COVERING SPACES OF 3-ORBIFOLDS

MARC LACKENBY

Abstract

Let O be a compact orientable 3-orbifold with non-empty singular locus and a finite volume hyperbolic structure. (Equivalently, the interior of O is the quotient of hyperbolic 3-space by a lattice in $\mathrm{PSL}(2, \mathbb{C})$ with torsion.) Then we prove that O has a tower of finite-sheeted covers $\{O_i\}$ with linear growth of p-homology, for some prime p. This means that the dimension of the first homology, with mod p coefficients, of the fundamental group of O_i grows linearly in the covering degree. The proof combines techniques from 3-manifold theory with group-theoretic methods, including the Golod-Shafarevich inequality and results about p-adic analytic pro-p groups. This has several consequences. Firstly, the fundamental group of O has at least exponential subgroup growth. Secondly, the covers $\{O_i\}$ have positive Heegaard gradient. Thirdly, we use it to show that a group-theoretic conjecture of Lubotzky and Zelmanov would imply that O has large fundamental group. This implication uses a new theorem of the author, which will appear in a forthcoming paper. These results all provide strong evidence for the conjecture that any closed orientable hyperbolic 3-orbifold with non-empty singular locus has large fundamental group. Many of the above results apply also to 3-manifolds commensurable with an orientable finite-volume hyperbolic 3-orbifold with non-empty singular locus. This includes all closed orientable hyperbolic 3-manifolds with rank two fundamental group, and all arithmetic 3-manifolds.

1. Introduction

A central topic in 3-manifold theory is concerned with a manifold's finite-sheeted covering spaces. The majority of effort in this field was initially focused on 3-manifolds that are finitely covered by manifolds that are well understood, in the hope that this would provide illuminating information about the original manifold. There are a number of excellent theorems in this direction, for example, the result due to Gabai, Meyerhoff and N. Thurston [5] that virtually hyperbolic 3-manifolds are hyperbolic. Attention has now turned to proving that hyperbolic

MSC (2000): 57N10; 30F40; 20E07

3-manifolds always have finite covers with 'nice' properties. The central unresolved question in this direction is the Virtually Haken Conjecture which proposes that any closed hyperbolic 3-manifold should be finitely covered by a Haken 3-manifold. There are even stronger versions of this conjecture, but they all remain wide open at present.

There are other questions about finite-sheeted covering spaces that arise naturally. A key one is: how many finite-sheeted covering spaces does a 3-manifold have, as a function of the covering degree? The answer is not known in general, not even asymptotically. This question fits naturally into an emerging area of group theory, which deals with 'subgroup growth'. This addresses the possible growth rates for the number of finite index subgroups of a group, as a function of their index. Linear groups play a central rôle in this theory, particularly lattices in Lie groups. However, discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ or equivalently the fundamental groups of orientable hyperbolic 3-orbifolds, remain poorly understood.

There are other natural questions that can be asked, which focus on the asymptotic behaviour of various properties of the finite-sheeted covering spaces. For example, one might wish to examine how the following quantities can grow, as a function of the covering degree: the rank of the fundamental group, the rank and order of the first homology, the Heegaard genus, the Cheeger constant and the first eigenvalue of the Laplacian. Indeed, a good understanding of these quantities is likely to lead to progress on the Virtually Haken Conjecture, by work of the author in [6].

The main tenet of this paper is that 3-orbifolds with non-empty singular locus form a more tractable class than 3-manifolds, when one is considering finite-sheeted covering spaces. In particular, for discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$, it is those with torsion that we can analyse most successfully.

Our first and central result is the following. It deals with $d_p(\,\cdot\,)$. This is defined to be the dimension of $H_1(\,\cdot\,;\mathbb{F}_p)$, where $\mathbb{F}_p$ is the field of prime p order, and $\cdot$ is either a group or a topological space. We say that a collection $\{G_i\}$ of finite-index subgroups of a finitely generated group G has *linear growth of mod p homology* if $\inf_i d_p(G_i)/[G : G_i] > 0$.

Theorem 1.1. *Let O be a compact orientable 3-orbifold with non-empty singular locus and a finite-volume hyperbolic structure. Then O has a tower of finite-sheeted covers $\ldots \rightarrow O_2 \rightarrow O_1 \rightarrow O$ where $\{\pi_1(O_i)\}$ has linear growth of mod p homology, for some prime p. Furthermore, one can ensure that the following properties also hold:*

(i) One can find such a sequence where each O_i is a manifold, and (when O is closed) another such sequence where each O_i has non-empty singular locus.

(ii) Successive covers $O_{i+1} \rightarrow O_i$ are regular and have degree p.

(iii) For infinitely many i, $O_i \rightarrow O_1$ is regular.

(iv) One can choose p to be any prime that divides the order of an element of $\pi_1(O)$.

A slightly stronger version of the main part of this theorem is as follows.

Theorem 1.2. *Any finitely generated, discrete, non-elementary subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *with torsion has a nested sequence of finite index subgroups with linear growth of mod p homology for some prime p.*

Linear growth of mod p homology is a strong conclusion, with several interesting consequences. For example, one can use it to find good lower bounds on the subgroup growth of the fundamental group of a 3-orbifold. The *subgroup growth function* $s_n(G)$, for a finitely generated group G, is defined to be the number of subgroups with index at most n. Of course, when O is an orbifold, $s_n(\pi_1(O))$ simply counts the number of covering spaces of O (with given basepoint) with degree at most n. It is said to have *polynomial growth* if there is some constant c such that $s_n(G) \leq n^c$ for each n. It has *at least exponential growth* if there is some constant $c > 1$ such that $s_n(G) \geq c^n$ for each n. Note that this is quite a strong form of exponential growth: many authors just insist that $s_n(G) \geq c^n$ for infinitely many n.

