ALTERNATING LINKS AND DEFINITE SURFACES

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Abstract. We establish a characterization of alternating links in terms of definite spanning surfaces. We apply it to obtain a new proof of Tait's conjecture that reduced alternating diagrams of the same link have the same crossing number and writhe. We also deduce a result of Banks and Hirasawa-Sakuma about Seifert surfaces for special alternating links. The appendix, written by Juhász and Lackenby, applies the characterization to derive an exponential time algorithm for alternating knot recognition.

MSC classes: 05C21, 05C50, 11H55, 57M15, 57M25, 57M27

1. Introduction.

"What is an alternating knot?" – Ralph Fox

A link diagram is alternating if its crossings alternate over and under around each link component, and a link is alternating if it admits an alternating diagram. The opening question due to Fox seeks a characterization of alternating links in terms intrinsic to the link complement [Lic97, p.32]. We establish such a characterization here in terms of definite spanning surfaces.

To describe it, a compact surface in a $\mathbb{Z}/2\mathbb{Z}$ homology sphere carries a natural pairing on its ordinary first homology group, mildly generalizing a definition by Gordon and Litherland [GL78, Section 2]. An alternating diagram of a non-split alternating link in S^3 yields an associated pair of black and white chessboard spanning surfaces for the link, and their pairings are respectively negative and positive definite. We establish the following converse:

Theorem 1.1. *Let L be a link in a $\mathbb{Z}/2\mathbb{Z}$ homology sphere with irreducible complement, and suppose that it bounds both a negative definite surface and a positive definite surface. Then L is a non-split alternating link in S^3, and it has an alternating diagram whose associated chessboard surfaces are isotopic rel boundary to the two given surfaces.*

The characterization given in Theorem 1.1 is compelling in that it leads to a geometric proof of part of Tait's conjectures, amongst other applications.

Theorem 1.2. *Any two connected, reduced, alternating diagrams of the same link have the same crossing number and writhe.*

A diagram is reduced if every crossing touches four distinct regions. Theorem 1.2 was originally proved independently by Kauffman, Murasugi, and Thistlethwaite (see [Kau87, Theorem 2.10], [Mur87, Theorem A] [Thi87, Theorem 1(i)]) for the crossing number and by Thistlethwaite [Thi88, Theorem 1] for the writhe. Their proofs used properties of the Jones and Kauffman polynomials, shortly following their discovery. By contrast, the proof we give is based on more classical topological constructions and some basic facts about flows on planar graphs.

A connected, oriented alternating diagram is special if one of the associated spanning surfaces is orientable. Seifert's algorithm outputs this surface when applied to such a diagram. An oriented alternating link is special if it has a special alternating diagram. Theorem 1.1 has the following straightforward consequence. The reverse direction follows from classic works by Crowell and Murasugi [Cro59, Mur58], while the forward direction was established more recently by Banks and Hirasawa-Sakuma using more intricate methods [Ban11, HS97].

Corollary 1.3. *A Seifert surface for a special alternating link L has minimum genus if and only if it is obtained by applying Seifert's algorithm to a special alternating diagram of L.*

Lidman (private communication) observed the following quick consequence of Theorem 1.1.

Corollary 1.4. *An amphichiral link with a definite spanning surface is alternating.*

Proof. If L bounds a definite surface S, then its mirror $\overline{L}$ bounds a definite surface $\overline{S}$ of the opposite sign. Assuming $L \simeq \overline{L}$, the result now follows from Theorem 1.1. ☐

Corollary 1.4 leads to another quick consequence. An almost-alternating diagram is one which becomes alternating upon changing one crossing.

Corollary 1.5. *An amphichiral knot that admits an almost-alternating diagram is alternating.*

Corollary 1.5 generalizes the fact that amphichiral Montesinos knots are alternating, which follows from the classification of Montesinos links [Bon79], [BZ03, Theorem 12.29 & Proposition 12.41].

Proof. The proof of Proposition 4.1 shows that for an almost-alternating link diagram, either the pairings on the two chessboard surfaces are singular, semidefinite, and have opposite signs, or else one of the chessboard surfaces is definite. The former cannot occur for a knot, since the discriminant of the pairing equals the knot determinant, which is non-zero. The result now follows from Corollary 1.4. ☐

András Juhász and Marc Lackenby applied Theorem 1.1 to the algorithmic detection of prime alternating knots. With their gracious permission, we include their result and proof.

Theorem 1.6. *There exists an algorithm that takes, as its input, a diagram of a prime knot K with c crossings, and determines whether K admits an alternating diagram. The running time is at most $\exp(kc^2)$ for some constant k.*

The decidability of whether a diagram presents an alternating knot was previously known. We thank the referee for suggesting the following line of argument. From the given diagram of K, compute the determinant $\det(K)$. By a theorem of Bankwitz [Ban30, Satz], if K has a reduced alternating diagram, then its crossing number is at most $\det(K)$. The algorithm begins by constructing all possible alternating diagrams with at most $\det(K)$ crossings. It then determines whether any of these is K, using the algorithm to decide whether two knots are equivalent, due to Haken [Hak68], Waldhausen [Wal78], Hemion [Hem79], and Matveev [Mat03]. An alternative method is to use the minimality of reduced alternating diagrams [Kau87, Mur87, Thi87] along with Coward and Lackenby's bound on the number of Reidemeister moves required to convert between two diagrams of a knot [CL14]. Both of these algorithms appear to take considerably longer than the bound in Theorem 1.6. The bound on the number of Reidemeister moves given by Coward and Lackenby is, as function of the initial crossing number c, a tower of exponentials with height κ^c for some constant κ, and so the running time for this algorithm is at least this large. There is no known explicit bound on the running time of the algorithm given by Haken, Waldhausen, Hemion, and Matveev, but it would seem to be hard to find a bound that was better than a tower of exponentials.

