

THE ASYMPTOTIC BEHAVIOUR OF HEEGAARD GENUS

MARC LACKENBY

1. INTRODUCTION

Heegaard splittings have recently been shown to be related to a number of important conjectures in 3-manifold theory: the virtually Haken conjecture, the positive virtual b_1 conjecture and the virtually fibred conjecture [3]. Of particular importance is the rate at which the Heegaard genus of finite-sheeted covering spaces grows as a function of their degree. This was encoded in the following definitions.

Let M be a compact orientable 3-manifold. Let $\chi_-^h(M)$ be the negative of the maximal Euler characteristic of a Heegaard surface for M . Let $\chi_-^{sh}(M)$ be the negative of the maximal Euler characteristic of a strongly irreducible Heegaard surface for M , or infinity if such a surface does not exist. Define the *infimal Heegaard gradient* of M to be

$$\inf_i \left\{ \frac{\chi_-^h(M_i)}{d_i} : M_i \text{ is a degree } d_i \text{ cover of } M \right\}.$$

The *infimal strong Heegaard gradient* of M is

$$\liminf_i \left\{ \frac{\chi_-^{sh}(M_i)}{d_i} : M_i \text{ is a degree } d_i \text{ cover of } M \right\}.$$

The following conjectures were put forward in [3]. According to Theorem 1.7 of [3], either of these conjectures, together with a conjecture of Lubotzky and Sarnak [4] about the failure of Property (τ) for hyperbolic 3-manifolds, would imply the virtually Haken conjecture for hyperbolic 3-manifolds.

Heegaard gradient conjecture. *A compact orientable hyperbolic 3-manifold has zero infimal Heegaard gradient if and only if it virtually fibres over the circle.*

Strong Heegaard gradient conjecture. *Any closed orientable hyperbolic 3-manifold has positive infimal strong Heegaard gradient.*

Some evidence for these conjectures was presented in [3]. More precisely, suitably phrased versions of these conjectures were shown to be true when one

restricts attention to cyclic covers dual to a non-trivial element of $H_2(M, \partial M)$, to reducible manifolds and (in the case of the strong Heegaard gradient conjecture) to congruence covers of arithmetic hyperbolic 3-manifolds.

A less quantitative version of the conjectures is simply that $\chi_-^{sh}(M_i)$ cannot grow too slowly as a function of d_i , and that if $\chi_-^h(M_i)$ does grow sufficiently slowly, then M is virtually fibred. These expectations are confirmed in the following result, which is the main theorem of this paper.

Theorem 1. *Let M be a closed orientable 3-manifold that admits a negatively curved Riemannian metric. Let $\{M_i \rightarrow M\}$ be a collection of finite regular covers with degree d_i .*

- (1) *If $\chi_-^h(M_i)/\sqrt{d_i} \rightarrow 0$, then $b_1(M_i) > 0$ for all sufficiently large i .*
- (2) *$\chi_-^{sh}(M_i)/\sqrt{d_i}$ is bounded away from zero.*
- (3) *If $\chi_-^h(M_i)/\sqrt[4]{d_i} \rightarrow 0$, then M_i fibres over the circle for all sufficiently large i .*

A slightly weaker form of Theorem 1(1) appeared in [3] as Corollary 1.4, with essentially the same proof. It is included here in order to emphasise its connection to the other two results.

The following corollary of Theorem 1(3) gives a necessary and sufficient condition for M to be virtually fibred in terms of the Heegaard genus of its finite covers. We say that a collection $\{M_i \rightarrow M\}$ of finite covers has *bounded irregularity* if the normalisers of $\pi_1 M_i$ in $\pi_1 M$ have bounded index in $\pi_1 M$.

Corollary 2. *Let M be a closed orientable 3-manifold with a negatively curved Riemannian metric, and let $\{M_i \rightarrow M\}$ be its finite-sheeted covers with degree d_i . Then the following are equivalent:*

- (1) *M_i is fibred for infinitely many i ;*
- (2) *in some subsequence with bounded irregularity, $\chi_-^h(M_i)$ is bounded;*
- (3) *in some subsequence with bounded irregularity, $\chi_-^h(M_i)/\sqrt[4]{d_i} \rightarrow 0$.*

Proof. (1) \Rightarrow (2). If some finite-sheeted cover \tilde{M} is fibred, then so is any finite cyclic cover M_i of \tilde{M} dual to the fibre. The normaliser $N(\pi_1 M_i)$ of $\pi_1 M_i$ in $\pi_1 M$ contains $\pi_1 \tilde{M}$, so $[\pi_1 M : N(\pi_1 M_i)]$ is bounded and hence these covers have

bounded irregularity. Also, $\chi_-^h(M_i)$ is bounded by twice the modulus of the Euler characteristic of the fibre, plus four.