Recall that an orbifold O is *geometric* if its interior is a quotient X/G, where X is a complete simply-connected homogeneous Riemannian manifold and G is a discrete group of isometries. There are 8 possible model geometries X for compact 3-orbifolds. Furthermore, when O is closed, it admits a geometry modelled on at most one such model space. Thurston's Orbifold Theorem ([1], [3]) states that a compact orientable 3-orbifold with non-empty singular locus and no bad 2-suborbifolds admits a 'canonical decomposition into geometric pieces'. Thus, it is natural to consider geometric 3-orbifolds. In the following result, we consider the subgroup growth of their fundamental groups.

Theorem 1.3. *Let O be a compact orientable geometric 3-orbifold with non-empty singular locus. Then, the subgroup growth of $\pi_1(O)$ is*

$$\begin{cases} \textit{polynomial}, & \textit{if } O \textit{ admits an } S^3,\ \mathbb{E}^3,\ S^2\times\mathbb{E}, \textit{ Nil or Sol geometry;} \\ \textit{at least exponential}, & \textit{otherwise.} \end{cases}$$

In fact, our techniques provide quite precise information about the number of subnormal subgroups of $\pi_1(O)$. Recall that a subgroup K of a group G is *subnormal*, denoted $K \lhd\lhd G$, if there is a finite sequence of subgroups $G = G_1 \rhd G_2 \rhd \dots \rhd G_n = K$, where each G_i is normal in G_{i-1}. Such groups arise naturally when one considers towers of regular covers. For a finitely generated group G, the number of subnormal subgroups of G with index at most n is denoted $s_n^{\lhd\lhd}(G)$. It is known that $s_n^{\lhd\lhd}(G)$ always grows at most exponentially (Theorem 2.3 of [16]). That is, there is a constant c depending on G such that $s_n^{\lhd\lhd}(G) \leq c^n$ for each n. We will show in Theorem 5.2 that, under the hypotheses of Theorem 1.3, $\pi_1(O)$ has either polynomial or exponential subnormal subgroup growth, depending on whether or not O admits an S^3, $\mathbb{E}^3$, $S^2 \times \mathbb{E}$, Nil or Sol geometry. In the hyperbolic case, this is rephrased as follows.

Theorem 1.4. *Any finitely generated, discrete, non-elementary subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *with torsion has exponential subnormal subgroup growth.*

Theorem 1.1 also provides some new information about the behaviour of the Heegaard Euler characteristic $\chi_-^h(O_i)$ of the manifold covering spaces O_i. Recall [6] that this is defined to be the negative of the largest possible Euler characteristic for a Heegaard surface for O_i. The *Heegaard gradient* of a sequence $\{O_i \to O\}$ of finite-sheeted (manifold) covers is defined to be $\inf_i \chi_-^h(O_i)/\mathrm{degree}(O_i \to O)$. Since $\chi_-^h(O_i)$ is bounded below by a linear function of $d_p(O_i)$, for any prime p, linear growth of mod p homology implies positive Heegaard gradient. Thus, Theorem 1.1 has the following immediate consequence.

Corollary 1.5. *Let O be a compact orientable 3-orbifold with non-empty singular* *locus and a finite-volume hyperbolic structure. Then O has a nested sequence of finite-sheeted manifold covers with positive Heegaard gradient.*

This is related to a proposed approach to the Virtually Haken Conjecture, which arises from the following theorem of the author [6]. (The theorem below is not stated precisely this way in [6], but this version is an immediate consequence of Theorem 1.4 of [6].)

Theorem 1.6. *Let M be a closed orientable irreducible 3-manifold, and let $\{M_i \to M\}$ be a nested sequence of finite-sheeted regular covering spaces of M. Suppose that*

(i) the Heegaard gradient of $\{M_i \to M\}$ is positive; and

(ii) $\pi_1(M)$ does not have Property (τ) with respect to $\{\pi_1(M_i)\}$.

Then M_i is Haken for all sufficiently large i.

Property (τ), referred to in the above theorem, is an important concept from group theory, with links to many areas of mathematics, including representation theory, graph theory and differential geometry [12]. Lubotzky and Sarnak [13] conjectured that a closed hyperbolic 3-manifold M should always have a sequence of finite-sheeted covering spaces where (ii) holds. A key question is whether this is true for the covers in Corollary 1.5. A positive answer would settle the Virtually Haken Conjecture for compact orientable hyperbolic 3-orbifolds with non-empty singular locus (by setting M in Theorem 1.6 to be the first manifold cover O_1). Indeed, the following recent theorem of the author would provide a much stronger conclusion. Recall that a group is *large* if it has a finite index subgroup that admits a surjective homomorphism onto a non-abelian free group.

Theorem 1.7. *Let G be a finitely presented group, let p be a prime and suppose that $G \geq G_1 \triangleright G_2 \triangleright \dots$ is a nested sequence of finite index subgroups, such that G_{i+1} is normal in G_i and has index a power of p, for each i. Suppose that $\{G_i\}$ has linear growth of mod p homology. Then, at least one of the following must hold:*

(i) G is large;

(ii) G has Property (τ) with respect to $\{G_i\}$.