Developments. At the time of writing, Joshua Howie independently obtained a characterization of alternating knots in terms of spanning surfaces and applied it towards the decidability of recognizing an alternating knot exterior [How15]. Using Theorem 2.1, it is easy to deduce the equivalence of Howie's condition on spanning surfaces with the one given in Theorem 1.1 in the case the ambient manifold is S^3. The recognition algorithm that Howie produces is quite different, as it works with a pair of spanning surfaces simultaneously, whereas the proof of Theorem 1.6 tests surfaces for definiteness one by one.

Organization. Section 2 reviews the work of Gordon and Litherland on their eponymous pairing and its applications to link signatures, and then points out how their definition and results generalize to the case of a $\mathbb{Z}/2\mathbb{Z}$ homology sphere. Section 3 defines definite surfaces and collects their basic properties. Section 4 applies this preparatory material in order to prove Theorem 1.1 and deduce Corollary 1.3. Section 5 develops the elementary theory of flows on planar graphs in order to deduce Theorem 1.2 from Theorem 1.1. Finally, the Appendix, written by Juhász and Lackenby, contains the proof of Theorem 1.6.

Convention. We use integer coefficients for all chain groups and homology groups unless stated otherwise.

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2. The Gordon-Litherland pairing.

Generalizing earlier work by several researchers [Goe33, KT76, Sei35, Tro62], Gordon and Litherland defined a symmetric bilinear pairing on the ordinary first homology group of a compact embedded surface in S^3 [GL78]. We recall their definition and their main results, and then we promote their work to the setting of a $\mathbb{Z}/2\mathbb{Z}$ homology sphere.

Let $Y = S^3$ and $S \subset Y$ be a compact, connected, embedded surface. The unit normal bundle to S embeds as a subspace $N(S) \subset Y \setminus S$ and carries a 2-to-1 covering map

$$p_S : N(S) \to S.$$

Given pair of homology classes $a, b \in H_1(S)$, represent them by embedded, oriented multi-curves $\alpha, \beta \subset S$. Define

$$\langle a, b\rangle_S = \mathrm{lk}(\alpha, p_S^{-1}(\beta)),$$

where lk denotes the linking number. Gordon and Litherland proved that the pairing $\langle\, ,\rangle_S$ establishes a well-defined, symmetric, bilinear pairing

$$\langle\, ,\rangle_S : H_1(S) \times H_1(S) \to \mathbb{Z}$$

[GL78, Theorem 3 and Proposition 9]. When S is orientable, the pairing coincides with the symmetrized Seifert pairing.

They also showed how to use the pairing $\langle\, ,\rangle_S$ to determine the signature of a link. Suppose that S is a spanning surface for a link L, meaning that $L = \partial S$. The components $K_1, \dots, K_m$ of L define projective homology classes $[K_1], \dots, [K_m] \in H_1(S)/\pm$. For a projective class $x \in H_1(S)/\pm$, let $|x|_S = \langle x, x\rangle_S$ denote its well-defined self-pairing. The value $\frac{1}{2}|[K_i]|_S$ equals the framing that S induces on K_i. Let $e(S)$ denote the *Euler number* $-\frac{1}{2}\sum_{i=1}^m |[K_i]|_S$. If L is oriented, then let $e(S, L) = -\frac{1}{2}|[L]|_S$. The two quantities are related by the identity $e(S, L) = e(S) - \mathrm{lk}(L)$, where $\mathrm{lk}(L)$ denotes the total linking number $\sum_{i<j} \mathrm{lk}(K_i, K_j)$. Lastly, let $\sigma(S)$ denote the signature of the pairing $\langle\, ,\rangle_S$.

Gordon and Litherland's result reads as follows in the case that $Y = S^3$ [GL78, Corollaries $5'$ and $5''$]. As we discuss below, it pertains more generally to the case of $\mathbb{Z}/2\mathbb{Z}$ homology sphere Y.

Theorem 2.1. *If S is a compact spanning surface for an unoriented link $L \subset Y$, then the quantity*

$$\sigma(S) + \frac{1}{2}e(S)$$

depends only on L, and it coincides with the Murasugi invariant $\xi(L)$ when $Y = S^3$. If L is oriented, then

$$\sigma(S) + \frac{1}{2}e(S, L)$$

depends only on L, and it coincides with the link signature $\sigma(L)$ when $Y = S^3$. □

The Murasugi invariant $\xi(L)$ is the average of the signatures of the different oriented links whose underlying unoriented link is L. Note that if S is a Seifert surface for an oriented link L, then $[L] = 0 \in H_1(S)/\pm$ and $\langle\, ,\rangle_S$ coincides with the symmetrized Seifert pairing. We therefore recover the familiar definition of the link signature in this case.

