The Dehn Surgery Problem

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Dehn surgery

This is a method for building 3-manifolds:

Start with a knot or link \( K \) in 3-sphere. Remove an open regular neighbourhood of \( K \), creating a 3-manifold \( M \) with boundary. Re-attach solid tori to \( M \), but in a different way. The resulting manifold is obtained by Dehn surgery on \( K \).
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An isotopy class of essential simple closed curves on $\partial N(K)$ is a slope.

The slopes on each component of $\partial N(K)$ are parametrised by fractions $p/q \in \mathbb{Q} \cup \{\infty\}$. 
Lickorish’s and Kirby’s theorems

**Theorem:** [Lickorish, Wallace] Any closed orientable 3-manifold is obtained by Dehn surgery on some link in $S^3$. 
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Dehn filling

Let

\( M \) = a compact orientable 3-manifold, with
\( \partial M \) = a collection of tori
\( s_1, \ldots, s_n \) = a collection slopes on \( \partial M \), one on each \( \partial \) component.

Then
\( M(s_1, \ldots, s_n) \) = the manifold obtained by attaching solid tori to \( M \) along the slopes \( s_1, \ldots, s_n \).

It is obtained from \( M \) by \textbf{Dehn filling}.

\textbf{General theme:} Properties of \( M \) should be inherited by \( M(s_1, \ldots, s_n) \), for ‘generic’ slopes \( s_1, \ldots, s_n \).
A major revolution in 3-manifold theory was the formulation (by Thurston) and proof (by Perelman) of:

**Geometrisation Conjecture**: ‘Any closed orientable 3-manifold has a canonical decomposition into geometric pieces.’
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Of these geometries, by the far the most important is hyperbolic geometry.
Hyperbolic structures

$M$ is hyperbolic if $M - \partial M = \mathbb{H}^3/\Gamma$ for some discrete group $\Gamma$ of isometries acting freely on $\mathbb{H}^3$.

Example: Seifert-Weber dodecahedral space

![Image of a dodecahedron with the word "GLUE" on it]
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The figure-eight knot
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$S^3 - K \cong$
We can realise these as (ideal) tetrahedra in $\mathbb{H}^3$:

This induces a hyperbolic structure on $S^3 - K$. 

---

\[ H^3 \]
Theorem: [Thurston] Let $K$ be a knot in $S^3$. Then $S^3 - K$ admits a hyperbolic structure if and only if $K$ is not a torus knot or a satellite knot:
Hyperbolic Dehn surgery

Suppose that $M$ has a hyperbolic structure, and that $\partial M$ is a single torus.

**Theorem:** [Thurston] For slopes $s$ on $\partial M$, with at most finitely many exceptions, $M(s)$ is also hyperbolic.
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The slopes $s$ for which $M(s)$ is not hyperbolic are called **exceptional**.
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The slopes $s$ for which $M(s)$ is not hyperbolic are called exceptional.

**The Dehn Surgery Problem:** How many exceptional slopes can there be?
The figure-eight knot

Let $M$ = the exterior of the figure-eight knot.

Theorem: [Thurston] $M$ has 10 exceptional slopes: $\{-4, -3, -2, -1, 0, 1, 2, 3, 4, \infty\}$.

Conjecture: [Gordon] For any hyperbolic 3-manifold $M$ with $\partial M$ a single torus, $M$ has at most 10 exceptional slopes. Moreover, the figure-eight knot exterior is the unique manifold with precisely 10.
The figure-eight knot

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**Theorem:** [L-Meyerhoff] For any hyperbolic 3-manifold $M$ with $\partial M$ a single torus, $M$ has at most 10 exceptional slopes.

But we still don’t know whether the figure-eight knot exterior is the unique such manifold.
History

This problem has received a lot of attention ...
Certainly \( \geq 200 \) papers
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Some approaches

**Approach 1:** Quantify Thurston’s arguments.
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Theorem: [Hodgson-Kerckhoff, Annals 2006] There are at most 60 exceptional slopes.

This was the first universal bound (ie independent of $M$) on the number of exceptional slopes.

They used the theory of cone manifolds.
Perelman’s proof of geometrisation ⇒

A closed orientable 3-manifold is hyperbolic if and only if it is irreducible, atoroidal, not Seifert fibred.
Topological methods

Perelman’s proof of geometrisation $\Rightarrow$

A closed orientable 3-manifold is hyperbolic if and only if it is irreducible, atoroidal, not Seifert fibred.

Irreducible: any embedded 2-sphere bounds a 3-ball.

Atoroidal: no $\pi_1$-injective embedded torus.

Seifert fibred: is foliated by circles.
Approach 2: Bound the intersection number of slopes $s_1$ and $s_2$ where $M(s_1)$ and $M(s_2)$ are reducible/toroidal/Seifert fibred.
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For example . . .