This is, in fact, a slightly abbreviated and slightly weaker version of the main theorem (Theorem 1.1) of [8]. Set G to be $\pi_1(O)$ and let G_i be $\pi_1(O_i)$, where $\{O_i \to O\}$ is one of the sequences of covering spaces in Theorem 1.1. We know from Theorem 1.1 that $\{G_i\}$ has linear growth of mod p-homology. Thus, the central question is: does G have Property (τ) with respect to $\{G_i\}$? We conjecture that we may pick $\{G_i\}$ so that it does not. In fact, we will see that this would follow from the following recent conjecture of Lubotzky and Zelmanov [14], which we have termed the GS-τ Conjecture.

Conjecture 1.8. (GS-τ Conjecture) *Let G be a group with finite presentation $\langle X|R\rangle$, and let p be a prime. Suppose that $d_p(G)^2/4 > |R| - |X| + d_p(G)$. Then G does not have Property (τ) with respect to some infinite nested sequence $\{G_i\}$ of normal subgroups with index a power of p.*

The point is that the pro-p completion of G has a pro-p presentation with $d_p(G)$ generators and $(|R| - |X| + d_p(G))$ relations. Thus, the condition in the GS-τ Conjecture asserts that the Golod-Shafarevich inequality does not hold for this pro-p presentation. The theory of such pro-p groups is advanced and there is some hope that it may be applied and developed to prove this conjecture. Indeed, the Golod-Shafarevich inequality will play a central rôle in this paper.

We will show that the following is a consequence of Theorems 1.1 and 1.7.

Theorem 1.9. *The GS-τ Conjecture implies that the fundamental group of every closed hyperbolic 3-orbifold with non-empty singular locus is large.*

It therefore seems that the evidence for the following conjecture is stacking up.

Conjecture 1.10. *The fundamental group of any closed hyperbolic 3-orbifold with non-empty singular locus is large.*

Indeed, Theorems 1.1 and 1.3 are already pointing in this direction. This is because linear growth of mod p homology for some nested sequence of finite-index subgroups and (at least) exponential subgroup growth are both strong properties of a group, which are enjoyed when the group is large.

It is worth pointing out that the GS-τ Conjecture is not the only possible approach to Conjecture 1.10. Another possible route, explained in [8], is via error-correcting codes.

Another piece of evidence for Conjecture 1.10 comes from work of the author in [7]. There, a sequence of results about 3-orbifolds is established, one of which is the following.

Theorem 1.11. (Theorem 3.6 of [7]) *Let O be a compact orientable 3-orbifold (with possibly empty singular locus), and let K be a knot in O, disjoint from its singular locus, such that $O - K$ admits a finite-volume hyperbolic structure. For any integer n, let $O(K,n)$ denote the 3-orbifold obtained from O by adjoining a singular component along K with order n. Then, for infinitely many values of n, $\pi_1(O(K,n))$ is large.*

Of course, Conjecture 1.10 is weaker than the old conjecture that the fundamental group of any closed hyperbolic 3-manifold is large, because any closed hyperbolic 3-orbifold is finitely covered by a hyperbolic 3-manifold, by Selberg's Lemma. But the main purpose of this paper is to demonstrate that covering spaces of 3-orbifolds with non-empty singular locus are more tractable than the manifold case, and so there is some chance that Conjecture 1.10 may be more likely to be true.

The slightly stronger version of Theorem 1.7 that appears in [8] has the following consequence. (See [8] for an explanation of this deduction.)

Theorem 1.12. *Let G be a finitely presented group, and let p be a prime. Suppose that G has an infinite nested sequence of subnormal subgroups $\{G_i\}$, each with index a power of p and with linear growth of mod p homology. Then G has such a sequence that also has Property (τ).*

Combining this with Theorem 1.1, we have the following corollary.

Theorem 1.13. *Let O be a compact orientable 3-orbifold with non-empty singular locus and a finite-volume hyperbolic structure. Then O has a nested sequence of finite-sheeted covers with Property (τ).*

It is worth pointing out that the above results remain true for hyperbolic 3-manifolds that are commensurable with hyperbolic 3-orbifolds with non-empty singular locus.

Theorem 1.14. *Let M be a 3-manifold that is commensurable with a compact orientable finite-volume hyperbolic 3-orbifold O with non-empty singular locus. Let p be a prime that divides the order of a torsion element of $\pi_1(O)$. Then $\pi_1(M)$ has exponential subnormal subgroup growth, and M has a nested sequence of finite-sheeted covers that have linear growth of mod p homology and have Property (τ).*

There are many examples of such 3-manifolds M. A large class of examples is the orientable hyperbolic 3-manifolds with rank two fundamental group, which are 2-fold regular covers of hyperbolic 3-orbifolds with non-empty singular locus (Corollary 5.4.2 of [19]).

Corollary 1.15. *If M a closed orientable hyperbolic 3-manifold with rank two fundamental group, then the conclusions of Theorems 1.14 apply to M, for $p = 2$.*

Another important class of 3-manifolds to which one may apply these results are arithmetic hyperbolic 3-manifolds. The following theorem appears in [9].

Theorem 1.16. *Let M be an arithmetic hyperbolic 3-manifold. Then M is commensurable with an arithmetic hyperbolic 3-orbifold, with fundamental group that contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Thus, we have the following result.