Now we turn to the case in which Y is an arbitrary $\mathbb{Z}/2\mathbb{Z}$ homology sphere. The preceding summary carries over to this setting, and we highlight the necessary alterations. The key modification is that in an oriented rational homology sphere Y, a pair of disjoint, oriented curves $K_1, K_2 \subset Y$ have a *rational* linking number $\mathrm{lk}(K_1, K_2)$. To describe it, take a rational Seifert surface S_1 whose boundary runs $q > 0$ times around K_1 and meets K_2 transversely.

Then set $\mathrm{lk}(K_1, K_2) = (S_1 \cdot K_2)/q$. A standard argument shows that this value is independent of the choice of rational Seifert surface, it is symmetric in K_1 and K_2, and it extends by linearity to a $\mathbb{Q}$-valued function on pairs of disjoint, oriented links in Y.

The Gordon-Litherland pairing in this setting is a pairing

$$H_1(S) \times H_1(S) \to \mathbb{Q}$$

defined exactly as above with respect to the rational linking number. The proof that it is well-defined and bilinear is straightforward, and the proof of [GL78, Proposition 9] applies directly to show that it is symmetric. The remaining definitions that go into the statement of Theorem 2.1 also apply directly to this setting without change. The proof of Theorem 2.1 given in [GL78, Section 6] applies as well, with two important notes. First, the notion of S^*-equivalence of spanning surfaces carries over without change, as does the proof of [GL78, Proposition 10], using the rational linking number. Second, Proof I of [GL78, Theorem 11] only uses the fact that S^3 is a $\mathbb{Z}/2\mathbb{Z}$ homology sphere. The salient point is that, in the notation of that proof, $V_0 \cup V_1$ is a (mod 2) 2-cycle, so there exist (mod 2) 3-chains $Y_0, Y_1 \subset S^3$ such that $S^3 = Y_0 \cup Y_1$ and $Y_0 \cap Y_1 = \partial Y_0 = \partial Y_1 = V_0 \cup V_1$. This decomposition is used implicitly in the assertions made there about the subspaces M and M'. As the same holds for any $\mathbb{Z}/2\mathbb{Z}$ homology sphere Y, the proof adapts simply by substituting Y for S^3.

We may take the invariant values $\xi(L)$ and $\sigma(L)$ appearing in Theorem 2.1 as the natural generalizations of the Murasugi invariant and the oriented link signature of a link L in a $\mathbb{Z}/2\mathbb{Z}$ homology sphere Y such that $[L] = 0 \in H_1(Y; \mathbb{Z}/2\mathbb{Z})$. We mention in closing that signatures of oriented links in rational homology spheres were studied in greater generality by Cha and Ko in [CK02].

3. Definite surfaces.

A compact, connected surface S in a $\mathbb{Z}/2\mathbb{Z}$ homology sphere is *definite* (either positive or negative) if its Gordon-Litherland pairing is.

Proposition 3.1. *If S is a definite surface with boundary L, then $b_1(S)$ is minimal over all spanning surfaces for L with the same Euler number as S. Moreover, if S' is such a surface with $b_1(S) = b_1(S')$, then S' is definite and of the same sign as S.*

Proof. The Gordon-Litherland formula implies that all such surfaces have the same signature, whose absolute value therefore bounds from below the first Betti number of any such surface. By definition, this bound is attained by a definite surface. □

Corollary 3.2. *A definite surface is incompressible.* □

Following Proposition 4.1 below, Corollary 3.2 generalizes [MT93, Prop 2.3], which treats the case of a chessboard surface associated with a reduced alternating diagram of a link in S^3.

Lemma 3.3. *If S is definite and $S' \subset S$ is a compact subsurface with connected boundary, then S' is definite.*

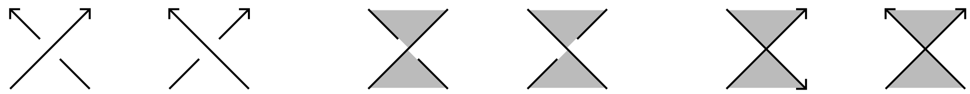

FIGURE 1. A positive and a negative crossing in an oriented link diagram; a type a and a type b crossing in a colored link diagram; and a type I and a type II crossing in an oriented, colored link projection.