(2) \Rightarrow (3). This is trivial, since d_i must tend to infinity.

(3) \Rightarrow (1). Since $[\pi_1 M : N(\pi_1 M_i)]$ is bounded in this subcollection, we may pass to a further subsequence where $N(\pi_1 M_i)$ is a fixed subgroup of $\pi_1 M$. Let \tilde{M} be the finite-sheeted cover of M corresponding to this subgroup. Then, the covers $\{M_i \rightarrow M\}$ in this subsequence give a collection $\{M_i \rightarrow \tilde{M}\}$ of finite regular covers such that $\chi_-^h(M_i)/[\pi_1 \tilde{M} : \pi_1 M_i]^{1/4} \rightarrow 0$. By Theorem 1(3), M_i is fibred for all sufficiently large i . \square

2. BACKGROUND MATERIAL

GENERALISED HEEGAARD SPLITTINGS

A Heegaard splitting of a closed orientable 3-manifold can be viewed as arising from a handle structure. If one builds the manifold by starting with a single 0-handle, then attaching some 1-handles, then some 2-handles and then a 3-handle, the manifold obtained after attaching the 0- and 1-handles is handlebody, as is the closure of its complement. Thus, the boundary of this submanifold is a Heegaard surface. Generalised Heegaard splittings arise from more general handle structures: one starts with some 0-handles, then adds some 1-handles, then some 2-handles, then 1-handles, and so on, in an alternating fashion, ending with some 3-handles. One then considers the manifold embedded in M consisting of the 0-handles and the first j batches of 1- and 2-handles. Let F_j be the boundary of this manifold, but discarding any 2-sphere components that bound 0- or 3-handles. After a small isotopy, so that these surfaces are all disjoint, they divide M into compression bodies. In fact, the surfaces $\{F_j : j \text{ odd}\}$ form Heegaard surfaces for the manifold $M - \bigcup\{F_j : j \text{ even}\}$. We term the surfaces F_j *even* or *odd*, depending on the parity of j .

More details about generalised Heegaard splittings can be found in [7] and [6]. The following theorem summarises some of the results from [7].

Theorem 3. *From any minimal genus Heegaard surface F for a closed orientable irreducible 3-manifold M , other than S^3 , one can construct a generalised Heegaard*

splitting $\{F_1, \dots, F_n\}$ in M with the following properties:

1. F_j is incompressible and has no 2-sphere components, for each even j ;
2. F_j is strongly irreducible for each odd j ;
3. F_j and F_{j+1} are not parallel for any j ;
4. $|\chi(F_j)| \leq |\chi(F)|$ for each j ;
5. $|\chi(F)| = \sum (-1)^j \chi(F_j)$.

Corollary 4. *Let M , F and $\{F_1, \dots, F_n\}$ be as in Theorem 3. Suppose that, in addition, M is not a lens space. Let \overline{F} be the surface obtained from $\bigcup_j F_j$ by replacing any components that are parallel by a single component. Then*

1. $|\chi(\overline{F})| \leq |\chi(\bigcup_j F_j)| < |\chi(F)|^2$;
2. \overline{F} has at most $\frac{3}{2}|\chi(F)|$ components.

Proof. Note first that no component of $\bigcup_j F_j$ is a 2-sphere. When j is even, this is (1) of Theorem 3. The same is true when j is odd, since the odd surfaces form Heegaard surfaces for the complement of the even surfaces, and M is not S^3 . Hence, none of the compression bodies H in the complement of $\bigcup_j F_j$ is a 3-ball.

We claim also that no H is a solid torus. For if it were, consider the compression body to which it is adjacent. If this were a product, some even surface would be compressible, contradicting (1) of Theorem 3. However, if it was not a product, then it is a solid torus, possibly with punctures. If it has no punctures, then M is a lens space, contrary to assumption. If it does, then some component of an even surface would be a 2-sphere, again contradicting (1). This proves the claim.