Corollary 1.17. *If M is an arithmetic hyperbolic 3-manifold, then the conclusions of Theorem 1.14 apply to M for $p = 2$.*

In this paper, we assume some familiarity with the basic theory of orbifolds. In particular, we take as given the following terminology: singular locus, covering space, fundamental group. For an explanation of these terms and introduction to orbifolds, we suggest [3] and [18] as helpful references.

2. Covering spaces without vertices

The main focus of this paper will be 3-orbifolds that admit a hyperbolic structure and that have non-empty singular locus. It turns out that it is easiest to deal with 3-orbifolds that have singular locus containing no vertices. Our goal in this section is to show that one can always arrange this to be the case, by first passing to a finite-sheeted cover.

Proposition 2.1. *Let O be a compact orientable hyperbolic 3-orbifold with non-empty singular locus. Then O has a finite-sheeted cover $\tilde{O}$ with singular locus that is non-empty and that contains no vertices. Moreover, for any prime p that divides the order of an element of $\pi_1(O)$, we may arrange that the order of each component of this singular locus is p. In addition, we may ensure that there is a degree p regular cover $M \to \tilde{O}$ where M is a manifold, and where the composite cover $M \to \tilde{O} \to O$ is regular.*

Proof. Since O is hyperbolic, its fundamental group is realised as a subgroup G of $\mathrm{PSL}(2, \mathbb{C})$. By Selberg's Lemma, G has a finite index normal subgroup K that is torsion free. Let M be the regular covering space of O corresponding to K; this is a manifold. Let μ be an element of G that has finite order. We may choose μ so that it has order p. We claim that the covering space $\tilde{O}$ of O corresponding to $K\langle\mu\rangle$ has non-empty singular locus that contains no vertices. Since K is normal in G, it is also normal in $K\langle\mu\rangle$. So, $M \to \tilde{O}$ is a regular cover, with covering group $K\langle\mu\rangle/K$, which is cyclic of order p. Hence, $\tilde{O}$ is the quotient of M by a finite order orientation-preserving diffeomorphism. The singular locus of $\tilde{O}$ is therefore a 1-manifold: it has no vertices. It is non-empty, because the fundamental group of $\tilde{O}$, namely $K\langle\mu\rangle$, contains μ which has finite order. Since the order of the diffeomorphism is p, which is prime, the order of every component of the singular locus is p. □

We now introduce some convenient terminology. If p is a prime and O is a 3-orbifold, then we let $\mathrm{sing}^c_p(O)$ denote those simple closed curve components of the singular locus of O with singularity order that is a multiple of p.

3. Covering spaces with large mod p homology

The goal of this section is to prove the following result.

Theorem 3.1. *Let O be an orientable finite-volume hyperbolic 3-orbifold. Let p be a prime such that $\mathrm{sing}^c_p(O)$ is non-empty. Then there is a finite-sheeted cover $\tilde{O} \to O$, such that $d_p(\pi_1(\tilde{O})) \geq 11$ and where $\mathrm{sing}^c_p(\tilde{O})$ is non-empty. In addition, we may ensure that there is a regular covering space $\tilde{O}' \to \tilde{O}$ which has degree p*

or 1 *and where the composite cover* $\tilde{O}' \to O$ *is regular.*

The significance of the number 11 in the above theorem will be made clear in Theorem 4.1 and its proof. The proof of Theorem 3.1 relies on the following result, which is due to Lubotzky, and is of independent interest.

Theorem 3.2. *Let O be an orientable finite-volume hyperbolic 3-orbifold (with possibly empty singular locus). Then, for any prime p,*

$$\sup\{d_p(K) : K \textit{ is a finite index normal subgroup of } \pi_1(O)\} = \infty.$$

The proof of this requires three theorems, which we quote. The first is a consequence of Nori-Weisfeiler's Strong Approximation Theorem and the Lubotzky Alternative (see Corollary 18 of Window 9 in [16] for example.)

Theorem 3.3. *Let G be a finitely generated linear group that is not virtually soluble. Then, for any prime p,*

$$\sup\{d_p(K) : K \textit{ is a finite index subgroup of } G\} = \infty.$$

The subgroups provided by this theorem need not be normal. In fact, there exist groups G satisfying the hypotheses of Theorem 3.3 but where

$$\sup\{d_p(K) : K \text{ is a finite index normal subgroup of } G\}$$

is finite, for infinitely many primes p. An example is $\mathrm{SL}(n, \mathbb{Z})$, for any $n \geq 3$ and any prime p not dividing n (see Proposition 1.4 of [11]). So, more work is necessary before Theorem 3.2 can be proved. The second result we quote can be found in [15].

Theorem 3.4. *Let G be a finitely generated group, let p be a prime and let $\hat{G}_{(p)}$ be the pro-p completion of G. Then the following are equivalent:*

1. *$\hat{G}_{(p)}$ is p-adic analytic;*
2. *the supremum of $d_p(K)$, as K ranges over all characteristic subgroups of G with index a power of p, is finite;*

3. *the supremum of $d_p(K)$, as K ranges over all normal subgroups of G with index a power of p, is finite.*

The relationship between p-adic analytic pro-p groups and 3-manifolds can be seen in the following result, which is due to Lubotzky [10].

Theorem 3.5. *Let M be a compact orientable 3-manifold and let p be a prime. If $d_p(M) \geq 4$, then the pro-p completion of $\pi_1(M)$ is not p-adic analytic.*

The following is elementary and fairly well known.