Proof. Since S is definite, any compact subsurface S'' is semidefinite: the self-pairings of its homology classes take only one sign, and the self-pairing vanishes precisely on the kernel of the inclusion-induced map $H_1(S'') \to H_1(S)$. Since S and $\partial S'$ are connected, S/S' is a (possibly empty) connected surface with boundary, so $0 = H_2(S/S') \approx H_2(S, S')$. The long exact sequence of the pair (S, S') now shows that the inclusion-induced map $H_1(S') \to H_1(S)$ injects. It follows that S' is definite. $\square$

Lemma 3.4. *Suppose that Y is a $\mathbb{Z}/2\mathbb{Z}$ homology sphere, $L \subset Y$ is a link, $X = Y \setminus \mathring{\nu}(L)$ is irreducible, and $S_\pm \subset Y$ are $\pm$-definite spanning surfaces for L. If $S_+ \cap X$ and $S_- \cap X$ are in minimal position, then $S_+ \cap S_- \cap X$ does not contain a simple closed curve of intersection.*

Proof. Suppose that $S_+ \cap S_- \cap X$ contains a simple closed curve γ. The tubular neighborhood $\nu(\gamma)$ is a disk bundle over S^1, and it is orientable, since Y is a $\mathbb{Z}/2\mathbb{Z}$ homology sphere. Thus, $\partial\nu(\gamma)$ is a torus, and $S_+ \cap \partial\nu(\gamma)$ and $S_- \cap \partial\nu(\gamma)$ are parallel on $\partial\nu(\gamma)$. Moreover, $S_\pm \cap \partial\nu(\gamma)$ is isotopic to $p^{-1}_{S_\mp}(\gamma)$ in $Y \setminus \gamma$. It follows that $0 \leq |\gamma|_{S_+} = |\gamma|_{S_-} \leq 0$. Therefore, γ is null-homologous in both S_+ and S_-. Let $S'_\pm \subset S_\pm$ denote the orientable subsurfaces with $\partial S'_\pm = \gamma$. These surfaces are respectively positive and negative definite by Lemma 3.3, and $\sigma(S'_+) = \sigma(S'_-) = \sigma(\gamma)$ because they are Seifert surfaces for the knot $\gamma \subset Y$. Therefore, $0 \leq b_1(S'_+) = \sigma(S'_+) = \sigma(S'_-) = -b_1(S'_-) \leq 0$, so S'_+ and S'_- are disks. By passing to an innermost disk, we may assume that S'_+ and S'_- have disjoint interiors, so their union is a sphere. Since X is irreducible, the sphere $S'_+ \cup S'_-$ bounds a ball in X. This ball guides an isotopy that reduces the number of components of $S_+ \cap S_- \cap X$, so $S_+ \cap X$ and $S_- \cap X$ were not in minimal position. The conclusion of the Lemma now follows. $\square$

4. PROOF OF THE CHARACTERIZATION.

The following Proposition characterizes alternating *diagrams* in terms of the definiteness of their associated chessboard surfaces. It plays a role in the proof of Theorem 1.1 to follow.

Proposition 4.1. *Let D denote a connected diagram of a link L, and let B and W denote its associated chessboard surfaces. Then D is alternating if and only if B and W are definite surfaces of opposite signs.*

Proof. Orient D. Referring to Figure 1, the value $\frac{1}{2}e(B, L)$ equals the number of crossings that are both of type b and type II minus the number of crossings that are both of type a and type II [GL78, Lemma 7]. Similarly, $\frac{1}{2}e(W, L)$ equals that number of crossings that are both of type a and type I minus the number of crossings that are both of type b and type I. Let

$a(D)$ and $b(D)$ denote the number of type a and type b crossings, respectively. It follows that

$$b(D) - a(D) = \frac{1}{2}e(B, L) - \frac{1}{2}e(W, L).$$

On the other hand, Theorem 2.1 gives

$$\frac{1}{2}e(B, L) - \frac{1}{2}e(W, L) = \sigma(W) - \sigma(B).$$

Taking the absolute value leads to

$$|b(D) - a(D)| = |\sigma(W) - \sigma(B)| \leq |\sigma(W)| + |\sigma(B)| \leq b_1(W) + b_1(B) = c(D),$$

where $c(D)$ denotes the crossing number of D. The last equality follows from an Euler characteristic calculation. Equality holds in the first inequality if and only if $\sigma(W)$ and $\sigma(B)$ have opposite signs, and equality holds in the second inequality if and only W and B are definite. Therefore, $|b(D) - a(D)| = c(D)$ if and only if W and B are definite and of opposite signs. On the other hand, this equality holds if and only if the connected diagram D is alternating. The statement of the Proposition now follows. □

Proof of Theorem 1.1. As in Lemma 3.4, set $X = Y \setminus \overset{\circ}{\nu}(L)$ and put $S_+ \cap X$ and $S_- \cap X$ in minimal position. Write $\partial X = \partial_1 X \cup \cdots \cup \partial_m X$ corresponding to $L = K_1 \cup \cdots \cup K_m$. The number of points of intersection in $S_+ \cap S_- \cap \partial_i X$ equals the difference in framings $\frac{1}{2}|[K_i]|_{S_+} - \frac{1}{2}|[K_i]|_{S_-}$. We stress that this difference is non-negative, due to the signs of the surfaces. The number of arc components of $S_+ \cap S_- \cap X$ equals half the sum of these differences, which is $c := \frac{1}{2}e(S_-) - \frac{1}{2}e(S_+)$.