We expand (5) of Theorem 3 as follows:

$$|\chi(F)| = \frac{-\chi(F_1)}{2} + \frac{\chi(F_2) - \chi(F_1)}{2} + \frac{\chi(F_2) - \chi(F_3)}{2} + \dots + \frac{-\chi(F_n)}{2}. \quad (*)$$

For any compression body H , other than a 3-ball, with negative boundary $\partial_- H$ and positive boundary $\partial_+ H$, $\chi(\partial_- H) - \chi(\partial_+ H)$ is even and non-negative. It is zero if and only if H is a product or a solid torus. Since F_j and F_{j+1} are not parallel for any j , each term in (*) is therefore at least one. So, $n+1$, the number

of terms on the right-hand side of (*), is at most $|\chi(F)|$. Hence,

$$|\chi(\bigcup_j F_j)| = \sum_j |\chi(F_j)| \leq n|\chi(F)| < |\chi(F)|^2.$$

The inequality $|\chi(\overline{F})| \leq |\chi(\bigcup_j F_j)|$ simply follows from the fact that we discard some components of $\bigcup_j F_j$ to form \overline{F} . This proves (1).

Now it is trivial to check that, for any compression body H , other than a 3-ball, solid torus or product, $|\partial H| \leq \frac{3}{2}(\chi(\partial_- H) - \chi(\partial_+ H))$. The number of components of \overline{F} is half the sum, over all complementary regions H of $\bigcup_j F_j$ that are not products, of $|\partial H|$. This is at most $\frac{3}{4}(\chi(\partial_- H) - \chi(\partial_+ H))$. But the sum, over all complementary regions H of $\bigcup_j F_j$, of $\frac{1}{2}(\chi(\partial_- H) - \chi(\partial_+ H))$ is the right-hand side of (*). Thus, we deduce that the number of components of \overline{F} is at most $\frac{3}{2}|\chi(F)|$, proving (2). \square

REALISATION AS MINIMAL SURFACES

One advantage of using generalised Heegaard splittings satisfying (1) and (2) of Theorem 3 is that minimal surfaces then play a rôle in the theory. The following theorem of Freedman, Hass and Scott [2] applies to the even surfaces.

Theorem 5. *Let S be an orientable embedded incompressible surface in a closed orientable irreducible Riemannian 3-manifold. Suppose that no two components of S are parallel, and that no component is a 2-sphere. Then there is an ambient isotopy of S so that afterwards each component is either a least area, minimal surface or the boundary of a regular neighbourhood of an embedded, least area, minimal non-orientable surface.*

We will apply the above result to the incompressible components of \overline{F} . If we cut M along these components, the remaining components form strongly irreducible Heegaard surfaces for the complementary regions. A theorem of Pitts and Rubinstein [5] now applies.

Theorem 6. *Let S_1 be a (possibly empty) embedded stable minimal surface in a closed orientable irreducible Riemannian 3-manifold M with a bumpy metric. Let S_2 be a strongly irreducible Heegaard surface for a complementary region of S_1 . Then there is an ambient isotopy, leaving S_1 fixed, taking S_2 to a minimal surface, or to the boundary of a regular neighbourhood of a minimal embedded*

non-orientable surface, with a tube attached that is vertical in the I -bundle structure on this neighbourhood.

Bumpy metrics were defined by White in [8]. After a small perturbation, any Riemannian metric can be made bumpy. Then we may ambiently isotope \bar{F} so that each component is as described in Theorems 5 and 6.

We will need some parts of the proof of Theorem 6, and not just its statement. Let X be the component of $M - S_1$ containing S_2 . Then, as a Heegaard surface, S_2 determines a sweepout of X . In any sweepout, there is a surface of maximum area, although it need not be unique. Let a be the infimum, over all sweepouts in this equivalence class, of this maximum area. Then Pitts and Rubinstein showed that there is a sequence of sweepouts, whose maximal area surfaces tend to an embedded minimal surface, and that the area of these surfaces tends to a . This minimal surface, or its orientable double cover if it is non-orientable, is isotopic to S_2 or to a surface obtained by compressing S_2 .

