The Dehn Surgery Problem

Marc Lackenby

19 June 2012

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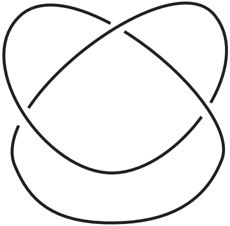

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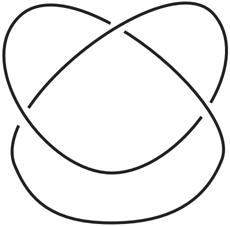

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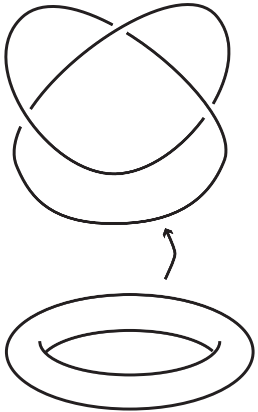

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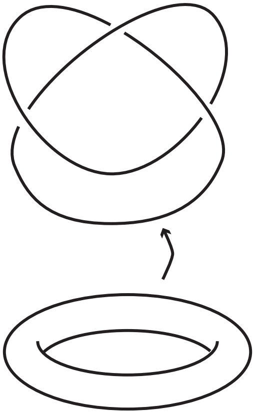
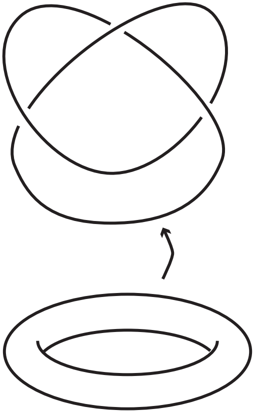
The resulting manifold is obtained by Dehn surgery on K.



Slopes

The essentially different ways of doing this are parametrised by the ‘slopes’ on $\partial N(K)$.



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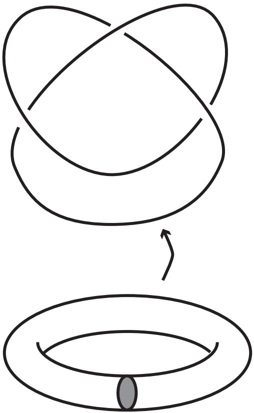

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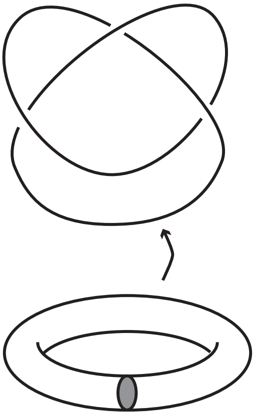

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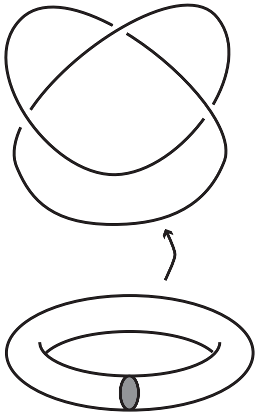

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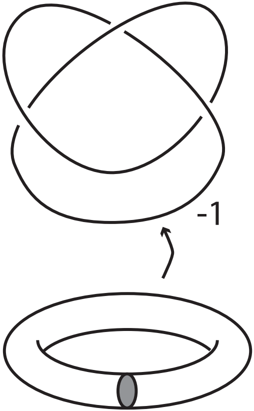
The slopes on each component of $\partial N(K)$ are parametrised by fractions $p/q \in \mathbb{Q} \cup \{\infty\}$.



Lickorish's and Kirby's theorems

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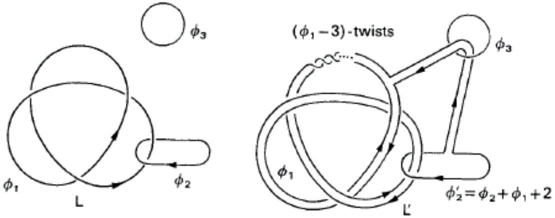

Dehn filling

Let

M = a compact orientable 3-manifold, with
∂M = a collection of tori
$s_1, \dots, s_n$ = a collection slopes on ∂M, one on each ∂ component.