An orientation on X induces an orientation on ∂X, and an orientation on each link component K_i induces orientations on $S_+ \cap \partial_i X$ and $S_- \cap \partial_i X$. Every intersection point between $S_+ \cap \partial X$ and $S_- \cap \partial X$ on ∂X has the same sign with respect to these orientations, since $\frac{1}{2}|[K_i]|_{S_+} - \frac{1}{2}|[K_i]|_{S_-} \geq 0$ for all i. An arc component of $S_+ \cap S_- \cap X$ extends to an arc $a \subset S_+ \cap S_-$ such that $a \cap L = \partial a$. Let A denote the union of these c arcs. It follows from the consistency of the signs of intersection that a neighborhood $\nu(a)$ is modeled on the neighborhood of a crossing in a link diagram, where the chessboard surfaces meet along an arc that runs between the over and under crossing. In particular, $\nu(a)$ has a product structure $D^2 \times I$ such that a is contained in $\{0\} \times I$ and the projection to D^2 maps $(S_+ \cup S_- - a) \cap \nu(a)$ homeomorphically to $D^2 - \{0\}$.

By Lemma 3.4, $S_+ \cap S_- \cap X$ does not contain any simple closed curves. Therefore, the 2-complex $S_+ \cup S_-$ is a 2-manifold away from A. From the decomposition of $\nu(S_+ \cup S_-)$ as the union $\nu(A) \cup \nu(S_+ \cup S_- - A)$, we see that $\nu(S_+ \cup S_-)$ can be identified with $\nu(S) \approx S \times I$ for some closed embedded surface $S \subset Y$. Moreover, the projection $\nu(S) \to S$ maps each arc $a \subset A$ to a distinct point in S, and it maps $S_+ \cup S_- - A$ homeomorphically to the complement of these points in S.

The intersection $S_+ \cap S_- = L \cup A$ has Euler characteristic $-c$. As in the proof of Proposition 4.1, Theorem 2.1 gives $c = \sigma(S_+) - \sigma(S_-) = b_1(S_+) + b_1(S_-)$. Thus,

$$\chi(S_+ \cup S_-) = \chi(S_+) + \chi(S_-) - \chi(S_+ \cap S_-) = (1 - b_1(S_+)) + (1 - b_1(S_-)) + c = 2,$$

and

$$\chi(S) = \chi(\nu(S)) = \chi(\nu(S_+ \cup S_-)) = \chi(S_+ \cup S_-) = 2$$

in turn. Since Y is orientable and $\chi(S) > 0$, it follows that S contains a sphere component S_0. The neighborhood $\nu(S_0) \subset \nu(S)$ meets L in a non-empty sublink L_0. Since $Y \setminus L$ is irreducible, each boundary component of $\nu(S_0) \approx S_0 \times I$ bounds a ball in $Y \setminus L$. Denoting the balls by B_1 and B_2, we obtain $Y = B_1 \cup \nu(S_0) \cup B_2 \approx S^3$, $L = L_0$, and $S = S_0$ is a sphere.

We now see that the projection $\nu(S) \to S$ gives a diagram D of L. The c double points of D are the images of the components of A, and the chessboard surfaces are isotopic rel boundary to S_+ and S_-. Since S_+ and S_- are definite and of opposite signs, Proposition 4.1 implies that D is alternating, and the characterization is complete. □

Proof of Corollary 1.3. Let L be a special alternating link, S a minimum genus Seifert surface for L, D a special alternating diagram of L, and S_D the surface obtained by applying Seifert's algorithm to D. Then S_D is one of the spanning surfaces associated with D, so it is definite by Proposition 4.1. Since $e(S) = e(S_D) = 0$ and S has minimum genus, Proposition 3.1 implies that $b_1(S) = b_1(S_D)$ and that S is definite. It follows that S_D has minimum genus, and by Theorem 1.1, that S is a spanning surface associated with some (special) alternating diagram of L. □

5. Lattices, graphs, and Tait's conjecture.

Let D denote a connected alternating diagram of a link L. Color its regions according to the convention that every crossing has type b. Let B and W denote its associated chessboard surfaces. By the proof of Proposition 4.1, B is negative definite and W is positive definite. The Gordon-Litherland pairing on either surface admits a natural interpretation as the lattice of integer-valued flows on a graph, as we now recall.

The surface W deformation retracts onto an undirected graph G that has a vertex in each white region and an edge through each crossing of D. This is the *Tait graph* of D. It has a planar embedding determined up to planar isotopy by D, and $c(D) = |E(G)|$. By the same construction, the surface B deformation retracts onto the planar dual G^*. If D is a connected diagram, then G is connected as well. The diagram D is reduced if and only if both of G and G^* are bridgeless.

Orient the edges of G arbitrarily to endow it with the structure of a 1-dimensional CW-complex. The chain group $C_1(G)$ inherits the structure of a standard Euclidean lattice by declaring the chosen oriented edge set to form a distinguished orthonormal basis. The *flow lattice* $F(G)$ is the sublattice $\ker(\partial) \subset C_1(G)$, where $\partial : C_1(G) \to C_0(G)$ denotes the boundary operator. Since $C_2(G) = 0$, we can identify the underlying abelian group of $F(G)$ with $H_1(G)$. Observe that $F(G)$ is an invariant of the undirected graph G and does not depend on the orientation of the edges chosen to construct it. The deformation retraction from W to G induces an isomorphism $H_1(W) \approx H_1(G)$. Gordon and Litherland showed that this isomorphism induces an isometry of lattices [GL78, Theorem 1]:

Theorem 5.1. *Let D denote an alternating diagram, W its white chessboard surface, and G its Tait graph. Then the deformation retraction from W to G induces an isometry between $(H_1(W), \langle\,,\rangle_W)$ and $F(G)$.* □

Similarly, the deformation retraction from B to G^* induces an isometry between $(H_1(B), -\langle\,,\rangle_B)$ and $F(G^*)$; we stress the negative sign taken on the intersection pairing on $H_1(B)$.

