \pm \sqrt{5}$.

References

[1] Jennings, S. A.: The structure of the group ring of a p-group over a modular field. Trans. Am. Math. Soc. **50** (1941), 175-185.

[2] Johnson D.L.: Presentations of Groups. 2nd edition, Cambridge University Press, 1990.

[3] http://people.maths.ox.ac.uk/lackenby/