Now, when M is negatively curved, one may use Gauss-Bonnet to bound the area of this surface. Suppose that $\kappa < 0$ is the supremum of the sectional curvatures of M . Then, as the surface is minimal, its sectional curvature is at most κ . Hence, by Gauss-Bonnet, its area is at most $2\pi|\chi(S_2)|/|\kappa|$. Thus, we have the following result.

Addendum 7. *Let S_1, S_2 and M be as in Theorem 6. Suppose that the sectional curvature of M is at most $\kappa < 0$. Then, for each $\epsilon > 0$, there is a sweepout of the component of $M - S_1$ containing S_2 , equivalent to the sweepout determined by S_2 , so that each surface in this sweepout has area at most $(2\pi|\chi(S_2)|/|\kappa|) + \epsilon$.*

One has a good deal of geometric control over minimal surfaces when M is negatively curved. As observed above, their area is bounded in terms of their Euler characteristic and the supremal sectional curvature of M . In fact, by ruling out the existence of long thin tubes in the surface, one has the following.

Theorem 8. *There is a function $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ with the following property. Let M be a Riemannian 3-manifold, whose injectivity radius is at least $\epsilon/2 > 0$, and whose sectional curvature is at most $\kappa < 0$. Let S be a closed minimal surface in M . Then there is a collection of at most $f(\kappa, \epsilon)|\chi(S)|$ points in S , such that*

the balls of radius $f(\kappa, \epsilon)$ about these points cover S . (Here, distance is measured using the path metric on S .)

This is proved in Proposition 6.1 of [3]. More precisely, formulas (1) and (2) there give the result.

THE CHEEGER CONSTANT OF MANIFOLDS AND GRAPHS

The *Cheeger constant* of a compact Riemannian manifold M is defined to be

$$h(M) = \inf \left\{ \frac{\text{Area}(S)}{\min\{\text{Volume}(M_1), \text{Volume}(M_2)\}} \right\},$$

where S ranges over all embedded codimension one submanifolds that divide M into M_1 and M_2 .

A central theme of [3] is that the Cheeger constant of a 3-manifold and its Heegaard splittings are intimately related. One example of this phenomenon is the following result.

Theorem 9. *Let M be a closed Riemannian 3-manifold. Let $\kappa < 0$ be the supremum of its sectional curvatures. Then*

$$h(M) \leq \frac{4\pi \chi_-^h(M)}{|\kappa| \text{Volume}(M)}.$$

This is essentially Theorem 4.1 of [3]. However, there, $\chi_-^h(M)$ is replaced by $c_+(M)$, which is an invariant defined in terms of the generalised Heegaard splittings of M . But the above inequality follows from an identical argument. We briefly summarise the proof.

From a minimal genus Heegaard splitting of M , construct a generalised Heegaard splitting $\{F_1, \dots, F_n\}$ satisfying (1) to (5) of Theorem 3. Let \overline{F} be the surface obtained from $\bigcup_j F_j$ by discarding multiple copies of parallel components. Apply the isotopy of Theorem 5 to the incompressible components of \overline{F} . Each complementary region corresponds to a component of the complement of the even surfaces, and therefore contains a component of some odd surface F_j . Label this region with the integer j , and let M_j be the union of the regions labelled j . There is some odd j such that the volumes of $M_1 \cup \dots \cup M_{j-2}$ and $M_{j+2} \cup \dots \cup M_n$ are each at most half the volume of M . Now, $F_j \cap M_j$ forms a strongly irreducible

Heegaard surface for M_j . Applying Addendum 7, we find for each $\epsilon > 0$, a sweepout of M_j , equivalent to that determined by F_j , by surfaces with area at most $(2\pi|\chi(F_j)|/|\kappa|) + \epsilon$. But, $|\chi(F_j)| \leq \chi_-^h(M)$, by (4) of Theorem 3. Some surface in this sweepout divides M into two parts of equal volume. So, as ϵ was arbitrary,

$$h(M) \leq \frac{4\pi \chi_-^h(M)}{|\kappa| \text{Volume}(M)}.$$

In this paper, we will consider the Cheeger constants of regular finite-sheeted covering spaces M_i of M . Here, M_i is given the Riemannian metric lifted from M . It is possible to estimate $h(M_i)$ in terms of graph-theoretic data, as follows.