We obtain the following addendum to Theorem 1.1.

Corollary 5.2. *The surfaces stipulated in Theorem 1.1 do not contain a homology class of self-pairing ± 1 if and only if the alternating diagram guaranteed by Theorem 1.1 is reduced.*

Proof. The elements of self-pairing 1 in $F(G)$ are the loops in G. A loop in G is dual to a bridge in G^*. Therefore, D is reduced if and only if $H_1(B)$ and $H_1(W)$ do not contain elements of self-pairing ± 1. □

An element v in a positive definite lattice L is *simple* if $\langle v, x\rangle \le \langle x, x\rangle$ for all $x \in L$, and it is *irreducible* if it is simple and $\langle v, x\rangle = \langle x, x\rangle$ if and only if $x = 0$ or v. Irreducibility and simplicity are isometry invariants.

A *cycle* in G is a subgraph homeomorphic to S^1. Cyclically orienting its edges gives an *oriented cycle*, which we may view as an element in $F(G)$. An *Eulerian subgraph* of G is an edge-disjoint union of cycles. Cyclically orienting the edges of the cycles in a decomposition of an Eulerian subgraph gives an *oriented Eulerian subgraph*, which we may again view as an element of $F(G)$.

The following result is elementary. The first assertion appears as [GR01, Theorem 14.14.4], and the second follows as well from its proof. Again, we stress that its statement holds for an undirected graph.

Proposition 5.3. *The irreducible elements in $F(G)$ are the oriented cycles in G, and the simple elements in $F(G)$ are the oriented Eulerian subgraphs of G.* □

Given Proposition 5.3, the proof of the following Lemma is elementary and left to the reader.

Lemma 5.4. *Suppose that C_i and C_j are oriented cycles in a graph. The following are equivalent:*

(1) $C_i + C_j$ is simple;
(2) C_i and C_j induce opposite orientations on every edge in $C_i \cap C_j$;
(3) $|E(C_i) \cap E(C_j)| = -\langle C_i, C_j\rangle$. □

Theorem 5.5. *If G and G' are connected, bridgeless planar graphs with isometric flow lattices, then $|E(G)| = |E(G')|$.*

In fact, more is true: G and G' are *2-isomorphic*, in the terminology of [Whi33]. This stronger assertion, which holds without assumption on planarity, is sometimes called the discrete Torelli theorem, and versions of it appear in [Art06, BdlHN97, CV10, Gre13, SW10]. The assumption on planarity and the weaker conclusion in Theorem 5.5 lead to the simple proof that we present.

Proof. An orientation on S^2 induces an orientation on the faces of G. Since G is connected and bridgeless, their oriented boundaries form a collection of oriented cycles $C_1, \ldots, C_f \subset G$. They generate $F(G)$ subject to the single relation $C_1 + \cdots + C_f = 0$. Since G is bridgeless, each oriented edge occurs once in some C_i, and we have

$$|E(G)| = \sum_{i<j} |E(C_i) \cap E(C_j)|.$$

It follows as well from Lemma 5.4 that $C_i + C_j$ is simple for all $i \neq j$.

Suppose that $F(G) \xrightarrow{\sim} F(G')$ is an isometry. The elements $C_1, \ldots, C_f$ are irreducible, so their images are oriented cycles $C'_1, \ldots, C'_f \subset G'$, and $C'_i + C'_j$ is simple for all $i \neq j$. It follows that no three distinct cycles C'_i, C'_j, C'_k have an edge in common, since two of them would have to induce the same orientation on it, in violation of Lemma 5.4. Therefore,

$$\sum_{i<j} |E(C'_i) \cap E(C'_j)| \leq |E(G')|.$$

On the other hand, Lemma 5.4 gives

$$\sum_{i<j} |E(C_i) \cap E(C_j)| = \sum_{i<j} -\langle C_i, C_j \rangle = \sum_{i<j} -\langle C'_i, C'_j \rangle = \sum_{i<j} |E(C'_i) \cap E(C'_j)|.$$

Combining the indented equations yields $|E(G)| \leq |E(G')|$. By symmetry, the statement of the Theorem follows. □

Proof of Theorem 1.2. Let D and D'' denote two connected, reduced, alternating diagrams of the same link L. Color them according to the convention that every crossing has type b. Let W denote the white chessboard surface for D and B'' the black chessboard surface for D''. By Theorem 1.1 and Corollary 5.2, there exists a reduced, alternating diagram D' whose white chessboard surface W' is isotopic to W and whose black chessboard surface B' is isotopic to B''. Let G denote the Tait graph of D and G' the Tait graph of D'. Since $W \simeq W'$, it follows from Theorem 5.1 that $F(G) \approx F(G')$. By Theorem 5.5, it follows that $c(D) = |E(G)| = |E(G')| = c(D')$. Similarly, $c(D') = c(D'')$, and the first part of the Theorem follows.