Then
$M(s_1, \dots, s_n)$ = the manifold obtained by attaching solid tori to M along the slopes $s_1, \dots, s_n$.

It is obtained from M by Dehn filling.

General theme: Properties of M should be inherited by $M(s_1, \dots, s_n)$, for 'generic' slopes $s_1, \dots, s_n$.

Geometrisation

A major revolution in 3-manifold theory was the formulation (by Thurston) and proof (by Perelman) of:

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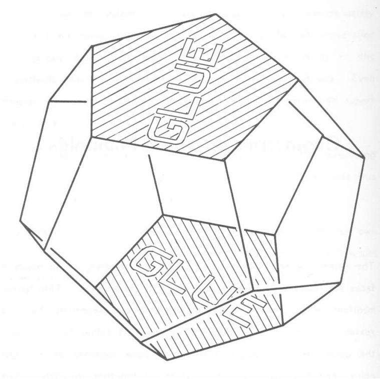
Of these geometries, by the far the most important is hyperbolic geometry.

Hyperbolic structures

M is hyperbolic if $M - \partial M = \mathbb{H}^3/\Gamma$ for some discrete group Γ of isometries acting freely on $\mathbb{H}^3$.

Example: Seifert-Weber dodecahedral space



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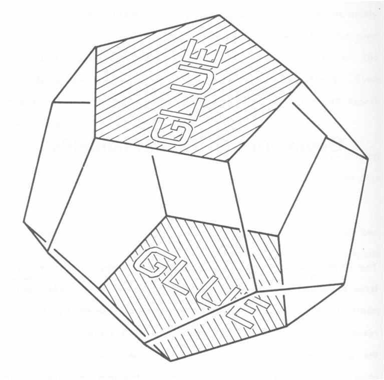
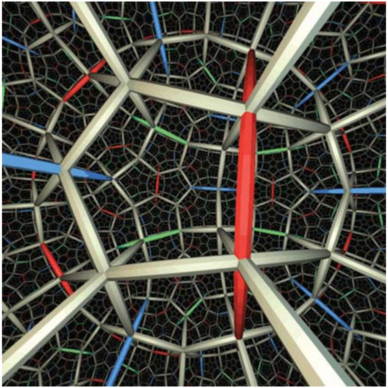
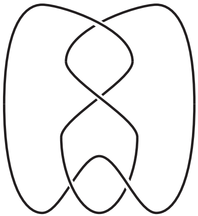




The figure-eight knot



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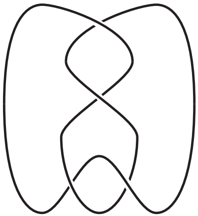
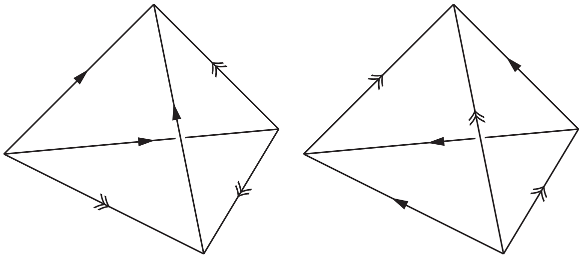
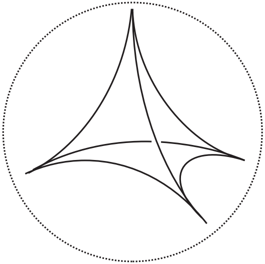


$S^3 - K \cong$



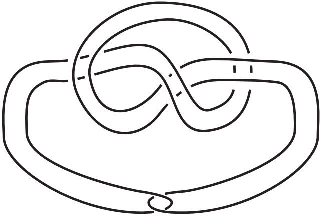
We can realise these as (ideal) tetrahedra in $\mathbb{H}^3$:



This induces a hyperbolic structure on $S^3 - K$.

