Section c Algebraic Topology

Hilary 2003

Question Sheet 3

1. Let $p\colon \tilde{X} \to X$ be a covering map. Give proofs of the following basic statements from the lectures:

   (i) p maps open sets to open sets;

   (ii) if $x_1,\ x_2 \in X$, then $p^{-1}(x_1)$ and $p^{-1}(x_2)$ have the same number of points;

   (iii) p is surjective;

   (iv) X has the quotient topology.

2. Construct a simply-connected covering of the following spaces:

   (i) the Möbius band;

   (ii) $S^2 \vee S^1$;

   (iii) $\mathbb{R}^2 - \{\text{point}\}$.

3. (i) We will define the *free group on two generators* a and b, denoted by $\langle a, b\rangle$. Its elements are words of finite length, using letters a, a^{-1}, b, b^{-1}, modulo the equivalence relation that identifies two words iff the differ by a finite sequence of the following moves:

   - if a letter and its inverse are adjacent in a word, they may be removed;
   - any letter and its inverse may be inserted into adjacent positions in a word.

   The group operation is simply writing one word after the other. Show that this does indeed form a group. (The empty word is allowed: it is the identity element.) Show also that any element is represented by a unique *reduced* word, which means that no letter and its inverse are adjacent in that word.

   (ii) Define an infinite graph G with a vertex for every element of $\langle a, b\rangle$, with the condition that two vertices, corresponding to elements g_1 and g_2, are joined by an edge iff

   $$g_1^{-1}g_2 \in \{a, a^{-1}, b, b^{-1}\}.$$

   Show that G is contractible.

   (iii) Construct a cover $G \to S^1 \vee S^1$, and hence prove that $\pi_1(S^1 \vee S^1) \cong \langle a, b\rangle$.

4. Construct coverings of $S^1 \vee S^1$ with fundamental groups that correspond to the following subgroups of $\langle a, b \rangle$:

   (i) $\langle a \rangle$

   (ii) $\{a^{n_1} b^{m_1} a^{n_2} b^{m_2} \ldots b^{m_k} : \sum n_i \text{ is even}\}$.

   (iii) the kernel of the homomorphism $\langle a, b \rangle \to \mathbb{Z} \times \mathbb{Z}$ that sends a to $(1, 0)$, and b to $(0, 1)$.