Proposition 3.6. *Let G be a finitely generated group, and let K be a normal subgroup. Then, $d_p(G) \leq d_p(K) + d_p(G/K)$.*

Proof. Let G' be $[G, G]G^p$, which is the subgroup generated by the commutators and p^{th} powers of G. Define K' similarly. Consider the exact sequence

$$1 \to \frac{KG'}{G'} \to \frac{G}{G'} \to \frac{G}{KG'} \to 1.$$

A set of elements in G/G' that maps to a generating set for G/KG', together with a generating set for KG'/G', forms a generating set for G/G'. Hence, writing $d(\ )$ for the minimal number of generators for a group,

$$d_p(G) = d(G/G') \leq d(KG'/G') + d(G/KG').$$

Now, $d(G/KG') = d_p(G/K)$. Also, KG'/G' is isomorphic to $K/(K \cap G')$, which is a quotient of K/K'. Hence, $d(KG'/G') \leq d(K/K') = d_p(K)$. The required inequality now follows. □

Proof of Theorem 3.2. Let O be a compact orientable 3-orbifold with a finite-volume hyperbolic structure. Then $\pi_1(O)$ is realised as a lattice in $\mathrm{PSL}(2, \mathbb{C})$. Therefore, by Selberg's Lemma, $\pi_1(O)$ has a finite index subgroup which is torsion free. This corresponds to a manifold covering space of O. By Theorem 3.3, this has a finite-sheeted cover M where $d_p(M) \geq 4$, say. Let $M' \to O$ be the covering corresponding to the intersection of all conjugates of $\pi_1(M)$ in $\pi_1(O)$. Then $M' \to M$ and $M' \to O$ are finite-sheeted regular covers.

By Theorem 3.5, the pro-p completion of $\pi_1(M)$ is not p-adic analytic. Hence, by Theorem 3.4, M has finite-sheeted covers M_i such that $\pi_1(M_i)$ is characteristic in $\pi_1(M)$ and has index a power of p, and where $d_p(M_i)$ tends to infinity. Let

M_i' be the cover of M corresponding to the subgroup $\pi_1(M') \cap \pi_1(M_i)$. This is the intersection of two normal subgroups of $\pi_1(M)$ and so is normal in $\pi_1(M)$. Hence, M_i' regularly covers M, M' and M_i. Now, by Proposition 3.6, $d_p(M_i') \geq d_p(M_i) - d_p(\pi_1(M_i)/\pi_1(M_i'))$. But $\pi_1(M_i)/\pi_1(M_i') = \pi_1(M_i)/(\pi_1(M_i) \cap \pi_1(M')) = \pi_1(M_i)\pi_1(M')/\pi_1(M')$, which is a subgroup of $\pi_1(M)/\pi_1(M')$. There are only finitely many such subgroups and hence their d_p is uniformly bounded above. So, $d_p(M_i')$ tends to infinity.

Now, $\pi_1(M')/\pi_1(M_i') = \pi_1(M')/(\pi_1(M_i) \cap \pi_1(M')) = \pi_1(M')\pi_1(M_i)/\pi_1(M_i)$, which is a subgroup of $\pi_1(M)/\pi_1(M_i)$. This is a finite p-group, and hence so is $\pi_1(M')/\pi_1(M_i')$. Thus, $\pi_1(M_i')$ corresponds to a finite index normal subgroup of the pro-p completion of $\pi_1(M')$. Since $d_p(M_i')$ tends to infinity, this pro-p completion is not p-adic analytic, by Theorem 3.4, and so, by Theorem 3.4, $\pi_1(M')$ has a sequence of characteristic subgroups K_i, each with index a power of p, and where $d_p(K_i)$ tends to infinity. Since these are characteristic in $\pi_1(M')$, which is normal in $\pi_1(O)$, they are therefore normal in $\pi_1(O)$. These are the required subgroups. $\square$

Note that, in the proof of Theorem 3.2, we only used once the hypothesis that $\pi_1(O)$ is the fundamental group of an orientable finite-volume hyperbolic 3-orbifold. This was used to show that it has a finite index subgroup that is not p-adic analytic. Once this is known, the remainder of the proof is purely group-theoretic.

Proof of Theorem 3.1. Let μ be an element of $\pi_1(O)$ that forms a meridian for a component C of $\mathrm{sing}_p^c(O)$. Then μ^n has order p for some positive integer n. Let K be a finite index normal subgroup of $\pi_1(O)$, as in Theorem 3.2, where $d_p(K)$ is large (more than $10p$ will suffice). Let $\tilde{O}'$ be the regular covering space of O corresponding to this subgroup.

If μ^n lies in K, then we set $\tilde{O}$ to be $\tilde{O}'$. Note that $\mathrm{sing}_p^c(\tilde{O})$ is non-empty because the inverse image of C in $\tilde{O}$ contains at least one singular component such that p divides its singularity order, as μ^n lies in K.

So, suppose that μ^n does not lie in K. We will consider the covering space $\tilde{O}$ corresponding to the subgroup $K\langle \mu^n \rangle$. Note that again $\mathrm{sing}_p^c(\tilde{O})$ is non-empty. Note also that $\tilde{O}' \to \tilde{O}$ has degree equal to the order of $K\langle \mu^n \rangle / K$, which is p.