Hyperbolic structures on knot complements

Theorem: [Thurston] Let K be a knot in S^3. Then $S^3 - K$ admits a hyperbolic structure if and only if K is not a torus knot or a satellite knot:



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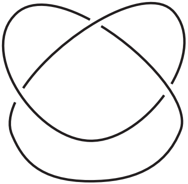

torus knot

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Suppose that M has a hyperbolic structure, and that ∂M is a single torus.

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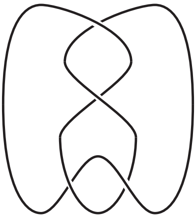
The Dehn Surgery Problem: How many exceptional slopes can there be?

The figure-eight knot



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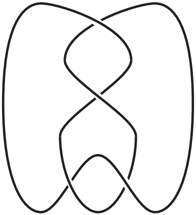

Let M = the exterior of the figure-eight knot.

Theorem: [Thurston] M has 10 exceptional slopes:

$$\{-4, -3, -2, -1, 0, 1, 2, 3, 4, \infty\}.$$

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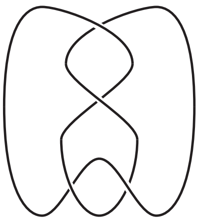

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Conjecture: [Gordon] For any hyperbolic 3-manifold M with ∂M a single torus, M has at most 10 exceptional slopes. Moreover, the figure-eight knot exterior is the unique manifold with precisely 10.

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But we still don't know whether the figure-eight knot exterior is the unique such manifold.

History

This problem has received a lot of attention ...
Certainly > 200 papers

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M. Culler, C. Gordon, J. Luecke, P. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) 125 (1987) 237–300.
D. Gabai, *Foliations and the topology of* 3*-manifolds. III.* J. Differential Geom. 26 (1987) 479–536.
S. Boyer, X. Zhang, *Finite Dehn surgery on knots.* J. Amer. Math. Soc. 9 (1996) 1005–1050
S. Boyer, X. Zhang, *On Culler-Shalen seminorms and Dehn filling,* Ann. of Math. (2) 148 (1998) 737–801.
M. Lackenby, *Word hyperbolic Dehn surgery*, Invent. math. 140 (2000) 243–282.
I. Agol, *Bounds on exceptional Dehn filling*, Geom. Topol. 4 (2000) 431–449.
S. Boyer, X. Zhang, *A proof of the finite filling conjecture,* J. Differential Geom. 59 (2001) 87–176.
C. Hodgson, S. Kerckhoff, *Universal bounds for hyperbolic Dehn surgery.* Ann. of Math. (2) 162 (2005) 367–421.

I. Agol, *Bounds on exceptional Dehn filling II.* Geom. Topol. 14 (2010) 1921–1940.

Some approaches

Approach 1: Quantify Thurston's arguments.

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Theorem: [Hodgson-Kerckhoff, Annals 2006] There are at most 60 exceptional slopes.

This was the first universal bound (ie independent of M) on the number of exceptional slopes.

They used the theory of cone manifolds.

Topological methods

Perelman's proof of geometrisation $\Rightarrow$

A closed orientable 3-manifold is hyperbolic if and only if it is irreducible, atoroidal, not Seifert fibred.

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A closed orientable 3-manifold is hyperbolic if and only if it is irreducible, atoroidal, not Seifert fibred.

Irreducible: any embedded 2-sphere bounds a 3-ball.

Atoroidal: no π_1-injective embedded torus.

Seifert fibred: is foliated by circles.

Topological methods

Approach 2: Bound the intersection number of slopes s_1 and s_2 where $M(s_1)$ and $M(s_2)$ are reducible/toroidal/Seifert fibred.

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Theorem: [Gordon-Luecke, Topology 1997] If $M(s_1)$ and $M(s_2)$ are reducible, then s_1 and s_2 have intersection number 1. Hence, $M(s)$ can be reducible for at most 3 slopes s.

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But analysing when $M(s_1)$ and $M(s_2)$ are both Seifert fibred is hard.

Negatively curved metrics

Approach 3: Construct a negatively curved metric on $M(s)$.