For the second part, orient the diagrams. Let $p(\cdot)$ and $n(\cdot)$ denote the number of positive and negative crossings in a diagram, respectively. By Theorem 2.1, [GL78, Lemma 7], and the fact that every crossing has type b, it follows that $\sigma(L) = \sigma(W) - p(D) = \sigma(W') - p(D')$. Since $W \simeq W'$, it follows that $p(D) = p(D')$. Since $c(D) = c(D')$ by the first part of the Theorem, it follows that $n(D) = n(D')$, as well. Therefore, D and D' have the same writhe $p(D) - n(D) = p(D') - n(D')$. Similarly, D' and D'' have the same writhe, and the second part of the Theorem follows. □

As remarked after the proof of Theorem 5.5, the graphs G and G' that arise in the proof of Theorem 1.2 are 2-isomorphic. It follows that Theorem 1.2 can be strengthened to the statement that two connected, reduced, alternating diagrams of the same link are actually mutants [Gre13, Proposition 4.4]. This conclusion follows as well from [Gre13, Theorem 1.1] under a weaker assumption, yet its proof relies on gauge theory in the guise of Floer homology or Donaldson's theorem.

6. Appendix: Algorithmic Detection of Alternating Knots. By András Juhász and Marc Lackenby

This appendix contains the proof of Theorem 1.6.

We are given a diagram D for a prime knot K with c crossings. If the knot K is alternating, then it has a reduced alternating diagram having $c' \leq c$ crossings, by a theorem of Kauffman, Murasugi, and Thistlethwaite [Kau87, Mur87, Thi87]. The chessboard surfaces S_1 and S_2 for this diagram have the following properties:

(1) $\chi(S_1) + \chi(S_2) = 2 - c' \geq 2 - c$;
(2) the Gordon-Litherland pairings of S_1 and S_2 are positive definite and negative definite, respectively;
(3) they are connected and π_1-injective, hence incompressible and boundary-incompressible [Aum56].

Conversely, by Theorem 1.1, the existence of spanning surfaces S_1 and S_2 satisfying (2) imply that the knot is alternating. We need to show how to find these surfaces.

We start by using D to construct a triangulation T of X, the exterior of K. The method of Hass, Lagarias and Pippenger [HLP99, Lemma 7.1] is convenient here. In that lemma, they construct a triangulated polyhedron in $\mathbb{R}^3$. The number of tetrahedra in this triangulation is bounded above by a linear function of c. Each simplex is straight in $\mathbb{R}^3$. A copy of K is a subset of the 1-skeleton lying in the interior of the polyhedron, and in fact, its vertical projection onto the x-y plane is a copy of D. According to [HLP99, Lemma 7.1], this triangulation can be constructed in time bounded above by a linear function of $c \log c$. One can then enlarge this polyhedron, by attaching simplices, so that its boundary is equal to the boundary of a tetrahedron. This can be achieved while maintaining a linear bound on the number of simplices, as a function of c, and keeping the property that each simplex is straight in $\mathbb{R}^3$. The running time for this step is bounded by a polynomial function of c. Then one can attach a 3-simplex to the boundary of this polyhedron to obtain a triangulation of the 3-sphere. By taking the second derived subdivision of this triangulation and then removing the interior of simplices incident to K, we obtain the required triangulation T. Note that its simplices are also all straight in $\mathbb{R}^3$, with the exception of one 3-simplex enclosing the point at infinity. It is also simple to arrange that the projection of each 2-simplex to the diagram D is an injection. In addition, a meridian of K can be realised as a subset Γ of the 1-skeleton.

We will consider surfaces that are normal with respect to T. We therefore recall various terms from normal surface theory. A much more complete survey can be found in the book of Matveev [Mat03]. A surface is *normal* if it intersects each tetrahedron in a collection of *triangles* and *squares*, as shown in Figure 3.10 of [Mat03] for example. In each tetrahedron, there are four possible types of triangle and three types of square. A properly embedded normal surface S is specified by its *vector* which simply lists the number of triangles and squares of each type appearing in S. The number of co-ordinates in this vector is therefore 7 times the number of tetrahedra. A vector with non-negative integral co-ordinates represents a properly embedded normal surface if and only if it satisfies a collection of linear equations, known as the *matching equations* [Mat03, Section 3.3.4], and it satisfies the *quadrilateral constraints* which require that no two distinct square types co-exist within the same tetrahedron. A normal surface S is said to be the *sum* of two other normal surfaces if the vector of S is the sum of the vectors of these other surfaces. A normal surface is said to be *fundamental* if it cannot be written as a sum of two non-empty normal surfaces.

We will need the following lemma.