We want to show that, when $d_p(K)$ is large, then so is $d_p(K\langle\mu^n\rangle)$. In particular, when $d_p(K) \geq 10p + 1$, then $d_p(K\langle\mu^n\rangle) \geq 11$. This is an immediate consequence of the following well known proposition (setting $H = K\langle\mu^n\rangle$). □

Proposition 3.7. *Let H be a finitely generated group, and let K be a subnormal subgroup with index a power of a prime p. Then*

$$d_p(K) - 1 \leq [H : K](d_p(H) - 1).$$

Proof. By a straightforward induction, we may reduce to the case where K is a normal subgroup of H. Let K' denote $[K, K]K^p$, and define H' similarly. Then K/K' is isomorphic to $H_1(K; \mathbb{F}_p)$. Now, K' is characteristic in K, which is normal in H, and hence K' is normal in H. Thus, H/K' is a finite p-group. It is a well known fact that any finite p-group has a generating set with size equal to the dimension of its first homology with $\mathbb{F}_p$-coefficients, in this case $d_p(H/K')$. But $d_p(H/K') = d_p(H/K'H') = d_p(H)$. Now, K/K' is a subgroup of H/K' with index $[H : K]$. Applying the Reidemeister-Schreier process to this subgroup, we obtain a generating set for K/K' with size $[H : K](d_p(H) - 1) + 1$. This is therefore an upper bound for $d_p(K)$. The required inequality follows. □

4. Linear growth of the number of singular components

We now come to the central result of this paper, from which all later analysis of hyperbolic 3-orbifolds will follow.

Theorem 4.1. *Let O be a compact orientable 3-orbifold, with boundary a (possibly empty) union of tori, and with singular locus that is a link. Let p be a prime that divides the order of a component C of the singular locus, and let $\langle\langle \pi_1(\partial N(C)) \rangle\rangle$ be the subgroup of $\pi_1(O)$ normally generated by $\pi_1(\partial N(C))$. Suppose that $d_p(\pi_1(O)) \geq 11$. Then $\pi_1(O)$ has an infinite nested sequence of finite index subgroups $\{G_i\}$ such that*

(i) each G_i contains $\langle\langle \pi_1(\partial N(C)) \rangle\rangle$;

(ii) each G_{i+1} is normal in G_i and has index p; and

(iii) infinitely many G_i are normal in $\pi_1(O)$ and have index a power of p.

The significance of this theorem comes from the following proposition.

Proposition 4.2. *Let O be a 3-orbifold, let p be a prime, and let C be some component of* $\mathrm{sing}_p^c(O)$. *Let $\tilde{O} \to O$ be a finite-sheeted cover, such that $\pi_1(\tilde{O})$ contains the subgroup of $\pi_1(O)$ normally generated by $\pi_1(\partial N(C))$. Then $|\mathrm{sing}_p^c(\tilde{O})| \geq$ $\mathrm{degree}(\tilde{O} \to O)$.*

Proof. The condition that $\pi_1(\tilde{O})$ contains the subgroup of $\pi_1(O)$ normally generated by $\pi_1(\partial N(C))$ forces the inverse image of $\partial N(C)$ in $\tilde{O}$ to be a disjoint union of copies of $\partial N(C)$. Each copy bounds a component of $\mathrm{sing}_p^c(\tilde{O})$. □

We will also need the following key fact, which will be used both in the proof of Theorem 4.1 and in its applications later.

Proposition 4.3. *Let O be a compact orientable 3-orbifold with singular locus consisting of a link. Then, for any prime p, $d_p(\pi_1(O)) \geq |\mathrm{sing}_p^c(O)|$.*

Proof. We may remove the components of the singular locus of O that have order coprime to p, and replace them by manifold points. This is because such an operation changes neither $d_p(\pi_1(O))$ nor $\mathrm{sing}_p^c(O)$. Thus, we may assume that every component of the singular locus lies in $\mathrm{sing}_p^c(O)$.

Let M be $O - \mathrm{int}(N(\mathrm{sing}_p^c(O))$. Then, it is a well-known consequence of Poincaré duality that $d_p(M)$ is at least $\frac{1}{2} d_p(\partial M)$, which is at least $|\mathrm{sing}_p^c(O)|$. One obtains $\pi_1(O)$ from $\pi_1(M)$ by adding relations that are each p^{th} powers of words. This does not affect d_p. We therefore obtain the required inequality. □

We will need the following well known theorem. It deals with groups Γ whose pro-p completion has a (minimal) presentation that does not satisfy the Golod-Shafarevich condition. A proof can be found in [10]. In fact, Theorem 1.2 of [10] is a significantly stronger result.

Theorem 4.4. *Let $\langle X|R\rangle$ be a finite presentation of a group Γ, and let p be a prime. Suppose that*

$$d_p(\Gamma)^2/4 > |R| - |X| + d_p(\Gamma).$$

Then Γ has an infinite nested sequence of subgroups $\Gamma = \Gamma_1 \geq \Gamma_2 \geq \dots$, where Γ_{i+1} is normal in Γ_i and has index p. Furthermore, infinitely many Γ_i are normal in Γ.

Remark. The normal subgroups of Γ can, in fact, be taken to be the p-lower central series, which is defined by setting $\Gamma_1 = \Gamma$ and $\Gamma_{j+1} = [\Gamma_j, \Gamma]\Gamma_j^p$. The quotients Γ_j/Γ_{j+1} are elementary abelian p-groups, and therefore we may interpolate between Γ_j and Γ_{j+1} by a sequence of subgroups, where each is normal in its predecessor and has index p.