Theorem: [Gromov-Thurston] For all slopes s on ∂M with at most 48 exceptions, $M(s)$ admits a negatively curved Riemannian metric.

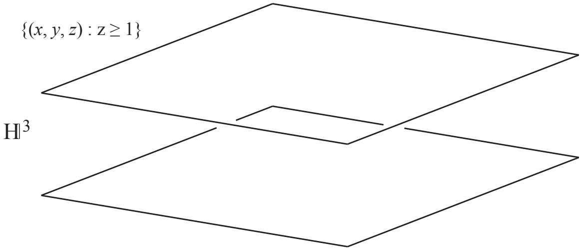
Perelman $\Rightarrow$ $M(s)$ then admits a hyperbolic structure.

The ends of a hyperbolic 3-manifold

Suppose ∂M is a non-empty collection of tori.

The ends of $M - \partial M$ are of the form $\partial M \times [1, \infty)$.

Geometrically, they are obtained from $\{(x, y, z) : z \geq 1\}$ in upper-half space, quotiented out by the action of $\mathbb{Z} \times \mathbb{Z}$ acting by parabolic isometries.



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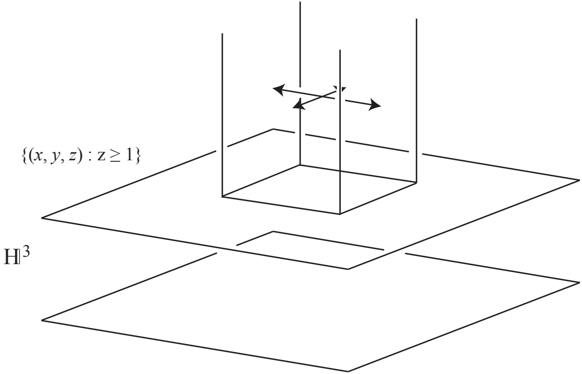
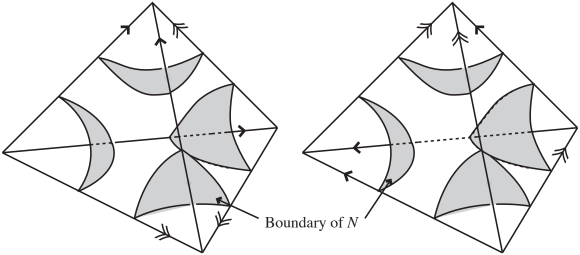


This is a called a *horoball neighbourhood* of the end.

Example: the figure-8 knot



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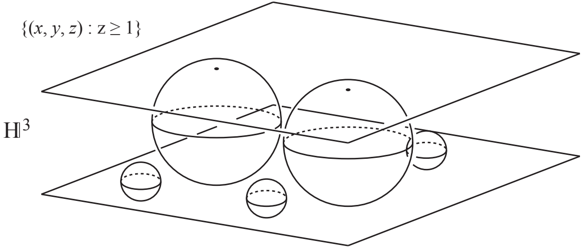

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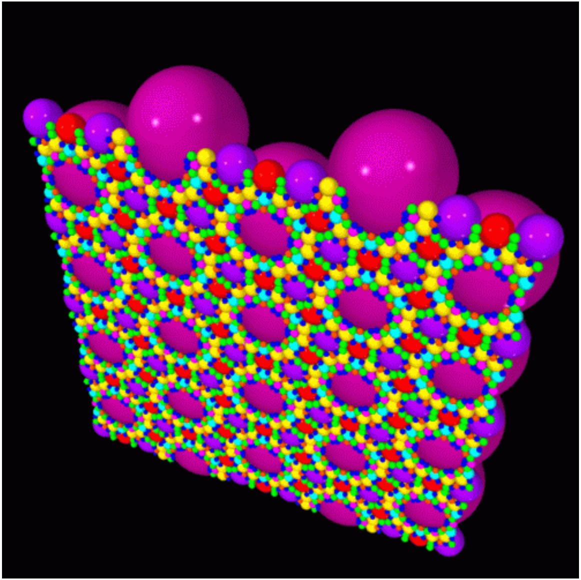
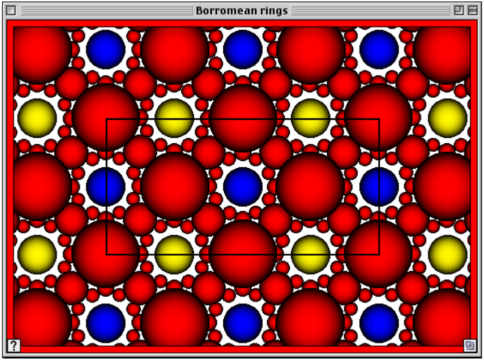


