Section c Algebraic Topology

Hilary 2003

Question Sheet 4

1. Construct a covering map from $\mathbb{R}^2$ to the Klein bottle K and use it show that $\pi_1(K)$ is isomorphic to the group whose elements are pairs (m,n) of integers, with the non-abelian group operation given by

$$(m,n) \star (p,q) = (m + (-1)^n p, n + q).$$

2. For any positive integer n, construct a covering space action of $\mathbb{Z}_n$ on S^3. Calculate the fundamental group of the quotient space.

3. (i) Show that there is no map $f\colon S^2 \to S^1$ such that $f(-x) = -f(x)$ for all $x \in S^2$. [*Hint: such a map would define a map* $g\colon \mathbb{R}P^2 \to \mathbb{R}P^1$.]

   (ii) Prove that for any continuous map $h\colon S^2 \to \mathbb{R}^2$, there is a point $x \in S^2$ such that $h(x) = h(-x)$. [*At any given time, there are always points on opposite sides of the earth with the same temperature and pressure.*]

4. Show that the group $SO(3)$ is homeomorphic to $D^3/\sim$, where $\sim$ is the relation that identifies antipodal points on S^2. Deduce that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$ and hence that $\pi_1(SO(3)) \cong \mathbb{Z}_2$. Find an explicit generator. [*From this, we deduce that the* $SO(3)$ *is covered by the 3-sphere. This, in fact, has a group structure - it is* $SU(2)$ *- and the covering map* $SU(2) \to SO(3)$ *is a homomorphism.*]

A space X is *locally path-connected* if, for any $x \in X$ and any neighbourhood V of x, there is an open set in V that contains x and that is path-connected.

5. (*Optional*) Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x)$ be a cover, and let $f\colon (Y, y_0) \to (X, x_0)$ be any map, where Y is path-connected and locally path-connected. Prove that a lift $\tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of f exists iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. [*Hint: Use the paths based at* y_0 *to define* $\tilde{f}$*; use local path-connectivity to prove that it is continuous.*] Deduce that, if X is locally path-connected, then any two universal covers of X are homeomorphic. [*In fact, a universal cover of* X *always exists if, in addition,* X *is semi-locally simply-connected, which means that each* $x \in X$ *has a neighbourhood* V *such that* $\pi_1(V) \to \pi_1(X)$ *induced by inclusion is trivial.*]