Proof of Theorem 4.1. Let O' be the orbifold with the same underlying manifold as O but with singular locus equalling $\mathrm{sing}_p^c(O)$. In other words, we remove from O those components of the singular locus with order coprime to p, and replace them by manifold points. Then there is a natural map $O \to O'$ which induces a surjective homomorphism $\pi_1(O) \to \pi_1(O')$ and an isomorphism $H_1(\pi_1(O); \mathbb{F}_p) \to H_1(\pi_1(O'); \mathbb{F}_p)$. In particular, $d_p(\pi_1(O')) \geq 11$. If we can show that the theorem holds for O', then it also holds for O. This is because the nested sequence of subgroups of $\pi_1(O')$ pulls back to give a corresponding sequence of subgroups of $\pi_1(O)$ with the required properties. Hence, by replacing O by O', we may assume that p divides the order of every component of the singular locus of O.

Removing the singular locus from O gives a compact orientable 3-manifold M with boundary a non-empty union of tori. Its fundamental group therefore has a presentation where the number of relations equals the number of generators minus 1. Let Γ be $\pi_1(O)/\langle\langle \pi_1(\partial N(C)) \rangle\rangle$. Then, Γ has a presentation $\langle X | R \rangle$ where $|R| - |X| = |\mathrm{sing}_p^c(O)|$, which is obtained from the presentation of $\pi_1(M)$ by killing a pair of generators for $\pi_1(\partial N(C))$, and adding a relation for each remaining component of the singular locus of O. By Proposition 4.3, $d_p(\pi_1(O)) \geq |\mathrm{sing}_p^c(O)|$, and hence $|R| - |X| \leq d_p(\pi_1(O))$. Now, $d_p(\Gamma) \geq d_p(\pi_1(O)) - 2$, since Γ is obtained from $\pi_1(O)$ by adding two relations. This implies that

$$\begin{aligned} d_p(\Gamma)^2/4 - d_p(\Gamma) + |X| - |R| &\geq d_p(\Gamma)^2/4 - 2d_p(\Gamma) - 2 \\ &\geq (d_p(\pi_1(O)) - 2)^2/4 - 2(d_p(\pi_1(O)) - 2) - 2 \\ &\geq 9^2/4 - 18 - 2 > 0. \end{aligned}$$

Note that the second and third inequalities hold because $x^2/4 - 2x - 2$ is an increasing function of x when $x \geq 4$. So, by Theorem 4.4, Γ has an infinite nested sequence of subgroups $\Gamma = \Gamma_1 \geq \Gamma_2 \geq \ldots$, where each Γ_{i+1} is normal in Γ_i and has index p. Furthermore, infinitely many Γ_i are normal in Γ. These pull back to give the required subgroups of $\pi_1(O)$. □

5. Growth of homology and subgroup growth

We can now prove the main theorem stated in the introduction.

Theorem 1.1. *Let O be a compact orientable 3-orbifold with non-empty singular locus and a finite-volume hyperbolic structure. Then O has a tower of finite-sheeted covers $\ldots \to O_2 \to O_1 \to O$ where $\{\pi_1(O_i)\}$ has linear growth of mod p homology, for some prime p. Furthermore, one can ensure that the following properties also hold:*

(i) One can find such a sequence where each O_i is a manifold, and (when O is closed) another such sequence where each O_i has non-empty singular locus.

(ii) Successive covers $O_{i+1} \to O_i$ are regular and have degree p.

(iii) For infinitely many i, $O_i \to O_1$ is regular.

(iv) One can choose p to be any prime that divides the order of an element of $\pi_1(O)$.

Proof. Consider first the main case, where O is closed. By Proposition 2.1, we may pass to a finite cover O' of O with non-empty singular locus, each component of which is a simple closed curve with order p. Moreover, we may ensure that O' has a degree p regular cover that is a manifold M. By Theorem 3.1, there is a finite-sheeted cover O'' of O', again where the singular locus is a non-empty collection of simple closed curves with order p, but also where $d_p(\pi_1(O'')) \geq 11$. Now apply Theorem 4.1 and Proposition 4.2 to deduce the existence of a tower of finite covers $O_i \to O''$ such that $|\mathrm{sing}^c_p(O_i)| \geq \mathrm{degree}(O_i \to O'')$. The covers $O_i \to O$ will be those required by the theorem. Proposition 4.3 gives that $d_p(\pi_1(O_i)) \geq |\mathrm{sing}^c_p(O_i)|$. Thus, $\{\pi_1(O_i)\}$ has linear growth of mod p homology, as required. By (ii) of Theorem 4.1, we may ensure that, for each i, $O_{i+1} \to O_i$ is regular and has degree p. By (iii) of Theorem 4.1, for infinitely many i, $O_i \to O''$ is regular, and hence so is $O_i \to O_1$.

We now show how to find another such sequence of covers consisting of manifolds. Now, O_i has a manifold cover M_i corresponding to the subgroup $\pi_1(O_i) \cap \pi_1(M)$. This is a regular cover, with degree equal to $[\pi_1(O_i) : \pi_1(O_i) \cap \pi_1(M)] = [\pi_1(O_i)\pi_1(M) : \pi_1(M)] = p$, since $\pi_1(O_i)\pi_1(M)/\pi_1(M)$ is a non-trivial subgroup of $\pi_1(O')/\pi_1(M)$, which is cyclic of order p. The fact that $\{\pi_1(M_i)\}$ has linear growth of mod p homology is a consequence of Proposition 3.6 (letting $G = \pi_1(O_i)$ and $K = \pi_1(M_i)$). Conclusions (ii) and (iii) of Theorem 1.1 apply to the sequence $\{M_i\}$ because they apply to $\{O_i\}$.