By analogy with the Cheeger constant for a Riemannian manifold, one can define the *Cheeger constant* $h(X)$ of a finite graph X . If A is a subset of the vertex set $V(X)$, ∂A denotes those edges with precisely one endpoint in A . Then $h(X)$ is defined to be

$$\inf \left\{ \frac{|\partial A|}{|A|} : A \subset V(X) \text{ and } 0 < |A| \leq |V(X)|/2 \right\}.$$

Proposition 10. *Let M be a compact Riemannian manifold. Let \mathcal{X} be a finite set of generators for $\pi_1 M$. Then there is a constant $k_1 \geq 1$ with the following property. If X_i is the Cayley graph of $\pi_1 M / \pi_1 M_i$ with respect to the generators \mathcal{X} , then*

$$k_1^{-1} h(X_i) \leq h(M_i) \leq k_1 h(X_i).$$

This is essentially contained in [1], but we outline a proof. Lemma 2.3 of [3] states that, if \mathcal{X} and \mathcal{X}' are two finite sets of generators for $\pi_1 M$, then there is a constant $k \geq 1$ with the following property. If X_i and X'_i are the Cayley graphs of $\pi_1 M / \pi_1 M_i$ with respect to \mathcal{X} and \mathcal{X}' , then

$$k^{-1} h(X_i) \leq h(X'_i) \leq k h(X_i).$$

Thus, for the purposes of proving Proposition 10, we are free to choose \mathcal{X} . We do this as follows. We pick a connected fundamental domain in the universal cover of M . The translates of this domain to which it is adjacent correspond to a finite set \mathcal{X} of generators for $\pi_1 M$. There is an induced fundamental domain in any

finite regular cover M_i of M . Its translates are in one-one correspondence with the group $\pi_1 M / \pi_1 M_i$. Two translates are adjacent if and only if one is obtained from the other by right-multiplication by an element in \mathcal{X} . Thus, the Cayley graph X_i should be viewed as a coarse approximation to M_i . Any subset A of $V(X_i)$, as in the definition of $h(X_i)$, therefore determines a decomposition of M_i . After a further modification, we may assume that this is along a codimension one submanifold. The existence of a constant k_1 such that $h(M_i) \leq k_1 h(X_i)$ is then clear. The other inequality is more difficult to establish. One needs to control the geometry of a codimension one submanifold S in M_i that is arbitrarily close to realising the Cheeger constant of M_i . This is achieved in the proof of Lemma 2 of [1].

CONSTRUCTING NON-TRIVIAL COCYCLES

Some new machinery has been developed in [3] that gives necessary and sufficient conditions on a finitely presented group to have finite index subgroups with infinite abelianisation. We describe some of the ideas behind this now.

Let C be a finite cell complex with a single 0-cell and in which every 2-cell is a triangle. Let G be its fundamental group, and let \mathcal{X} be the generators arising from the 1-cells. Associated with any finite index normal subgroup H_i of G , there is a finite-sheeted covering space C_i of C . Its 1-skeleton X_i is the Cayley graph of G/H_i with respect to \mathcal{X} . The following theorem is an expanded form of Lemma 2.4 of [3] and has exactly the same proof. It will play a key rôle in this paper.

Theorem 11. *Suppose that $h(X_i) < \sqrt{2/(3|V(X_i)|)}$. Let A be any non-empty subset of $V(X_i)$ such that $|\partial A|/|A| = h(X_i)$ and $|A| \leq |V(X_i)|/2$. Then there is a 1-cocycle c on C_i that is not a coboundary. Its support is a subset of the edges of ∂A , and it takes values in $\{-1, 0, 1\}$. As a consequence, H_i has infinite abelianisation.*

3. THE PROOF OF THE MAIN THEOREM

We start with a closed orientable 3-manifold M admitting a negatively curved Riemannian metric. After a small perturbation, we may assume that the metric is bumpy. Let $\kappa < 0$ be the supremum of its sectional curvatures. Pick a 1-vertex

triangulation T of M . The edges of T , when oriented in some way, form a set \mathcal{X} of generators for $\pi_1(M)$. Let K be the 2-skeleton of the complex dual to T .

We will consider a collection $\{M_i \rightarrow M\}$ of finite regular covers of M , having the properties of Theorem 1. In particular, we will assume (at least) that $\chi_-^h(M_i)/\sqrt{d_i} \rightarrow 0$. (Note that this is justified when proving Theorem 1(2), by passing to a subsequence, and using the fact that $\chi_-^h(M_i) \leq \chi_-^{sh}(M_i)$.) The triangulation T and 2-complex K lift to T_i and K_i , say, in M_i . The 1-skeleton of T_i forms the Cayley graph X_i of $G_i = \pi_1 M/\pi_1 M_i$ with respect to \mathcal{X} .