**Theorem:** [Gordon-Luecke, Topology 1997] If $M(s_1)$ and $M(s_2)$ are reducible, then $s_1$ and $s_2$ have intersection number 1. Hence, $M(s)$ can be reducible for at most 3 slopes $s$. 
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But analysing when $M(s_1)$ and $M(s_2)$ are both Seifert fibred is hard.
Approach 3: Construct a negatively curved metric on $M(s)$.

Theorem: [Gromov-Thurston] For all slopes $s$ on $\partial M$ with at most 48 exceptions, $M(s)$ admits a negatively curved Riemannian metric.

Perelman $\Rightarrow$ $M(s)$ then admits a hyperbolic structure.
The ends of a hyperbolic 3-manifold

Suppose $\partial M$ is a non-empty collection of tori.

The ends of $M - \partial M$ are of the form $\partial M \times [1, \infty)$.

Geometrically, they are obtained from $\{(x, y, z) : z \geq 1\}$ in upper-half space, quotiented out by the action of $\mathbb{Z} \times \mathbb{Z}$ acting by parabolic isometries.
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This is called a horoball neighbourhood of the end.
Example: the figure-8 knot
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Horoball diagrams

Looking down from infinity in hyperbolic space, one gets a view of horoballs:
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Looking down from infinity in hyperbolic space, one gets a view of horoballs:

This is actually from the hyperbolic structure on the Borromean rings.

It was produced by the computer program Snappea.
The length of a slope

Let $N$ = a maximal horoball neighbourhood of the end.

$\partial N$ is a Euclidean torus.

Each slope $s$ on $\partial N$ is realised by a geodesic in $\partial N$.

The length $L(s)$ of $s$ is the length of this geodesic.
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The length $L(s)$ of $s$ is the length of this geodesic.

One can visualise this as the translation length of the corresponding parabolic acting on $\{(x, y, z) : z = 1\}$.
Theorem: [Gromov-Thurston] If $L(s) > 2\pi$, then $M(s)$ admits a negatively curved Riemannian metric.

The proof is very clever, but surprisingly straightforward!
The $2\pi$ theorem

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So, to bound the number of exceptional slopes, all we need is . . .

**Theorem:** [Gromov-Thurston] At most 48 slopes $s$ have $L(s) \leq 2\pi$. 
Bounding slope length

**Lemma:** [Thurston] Each slope has length \( \geq 1 \).

**Proof:**

Let \( \tilde{N} \) be the inverse image of \( N \) in \( \mathbb{H}^3 \).

We may arrange that one component of \( \tilde{N} \) is \( \{(x, y, z) : z \geq 1\} \).

Since \( N \) is maximal, another component of \( \tilde{N} \) touches it.

Then parabolic translations must translate by at least one, otherwise the interior of two components of \( \tilde{N} \) would intersect.

\[ \{(x, y, z) : z \geq 1\} \]

\[ \mathbb{H}^3 \]
The number of slopes with length at most $2\pi$

From Thurston’s Lemma, it is clear that there is an upper bound on the number of slopes with length at most $2\pi$. 

\[
\geq \frac{1}{2\pi} \geq 1
\]
Intersection numbers

For two slopes $s_1$ and $s_2$, let $\Delta(s_1, s_2)$ denote the modulus of their intersection number.
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**Lemma:** For two slopes $s_1$ and $s_2$,

$$\Delta(s_1, s_2) \leq \frac{L(s_1) \cdot L(s_2)}{\text{Area}(\partial \mathcal{N})}.$$
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**Lemma:** For two slopes $s_1$ and $s_2$,

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In our case,

$$\Delta(s_1, s_2) \leq \frac{(2\pi)^2}{\sqrt{3}/2} < 46.$$
The number of slopes with length at most $2\pi$

**Lemma**: [Agol] Let $S$ be a collection of slopes, such that any two have intersection number at most $\Delta$. Let $p$ be any prime more than $\Delta$. Then $|S| \leq p + 1$.

Setting $p = 47$ gives our bound.
How to improve the bound?

Recall

$$\Delta(s_1, s_2) \leq \frac{L(s_1) L(s_2)}{\text{Area}(\partial N)}.$$
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1. Increase the lower bound on $A = \text{Area}(\partial N)$.

2. Decrease the critical slope length below $2\pi$. 

[Adams, 1987] $A \geq \sqrt{3} \sim 1.732$

[Cao-Meyerhoff, 2001] $A \geq 3.35$

[Gabai-Meyerhoff-Milley, 2009] $A \geq 3.7$ (under some hypotheses)
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[Gabai-Meyerhoff-Milley, 2009] \( A \geq 3.7 \) (under some hypotheses)
The 6 theorem

**Theorem:** [L, Agol] If $L(s) > 6$, then $M(s)$ is irreducible, atoroidal, not Seifert fibred and has infinite, word hyperbolic fundamental group.

Perelman $\Rightarrow M(s)$ is then hyperbolic.

So, the critical slope length is reduced from $2\pi$ to 6.
Proving the 6 theorem

Suppose that $M(s)$ is reducible, and so contains a 2-sphere $S$ that doesn’t bound a ball.
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We may arrange that $S$ intersects the surgery solid torus in meridian discs.