Horoball diagrams

Looking down from infinity in hyperbolic space, one gets a view of horoballs:



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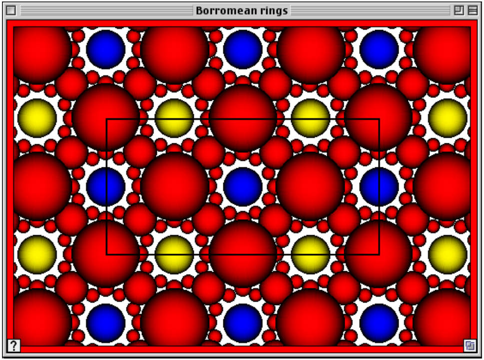

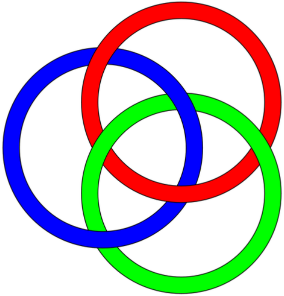

This is actually from the hyperbolic structure on the Borromean rings.

It was produced by the computer program Snappea.

The length of a slope

Let $N =$ a maximal horoball neighbourhood of the end.

∂N is a Euclidean torus.

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The length $L(s)$ of s is the length of this geodesic.

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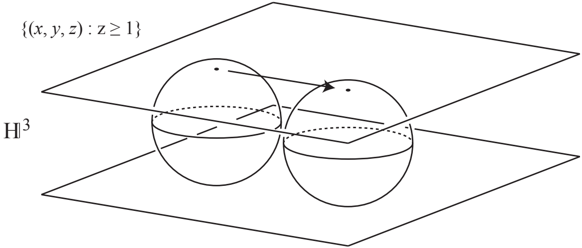
One can visualise this as the translation length of the corresponding parabolic acting on $\{(x, y, z) : z = 1\}$.



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So, to bound the number of exceptional slopes, all we need is ...

Theorem: [Gromov-Thurston] At most 48 slopes s have $L(s) \leq 2\pi$.

Bounding slope length

Lemma: [Thurston] Each slope has length ≥ 1.

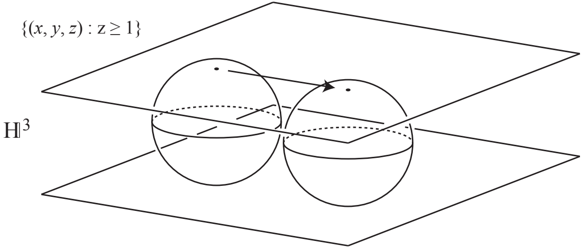
Proof:

Let $\tilde{N}$ be the inverse image of N in $\mathbb{H}^3$.

We may arrange that one component of $\tilde{N}$ is $\{(x, y, z) : z \geq 1\}$.

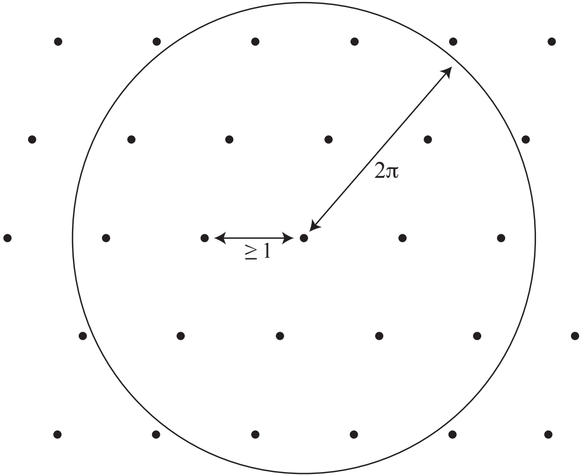
Since N is maximal, another component of $\tilde{N}$ touches it.