According to Theorem 9,

$$h(M_i) \leq \frac{4\pi}{|\kappa|} \frac{\chi_-^h(M_i)}{\text{Volume}(M_i)} = \frac{4\pi}{|\kappa| \text{Volume}(M)} \frac{\chi_-^h(M_i)}{d_i},$$

By Proposition 10, there is a constant $k_1 \geq 1$ independent of i such that $h(X_i) \leq k_1 h(M_i)$. Setting

$$k_2 = \frac{4\pi k_1}{|\kappa| \text{Volume}(M)},$$

we deduce that

$$h(X_i) \leq k_2 \frac{\chi_-^h(M_i)}{d_i}.$$

Let $V(X_i)$ be the vertex set of X_i . Let A be a non-empty subset of $V(X_i)$ such that $|\partial A|/|A| = h(X_i)$ and $|A| \leq |V(X_i)|/2 = d_i/2$. By Theorem 11, when $h(X_i) < \sqrt{2/(3d_i)}$, T_i admits a 1-cocycle c that is not a coboundary. Since $h(X_i) \leq k_2 \chi_-^h(M_i)/d_i$, and we are assuming (at least) that $\chi_-^h(M_i)/\sqrt{d_i} \rightarrow 0$, then such a cocycle exists for all sufficiently large i . This establishes (1) of the Theorem 1.

Theorem 11 states that c takes values in $\{-1, 0, 1\}$, and its support is a subset of the edges of ∂A . Dual to this cocycle is a transversely oriented normal surface S in T_i which is homologically non-trivial. Remove any 2-sphere components from S . This is still homologically non-trivial, since all 2-spheres in M_i are inessential, as M_i is negatively curved. The intersection of S with the 2-skeleton of T_i is a graph in S whose complementary regions are triangles and squares. Let $V(S)$ and $E(S)$ be its vertices and edges. Its vertices are in one-one correspondence with the edges of T_i in the support of c . So,

$$|V(S)| \leq |\partial A| = |A| h(X_i) \leq d_i h(X_i)/2 \leq k_2 \chi_-^h(M_i)/2.$$

The valence of each vertex is at most the maximal valence of an edge in T , k_3 , say. So,

$$|\chi(S)| < |E(S)| \leq |V(S)|k_3/2 \leq k_2k_3\chi_-^h(M_i)/4.$$

Setting $k_4 = k_2k_3/4$, we have deduced the existence of a homologically non-trivial, transversely oriented, properly embedded surface S with $|\chi(S)| \leq k_4\chi_-^h(M_i)$ and with no 2-sphere components. By compressing S and removing components if necessary, we may assume that S is also incompressible and connected. Thus, we have proved the following result.

Theorem 12. *Let M be a closed orientable 3-manifold with a negatively curved Riemannian metric. Then there is a constant $k_4 > 0$ with the following property. Let $\{M_i \rightarrow M\}$ be a collection of finite regular covers, with degree d_i . If $\chi_-^h(M_i)/\sqrt{d_i} \rightarrow 0$, then, for all sufficiently large i , M_i contains an embedded, connected, oriented, incompressible, homologically non-trivial surface S such that $|\chi(S)| \leq k_4\chi_-^h(M_i)$.*

By a theorem of Freedman, Hass and Scott [2] (Theorem 5 in this paper), there is an ambient isotopy taking S to a minimal surface. We therefore investigate the coarse geometry of minimal surfaces in M_i .

Set $\epsilon/2$ to be the injectivity radius of M . Let $f(\kappa, \epsilon)$ be the function from Theorem 8. Let \tilde{K} be the lift of the 2-complex K to the universal cover of M . Let k_5 be the maximum number of complementary regions of \tilde{K} that lie within a distance $f(\kappa, \epsilon)$ of any point, and let $k_6 = f(\kappa, \epsilon)k_5$.