Then $S \cap M$ is a planar surface $P$, with each boundary component having slope $s$. 

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Then $P$ can isotoped to a minimal surface in $M$. 

Proving the 6 theorem

For any minimal surface, its intrinsic curvature $\kappa$ is at most the curvature of the ambient space. So, $\kappa \leq -1$.

Gauss-Bonnet: $2\pi \chi(P) = \int P \kappa \leq \int P - 1 = -\text{Area}(P)$.

ie. $\text{Area}(P) \leq 2\pi (|\partial P| - 2)$.

If we can show that each boundary component of $P$ contributes at least $2\pi$ to the area of $P$, we'll have a contradiction.
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Consider the inverse image of $P$ in $\mathbb{H}^3$.
Part of this lies in the horoball $\{(x, y, z) : z \geq 1\}$. 

\[ \text{Area} \geq L(s) \]

So, if $L(s) > 2\pi$, we have a contradiction.
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So, if $L(s) > 2\pi$, we have a contradiction.
Proving the 6 theorem

We haven’t used all the area of $P$!

In fact, each component of $\partial P$ contributes at least $(2\pi/6)\cdot L(s)$ to the area of $P$. If $L(s) > 6$, then $M(s)$ is irreducible. The other conclusions (atoroidality etc) are proved similarly.
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Down to 12 exceptional slopes

Using that cusp area is at least 3.35 [Cao-Meyerhoff] and that critical slope length is 6, we proved:

**Theorem:** [Agol, L 2000] For any hyperbolic 3-manifold $M$ with $\partial M$ a single torus, $M$ has at most 12 exceptional slopes.
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**Theorem:** [Agol, L 2000] For any hyperbolic 3-manifold $M$ with $\partial M$ a single torus, $M$ has at most 12 exceptional slopes.

But reducing 12 down to 10 is very hard . . .
A difficulty

Unfortunately, there is an example, due to Agol, of a manifold $M$ s.t.
1. It has an exceptional slope $s$ with length 6.
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Unfortunately, there is an example, due to Agol, of a manifold $M$ s.t.
1. It has an exceptional slope $s$ with length 6.
2. It has 12 slopes with length at most 6.
Fortunately, this manifold falls into a well-understood family of exceptions.

**Theorem:** [Gabai, Meyerhoff, Milley] The cusp area of $M$ is at least $3.7$ unless $M$ is obtained by Dehn filling some of the boundary components of one of the manifolds

| $m412$ | $s596$ | $s647$ | $s774$ | $s776$ | $s780$ | $s785$ | $s898$ | $s959$ |

The notation is from the ‘census’ of hyperbolic 3-manifolds.
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The notation is from the ‘census’ of hyperbolic 3-manifolds.

Using this and the 6 theorem, you can get a bound of **11** exceptional surgeries.
Work of Cao and Meyerhoff

They consider the horoball diagram of $M$:

They let $e_2 \geq 1$ be the (Euclidean diameter)$^{-1/2}$ of the second largest horoballs.

If $e_2$ is close to 1, then we get lots of large horoballs and so large cusp area.
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If $e_2$ is close to 1, then we get lots of large horoballs and so large cusp area.

If $e_2$ is large, then the largest horoballs must be well-separated.

A computer calculation, ranging over all values of $e_2$, gives that $A \geq 3.35$. 
They consider the second, third and fourth largest horoballs!
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One can deduce that $A \geq 3.7$ unless certain configurations of horoballs arise.
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In these cases, one can show that $M$ is obtained by Dehn filling some of the boundary components of one of the manifolds

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How to solve the Dehn surgery problem

Improve the 6 theorem:

**Theorem**: Suppose that

\[ L(s) > \frac{\pi e_2}{\arcsin(e_2/2)} \]

if \( e_2 \leq \sqrt{2} \), and that

\[ L(s) > \frac{2\pi \sqrt{1 - e_2^{-2}}}{\arcsin(\sqrt{1 - e_2^{-2}})} \]

if \( e_2 > \sqrt{2} \). Then, \( M(s) \) is hyperbolic.
How to solve the Dehn surgery problem
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Repeat the analysis of Gabai, Milley and Meyerhoff, but specifically search for the number of slopes with length less than the given bound.

Must also use a range of geometric arguments to rule out certain horoball configurations.
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Repeat the analysis of Gabai, Milley and Meyerhoff, but specifically search for the number of slopes with length less than the given bound.

Must also use a range of geometric arguments to rule out certain horoball configurations.

The computer calculation takes about 2 days.

It only just works!
Can we trust the computer?

Yes!

The computer code can be checked by hand, much like checking a proof. The possibility that errors arise due to the use of floating point arithmetic can be ruled out. One replaces each real number by an interval, which is effectively the number with an error bar. Use is also made of the computer program Snap which is a version of Snappea that uses exact arithmetic.
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