Then parabolic translations much translate by at least one, otherwise the interior of two components of $\tilde{N}$ would intersect.



The number of slopes with length at most 2π

From Thurston's Lemma, it is clear that there is an upper bound on the number of slopes with length at most 2π.



Intersection numbers

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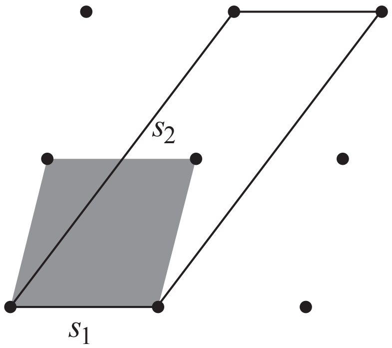

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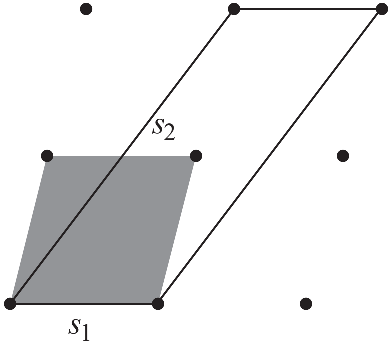
In our case,
$\Delta(s_1, s_2) \leq \frac{(2\pi)^2}{\sqrt{3}/2} < 46.$



The number of slopes with length at most 2π

Lemma: [Agol] Let S be a collection of slopes, such that any two have intersection number at most Δ. Let p be any prime more than Δ. Then $|S| \leq p + 1$.

Setting $p = 47$ gives our bound.

How to improve the bound?

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[Cao-Meyerhoff, 2001] $A \geq 3.35$

How to improve the bound?

Recall

$$\Delta(s_1, s_2) \leq \frac{L(s_1)\ L(s_2)}{\mathrm{Area}(\partial N)}.$$

1. Increase the lower bound on $A = \mathrm{Area}(\partial N)$.
2. Decrease the critical slope length below 2π.

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[Cao-Meyerhoff, 2001] $A \geq 3.35$
[Gabai-Meyerhoff-Milley, 2009] $A \geq 3.7$ (under some hypotheses)

The 6 theorem

Theorem: [L, Agol] If $L(s) > 6$, then $M(s)$ is irreducible, atoroidal, not Seifert fibred and has infinite, word hyperbolic fundamental group.

Perelman $\Rightarrow$ $M(s)$ is then hyperbolic.

So, the critical slope length is reduced from 2π to 6.

Proving the 6 theorem

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Then P can isotoped to a minimal surface in M.

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$$\mathrm{Area}(P) \leq 2\pi(|\partial P| - 2).$$

If we can show that each boundary component of P contributes at least 2π to the area of P, we'll have a contradiction.

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Part of this lies in the horoball $\{(x, y, z) : z \geq 1\}$.

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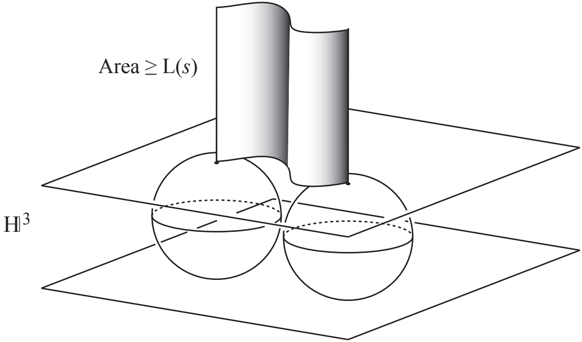