Lemma 6.1. *The surfaces S_1 and S_2 can be realised as normal surfaces with respect to T. Each is a sum of at most c fundamental normal surfaces. The number of normal triangles and squares in S_1 and S_2 is at most $e^{k_1 c}$ for some constant k_1.*

Proof. This is a fairly well-known application of normal surface theory. Recall that $\Gamma \subset \partial X$ is a subset of the 1-skeleton representing a meridian of K. We give the manifold X the boundary pattern Γ, in the sense of [Mat03, Definition 3.3.9]. Then (X, Γ) is simple in the sense of [Mat03, Definition 6.3.16], i.e., it is irreducible, boundary irreducible, and contains no essential tori and annuli (where all these terms are to be interpreted in a suitable sense in the presence of boundary pattern). This is because of Menasco's theorem that the exterior of a prime alternating knot contains no essential torus and the only essential annuli arise as the obvious annuli for a $(2, n)$ torus knot, but these necessarily intersect Γ [Men84, Corollary 2]. In [Mat03, Definition 6.3.8], Matveev defines the p-complexity of the normal surface S_i to be $-\chi(S_i)+|S_i \cap \Gamma| = -\chi(S_i)+1$, and this is at most a linear function of c, by (1) above. By [Mat03, Theorem 6.3.17], one can construct a finite list of normal surfaces with the property that any 2-sided properly embedded incompressible, boundary-incompressible, connected surface with at most this p-complexity is strongly equivalent to one in this list. Here, strongly equivalent just means that there is a homeomorphism of the pair (X, Γ) taking it to one of these normal surfaces. Now, the surfaces S_i need not be 2-sided, but the proof of [Mat03, Theorem 6.3.17] gives that S_i is a sum $\sum \lambda_j F_j$, where each F_j is a fundamental normal surface with non-negative p-complexity and each λ_j is a positive integer. In fact, the only F_j that appear in the sum have positive p-complexity. (No normal tori appear in the sum, using [Mat03, Proposition 6.3.21].) Hence, the number of summands for S_i is at most the p-complexity, which is at most c by (1). By a result of Hass and Lagarias [HL01, Lemma 3.2], the number of triangles and squares in a fundamental normal surface is at most $e^{k_2 t}$, where t is the number of tetrahedra and k_2 is a constant. Hence, this gives the final part of the Lemma. □

Proof of Theorem 1.6. The algorithm simply constructs all the normal surfaces in T consisting of at most $e^{k_1 c}$ triangles and squares, as in Lemma 6.1. The number of triangle and square types in T is 7 times the number of tetrahedra, and hence at most a linear function of c. Therefore, the number of possible normal surfaces we must consider is at most $\exp(k_3 c^2)$ for some constant k_3.

For each surface S_i, one has an explicit decomposition of the surface into triangles and squares, and hence an explicit cell structure. Using this, one can decide whether the surface has boundary a single curve with integral slope, and hence whether it extends to a spanning surface for K. The running time for this is bounded by a polynomial function of the number of 1-cells in ∂S_i and of the number of triangles in the triangulation of ∂X, and therefore by $e^{k_4 c}$ for some constant k_4. If S_i does not extend to a spanning surface, the algorithm discards this surface. Also, if S_i is disconnected or does not satisfy $|\chi(S_i)| \leq c$, then it is discarded.

Using the cell structure, one can find a spanning set for $H_1(S_i)$ in the 1-skeleton of S_i, by picking a maximal tree in the 1-skeleton, and then using loops that each run in the tree from a basepoint to the start of an edge not in the tree, then along the edge, and then back along the tree to the basepoint. The running time for this is bounded above by a polynomial function of the number of 1-cells in S_i. One can then reduce this spanning set of $H_1(S_i)$ to a basis for $H_1(S_i)$ using linear algebra. The running time is bounded by a polynomial function of the number of elements of the spanning set, the number of 1-cells and 2-cells of S_i, and of the number of digits of each co-ordinate of the vector representing S_i. Hence, the running time is at most $e^{k_5 c}$, for some constant k_5. The size of this basis is at most a linear function of c, because of (1). Each basis element is an embedded loop in the 1-skeleton of S_i, and hence lies within the 2-skeleton of the triangulation T.

We build a matrix representing the Gordon-Litherland pairing, with entries constructed as follows. We consider all pairs of loops α and β in the given basis for $H_1(S_i)$ and compute the linking number of α and $p_{S_i}^{-1}(\beta)$, in the terminology of Section 2. We can compute this linking number by projecting α and $p_{S_i}^{-1}(\beta)$ to the diagram D and counting their crossings with appropriate signs. Each 1-cell of $\alpha \cup p_{S_i}^{-1}(\beta)$ projects to a straight arc in the diagram, and so the number of crossings that we must count is at most the product of the number of 1-cells of α and the number of 1-cells of $p_{S_i}^{-1}(\beta)$. Hence, $e^{k_6 c}$, for a suitable constant k_6, is an upper bound for the modulus of the linking number and for the running time of this part of the algorithm.

Then we can determine, in time that is at most a polynomial function of c, whether this matrix representing the Gordon-Litherland pairing is positive or negative definite. By considering all properly embedded normal surfaces in T satisfying the bound of Lemma 6.1 and that have boundary a single curve with integral slope, we can therefore determine whether K has both positive definite and negative definite spanning surfaces. □

We plan to address the case of composite knots in a future paper. There, we will give an exponential-time algorithm that determines whether a (possibly composite) knot admits an alternating diagram.

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