Section c Algebraic Topology

Hilary 2003

Question Sheet 2

1. Show that for a space X, the following three conditions are equivalent:

   (i) Every map $S^1 \to X$ is homotopic to a constant map.

   (ii) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

   (iii) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

   Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

2. Let X and Y be spaces with basepoints x_0 and y_0. Show that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

   Hence, the fundamental group of a torus is $\mathbb{Z} \times \mathbb{Z}$. Identify generators a and b. Draw a figure giving the homotopy between $a.b$ and $b.a$.

3. For $n > 2$, prove that no two of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^n$ are homeomorphic.

4. Show that there is no retraction of a Möbius band onto its boundary.

5. Let G be a space with a basepoint e and a continuous map $m\colon G \times G \to G$ such that $m(g, e) = m(e, g) = g$ for all $g \in G$. (Topological groups arise as examples of such spaces, but neither associativity nor the existence of inverses are important here.) If u and v are loops based at e, show that $u.v$ and $v.u$ are each homotopic rel ∂I to the loop $t \mapsto m(u(t), v(t))$. Deduce that $\pi_1(G, e)$ is commutative.