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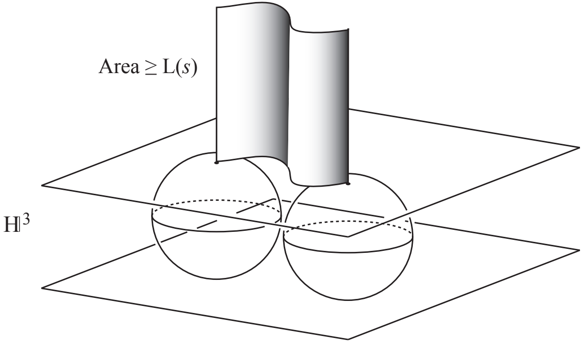

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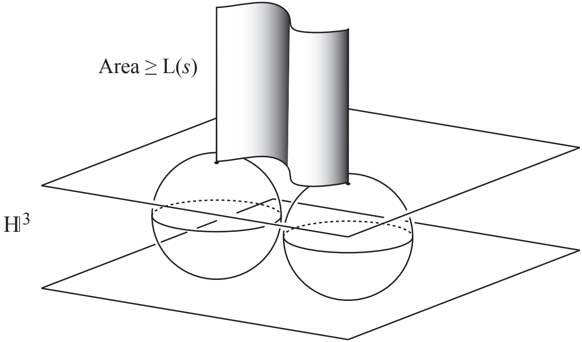

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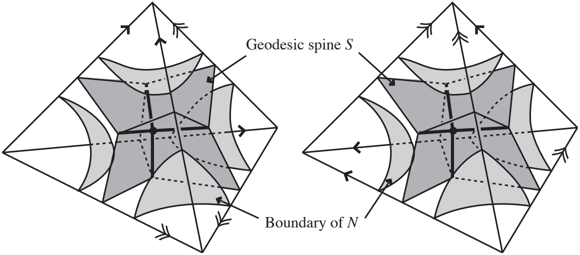
So, if $L(s) > 2\pi$, we have a contradiction.

Proving the 6 theorem

We haven't used all the area of P!



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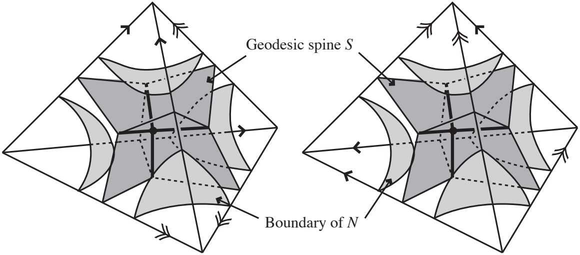

In fact, each component of ∂P contributes at least $(2\pi/6)\ L(s)$ to the area of P.

So, if $L(s) > 6$, then $M(s)$ is irreducible.

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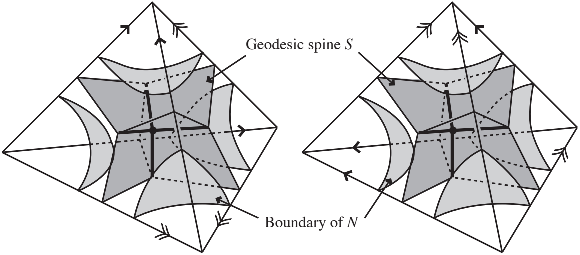

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The other conclusions (atoroidality etc) are proved similarly.

Down to 12 exceptional slopes

Using that cusp area is at least 3.35 [Cao-Meyerhoff] and that critical slope length is 6, we proved:

Theorem: [Agol, L 2000] For any hyperbolic 3-manifold M with ∂M a single torus, M has at most 12 exceptional slopes.

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But reducing 12 down to 10 is very hard ...

A difficulty

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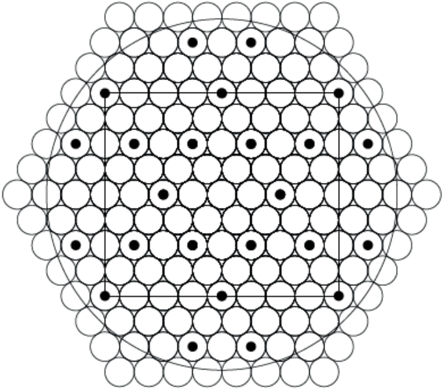
2. It has 12 slopes with length at most 6.