Lemma 13. *Let S be a minimal surface in M_i . Then S intersects at most $k_6|\chi(S)|$ complementary regions of K_i . Hence, running through any such region, there are at most $k_6|\chi(S)|$ translates of S under the covering group action of G_i .*

Proof. By Theorem 8, the number of balls of radius $f(\kappa, \epsilon)$ required to cover S is at most $f(\kappa, \epsilon)|\chi(S)|$. The centre of each of these balls has at most k_5 complementary regions of K_i within a distance $f(\kappa, \epsilon)$. So, S intersects at most $k_6|\chi(S)|$ complementary regions of K_i . Each such region corresponds to an element of G_i . To prove the second half of the lemma, we may concentrate on the region corresponding to the identity. Then a translate gS runs through here, for some g in G_i , if and only if S runs through the region corresponding to g^{-1} . Thus, there can be at most $k_6|\chi(S)|$ such g . \square

Proof of Theorem 1(2). Let F be a strongly irreducible Heegaard surface in M_i with $|\chi(F)| = \chi_-^{sh}(M_i)$. By Theorem 6, there is an ambient isotopy taking it either to a minimal surface or to the double cover of a minimal non-orientable surface, with a small tube attached. So, by Lemma 13, F intersects at most $k_6\chi_-^{sh}(M_i)$ complementary regions of K_i . Hence, by Lemma 13, the number of copies of S that F intersects is at most $(k_6\chi_-^{sh}(M_i))(k_6k_4\chi_-^h(M_i))$. This is less than d_i if $\chi_-^{sh}(M_i)/\sqrt{d_i}$ is sufficiently small. So there is a translate of S which misses F . It then lies in a complementary handlebody of F . But this is impossible, since S is incompressible. So, $\chi_-^{sh}(M_i)/\sqrt{d_i}$ is bounded away from zero. \square

Proof of Theorem 1(3). For ease of notation, let $x = \chi_-^h(M_i)$. Let $\{F_1, \dots, F_n\}$ be a generalised Heegaard splitting for M_i , satisfying (1) - (5) of Theorem 3, obtained from a minimal genus Heegaard splitting. Replace any components of $F_1 \cup \dots \cup F_n$ that are parallel by a single component, and let \overline{F} be the resulting surface. Isotope \overline{F} so that each component is as in Theorems 5 or 6. Corollary 4 states that $|\chi(\overline{F})| < x^2$. By Lemma 13, the number of complementary regions of K_i that can intersect \overline{F} is at most k_6x^2 . Let D be the corresponding subset of G_i .

Similarly, let C be the subset of G_i that corresponds to those complementary regions of K_i which intersect S . By Lemma 13, $|C| \leq k_6|\chi(S)| \leq k_6k_4x$.

We claim that, when i is sufficiently large, there are at least $9x/2$ disjoint translates of S under G_i that are also disjoint from \overline{F} . Let $m = 9x/2$. If the claim is not true, then for any m -tuple (g_1S, \dots, g_mS) of copies of S (where $g_j \in G_i$ for each j), either at least two intersect or one copy intersects \overline{F} . In the former case, $g_jc_1 = g_kc_2$, for some c_1 and c_2 in C , and for $1 \leq j < k \leq m$. Hence, $g_k^{-1}g_j \in CC^{-1}$. In the latter case, $g_jc_1 = d$ for some c_1 in C and d in D , and so $g_j \in DC^{-1}$. Thus, the sets $q_{jk}^{-1}(CC^{-1})$ and $p_j^{-1}(DC^{-1})$ cover $(G_i)^m$, where q_{jk} and p_j are the maps

$$\begin{aligned} q_{jk}: (G_i)^m &\rightarrow G_i \\ (g_1, \dots, g_m) &\mapsto g_k^{-1}g_j \end{aligned}$$

$$\begin{aligned} p_j: (G_i)^m &\rightarrow G_i \\ (g_1, \dots, g_m) &\mapsto g_j, \end{aligned}$$

for $1 \leq j < k \leq m$. The former sets $q_{jk}^{-1}(CC^{-1})$ each have size $|G_i|^{m-1}|CC^{-1}|$,

and the latter sets $p_j^{-1}(DC^{-1})$ have size $|G_i|^{m-1}|DC^{-1}|$. So,

$$|G_i|^m \leq \binom{m}{2} |G_i|^{m-1}|C|^2 + m|G_i|^{m-1}|C||D|.$$

This implies that

$$d_i = |G_i| \leq \binom{m}{2} (k_6 k_4 x)^2 + m(k_6 k_4 x)(k_6 x^2).$$

The right-hand side has order x^4 as $i \rightarrow \infty$. However, $x/\sqrt[4]{d_i} \rightarrow 0$, which is a contradiction, proving the claim.

