Section c Algebraic Topology

Hilary 2003

Question Sheet 7

1. Express the following spaces as cell complexes and prove that, in each case, you have used the minimum number of cells:

   (i) $S^1 \vee S^1$;

   (ii) S^n;

   (iii) T^2;

   (iv) the Klein bottle.

2. Prove that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell, where the attaching map is the covering map $p: S^{n-1} \to \mathbb{R}P^{n-1}$. Prove by induction on n, that

$$H_m(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } m = 0; \\ \mathbb{Z}_2 & \text{if } 0 < m < n \text{ and } m \text{ is odd;} \\ \mathbb{Z} & \text{if } m = n \text{ and } n \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

   At the same time, prove that $p_*: H_n(S^n) \to H_n(\mathbb{R}P^n)$ is the $\times 2$ map if n is odd, and the zero map if n is even.

3. (*Quite long*) Let $p: \tilde{X} \to X$ be a covering map, such that $p^{-1}(x)$ has a finite number (d, say) of points for each $x \in X$. Prove that, if X is a finite cell complex, then so is $\tilde{X}$. Show that $\chi(\tilde{X}) = d\, \chi(X)$. Deduce that if $p: \mathbb{R}P^2 \to X$ is a covering map, where X is a finite cell complex, then p is a homeomorphism.

4. Let X and Y be finite cell complexes. Prove that $\chi(X \times Y) = \chi(X)\chi(Y)$.

5. Use the Mayer-Vietoris theorem to find the homology groups of the Klein bottle [decompose it into two copies of $S^1 \times I$].

6. The *suspension* SX of a space X is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point, and $X \times \{1\}$ to another point. (Thus, $SS^n \cong S^{n+1}$.) Use the Mayer-Vietoris theorem to show that $H_n(SX) = H_{n-1}(X)$ for $n \geq 2$.

7. Let $G_1, \ldots, G_n$ be any finite sequence of finitely generated abelian groups. Show that there is a path-connected space K, with $H_i(K) \cong G_i$ for $1 \leq i \leq n$ and $H_i(K) = 0$ for $i > n$.