Section c Algebraic Topology

Hilary 2002

Question Sheet 6

1. An exact sequence of abelian groups $0 \longrightarrow G \xrightarrow{\alpha} H \xrightarrow{\beta} K \longrightarrow 0$ *splits* if either

   (i) there is a homomorphism $\gamma \colon K \to H$ such that $\beta \circ \gamma = \mathrm{id}_K$, or

   (ii) there is a homomorphism $\delta \colon H \to G$ such that $\delta \circ \alpha = \mathrm{id}_G$.

   Show that in this case $H \cong G \oplus K$. Show that if $K = \oplus_{i=1}^{n} \mathbb{Z}$, then the sequence splits.

2. For each of the following exact sequences of abelian groups and homomorphisms, say as much as you can about the unknown group G and/or the unknown homomorphism α:

   (i) $0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$

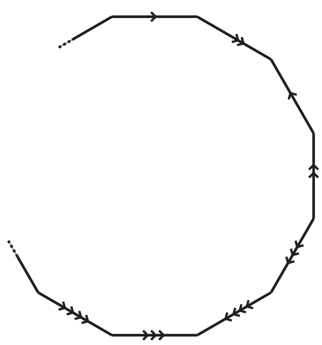
   (ii) $0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow 0$

   (iii) $0 \longrightarrow G \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$

   (iv) $0 \longrightarrow \mathbb{Z}_3 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0.$

3. Calculate the homology groups of $\bigvee_{i=1}^{k} S^1$, the space obtained from k circles by picking one point in each circle and identifying these k points to a single point.

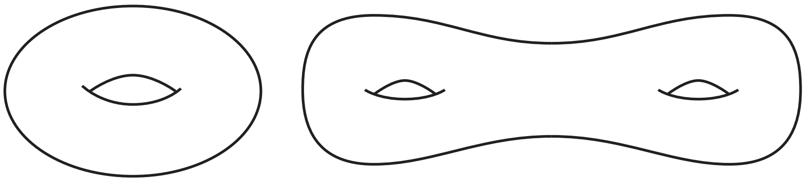
4. Let F_k be the space obtained from a $4k$-sided polygon by identifying the sides as shown:



   Let U be an open disc embedded in the interior of the polygon. By considering the pair $(F_k, F_K - U)$, prove that

$$H_n(F_k) = \begin{cases} \mathbb{Z} & \text{if } n = 0; \\ \oplus_{i=1}^{2k} \mathbb{Z} & \text{if } n = 1; \\ \mathbb{Z} & \text{if } n = 2; \\ 0 & \text{otherwise.} \end{cases}$$

[It is not hard to prove that F_k is homeomorphic to the 'k-holed torus', shown below. A famous theorem asserts that any compact 'orientable' surface is homeomorphic to one of the F_k. This question proves that they are all distinct.]



F_1 F_2

5. By using the method of the above question, calculate the homology groups of $\mathbb{R}P^2$.