Consider these $9x/2$ copies of S . Each lies in the complement of \overline{F} , which is a collection of compression bodies. Since S is incompressible and connected, each copy of S must be parallel to a component of \overline{F} . By Corollary 4(2), \overline{F} has at most $3x/2$ components. So, at least 3 copies of S are parallel, and at least 2 of these are coherently oriented. The proof is now completed by the following lemma. \square

Lemma 14. *Let S be a connected, embedded, oriented, incompressible, non-separating surface in a closed orientable 3-manifold M_i . Suppose that the image of S under some finite order orientation-preserving homeomorphism h of M_i is disjoint from S , parallel to it and coherently oriented. Then M_i fibres over the circle with fibre S .*

Proof. Let Y be the manifold lying between S and $h(S)$. It is copy of $S \times I$, with S and $h(S)$ corresponding to $S \times \{0\}$ and $S \times \{1\}$. Take a countable collection $\{Y_n : n \in \mathbb{Z}\}$ of copies of this manifold. Glue $S \times \{1\}$ in Y_n to $S \times \{0\}$ in Y_{n+1} , via h^{-1} . The resulting space Y_∞ is a copy of $S \times \mathbb{R}$. Let H be the automorphism of this space taking Y_n to Y_{n+1} for each n , via the ‘identity’. Let $p: Y_0 \rightarrow Y$ be the identification homeomorphism. Extend this to a map $p: Y_\infty \rightarrow M_i$ by defining $p|_{Y_n}$ to be $h^n p H^{-n}$.

We claim that this is a covering map. It may expressed as a composition $Y_\infty \rightarrow Y_\infty/\langle H^N \rangle \rightarrow M_i$, where N is the order of h . The first of these maps is obviously a covering map. The second is also, since it is a local homeomorphism and $Y_\infty/\langle H^N \rangle$ is compact. Hence, p is a covering map.

By construction, $h^n(S)$ lifts homeomorphically to $Y_{n-1} \cap Y_n$, for each n . Hence, the inverse image of S in Y_∞ includes all translates of $Y_{-1} \cap Y_0$ under

$\langle H^N \rangle$. These translates divide Y_∞ into copies of $S \times I$. Since $p^{-1}(S)$ is incompressible and any closed embedded incompressible surface in $S \times I$ is horizontal, we deduce that $p^{-1}(S)$ divides Y_∞ into a collection of copies of $S \times I$. The restriction of p to one of these components Z is a covering map to a component of $M_i - S$. But $M_i - S$ is connected, as S is connected and non-separating. So, p maps Z surjectively onto $M_i - S$. By examining this map near S , we see that it is degree one and hence a homeomorphism. Therefore, M_i is obtained from a copy of $S \times I$ by gluing its boundary components homeomorphically. So, M_i fibres over the circle with fibre S . \square

REFERENCES

1. R. BROOKS, *The spectral geometry of a tower of coverings*, J. Differential Geom. **23** (1986) 97–107.
2. M. FREEDMAN, J. HASS AND P. SCOTT, *Least area incompressible surfaces in 3-manifolds*, Invent. Math. **71** (1983) 609–642.
3. M. LACKENBY, *Heegaard splittings, the virtually Haken conjecture and Property (τ)* , Preprint.
4. A. LUBOTZKY, *Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem*, Ann. Math. (2) **144** (1996) 441–452.
5. J. PITTS AND J. H. RUBINSTEIN, *Existence of minimal surfaces of bounded topological type in three-manifolds*. Miniconference on geometry and partial differential equations (Canberra, 1985), 163–176.
6. M. SCHARLEMANN, *Heegaard splittings*, Handbook of Geometric Topology (Elsevier, 2002), 921–953.
7. M. SCHARLEMANN AND A. THOMPSON, *Thin position for 3-manifolds*, Geometric topology (Haifa, 1992), 231–238, Contemp. Math., 164.
8. B. WHITE, *The space of minimal submanifolds for varying Riemannian metrics*, Indiana Math. Journal **40** (1991) 161–200.

Mathematical Institute, Oxford University,
24-29 St Giles', Oxford OX1 3LB, UK.