Section c Algebraic Topology

Hilary 2002

Question Sheet 1

1. Let $\alpha: S^n \to S^n$ be the antipodal map (defined by $\alpha(x) = -x$). Prove that α is homotopic to the identity if n is odd. [Later in the course it will be shown that α is not homotopic to the identity if n is even.]

2. For any two maps $f, g: X \to S^n$ such that $f(x) \neq -g(x)$ for all $x \in X$, show that $f \simeq g$.

3. Let X be a contractible space and let Y be any space. Show that

   (i) X is path-connected;

   (ii) $X \times Y$ is homotopy equivalent to Y;

   (iii) any two maps from Y to X are homotopic;

   (iv) if Y is path-connected, any two maps from X to Y are homotopic.

The *wedge* $X \vee Y$ of two spaces X and Y, containing basepoints x and y, is the space obtained from the disjoint union of X and Y by identifying x and y. Often, the resulting space is independent of the choice of basepoints, in which case there is no need to specify them.

4. Prove that the following spaces are homotopy equivalent:

   (i) $S^1 \vee S^1$,

   (ii) the torus with one point removed,

   (iii) $\mathbb{R}^2$ minus two points.

   [*Precise formulae are not required; instead describe the maps using pictures.*]

5. *(Optional)* For a map $f: S^n \to X$, let $X \cup_f D^{n+1}$ be the space obtained from the disjoint union of X and D^{n+1} by identifying x and all points in $f^{-1}(x)$, for all $x \in X$. Show that if f and g are homotopic maps $S^n \to X$, then $X \cup_f D^{n+1}$ and $X \cup_g D^{n+1}$ are homotopy equivalent. Deduce that the space (known as the 'dunce cap') obtained by identifying the three sides of a triangle, as shown overleaf, is contractible.