Work of Gabai, Meyerhoff and Milley

Fortunately, this manifold falls into a well-understood family of exceptions.

Theorem: [Gabai, Meyerhoff, Milley] The cusp area of M is at least 3.7 unless M is obtained by Dehn filling some of the boundary components of one of the manifolds

$m412$	$s596$	$s647$	$s774$	$s776$	$s780$	$s785$	$s898$	$s959$

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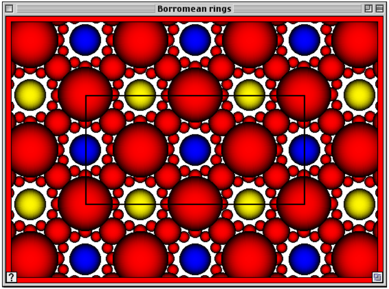
Using this and the 6 theorem, you can get a bound of 11 exceptional surgeries.

Work of Cao and Meyerhoff

They consider the horoball diagram of M:



They let $e_2 \geq 1$ be the (Euclidean diameter)$^{-1/2}$ of the second largest horoballs.

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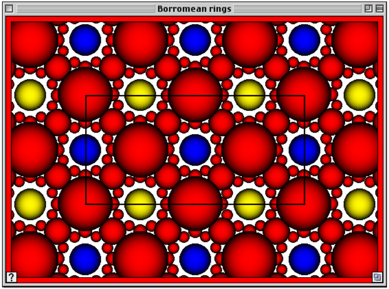

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If e_2 is close to 1, then we get lots of large horoballs and so large cusp area.

If e_2 is large, then the largest horoballs must be well-separated.

A computer calculation, ranging over all values of e_2, gives that $A \geq 3.35$.

Work of Gabai, Meyerhoff and Milley

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In these cases, one can show that M is obtained by Dehn filling some of the boundary components of one of the manifolds

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How to solve the Dehn surgery problem

Improve the 6 theorem:

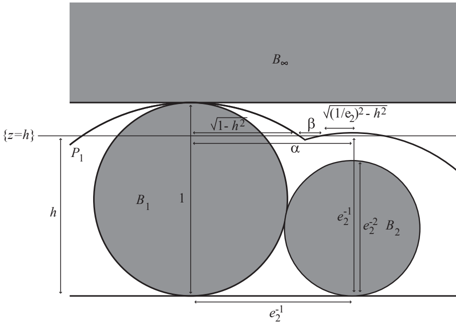
Theorem: Suppose that

$$L(s) > \frac{\pi e_2}{\arcsin(e_2/2)}$$

if $e_2 \leq \sqrt{2}$, and that

$$L(s) > \frac{2\pi\sqrt{1 - e_2^{-2}}}{\arcsin(\sqrt{1 - e_2^{-2}})}$$

if $e_2 > \sqrt{2}$. Then, $M(s)$ is hyperbolic.



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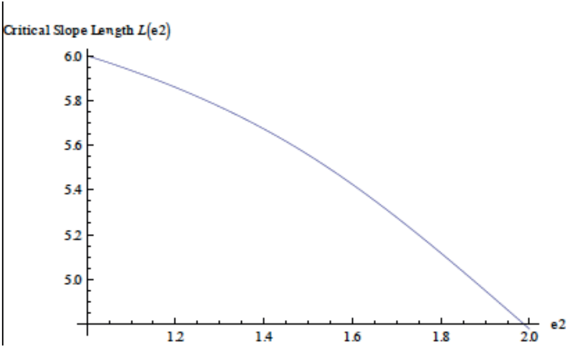

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Repeat the analysis of Gabai, Milley and Meyerhoff, but specifically search for the number of slopes with length less than the given bound.

Must also use a range of geometric arguments to rule out certain horoball configurations.

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Must also use a range of geometric arguments to rule out certain horoball configurations.

The computer calculation takes about 2 days.

It only just works!

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Use is also made of the computer program Snap which is a version of Snappea that uses exact arithmetic.