Suppose now that O has non-empty boundary. By Selberg's Lemma, we may pass to a finite-sheeted regular manifold cover M. By a result of Cooper, Long and Reid (Theorem 1.3 of [4], see also [7], [2] and [17]), $\pi_1(M)$ is large: it has a finite-index normal subgroup that admits a surjective homomorphism onto a free non-abelian group F. Let O_1 be the covering space corresponding to this subgroup. Now, F has a nested sequence of finite-index subgroups $\{F_i\}$ with linear growth of mod p homology. We may ensure that each F_{i+1} is normal in F_i with index p, that $F_1 = F$, and that infinitely many F_i are normal in F. The inverse image of these subgroups in $\pi_1(O)$ correspond to the required covering spaces O_i of O. □

A slight extension of Theorem 1.1 is the following.

Theorem 1.2. *Any finitely generated, discrete, non-elementary subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *with torsion has a nested sequence of finite index subgroups with linear growth of mod p homology for some prime p.*

Proof. Let O be the quotient of $\mathbb{H}^3$ by this subgroup. This is a hyperbolic 3-orbifold with non-empty singular locus. When O has finite volume, Theorem 1.1 gives a tower of finite-sheeted covering spaces $\{O_i\}$, and the subgroups of $\pi_1(O)$ corresponding to these covers are the required nested sequence. Suppose now that O has infinite volume. Selberg's Lemma, together with the assumption that $\pi_1(O)$ is finitely generated, implies that $\pi_1(O)$ has a finite-index normal subgroup that is torsion free. It is therefore isomorphic to the fundamental group of a compact orientable irreducible 3-manifold M with non-empty boundary. By the result of Cooper, Long and Reid (Theorem 1.3 of [4]), $\pi_1(M)$ is large, unless M is finitely covered by an I-bundle over a surface with non-negative Euler characteristic. But, $\pi_1(O)$ is then elementary, contrary to assumption. Thus, as argued in the bounded case in the proof of Theorem 1.1, this implies the existence of the required sequence of finite index subgroups. □

We are now in a position to prove Theorem 1.3 in the hyperbolic case.

Theorem 1.3. *Let O be a compact orientable geometric 3-orbifold with non-empty singular locus. Then, the subgroup growth of $\pi_1(O)$ is*

$$\begin{cases} \textit{polynomial}, & \textit{if } O \textit{ admits an } S^3, \mathbb{E}^3, S^2 \times \mathbb{E}, \textit{Nil or Sol geometry;} \\ \textit{at least exponential}, & \textit{otherwise.} \end{cases}$$

Proof (hyperbolic case). Let O be a compact orientable 3-orbifold that has non-empty singular locus and that is hyperbolic. Recall that this means that the interior of O admits a complete hyperbolic structure, which may have finite or infinite volume. If $\pi_1(O)$ is elementary, then O also admits a Euclidean structure and $\pi_1(O)$ has polynomial subgroup growth. (See Section 8 for this deduction.) If $\pi_1(O)$ is non-elementary and O has infinite volume, then, as in the proof of Theorem 1.2, $\pi_1(O)$ is large, and so has (faster than) exponential subgroup growth.

Thus, we may assume that O has finite volume. By Theorem 1.1, there is an infinite nested sequence of finite-sheeted covers $\{O_i\}$ where $\pi_1(O_i)$ has linear growth of mod p homology for some prime p, and where the degree of $O_{i+1} \to O_i$ is p for each i. The fact that $\pi_1(O)$ has at least exponential subgroup growth then follows from the following proposition. □

Proposition 5.1. *Let G be a finitely generated group, and let p be a prime. Suppose that G has an infinite nested sequence $\{G_i\}$ of finite index subgroups with linear growth of mod p homology. Suppose also that the index $[G_i : G_{i+1}]$ is bounded above independent of i. Then G has at least exponential subgroup growth. Furthermore, if each G_i is subnormal in G, then G has exponential subnormal subgroup growth.*

Proof. The number of normal subgroups of G_i with index p is $(p^{d_p(G_i)}-1)/(p-1)$. This is a lower bound for the number of subgroups of G with index $p[G : G_i]$. Adjoining G_i into this count, we deduce that for $n = p[G : G_i]$,

$$s_n(G) \geq \frac{p^{d_p(G_i)} - 1}{p - 1} + 1 > p^{d_p(G_i)-1}.$$

We are assuming that $\inf_i d_p(G_i)/[G : G_i]$ is some positive number λ. Hence, $s_n(G) \geq p^{\lambda[G:G_i]-1}$.

We need to find a lower bound on $s_n(G)$ for arbitrary positive integers n. This is where we use the assumption that $[G_i : G_{i+1}]$ is bounded above by some constant k. Thus, if we let $[[n]]$ denote the largest integer less than or equal to n of the form $p[G : G_i]$, we have the inequality $[[n]] \geq n/k$. Therefore,

$$\liminf_n \frac{\log s_n(G)}{n} \geq \liminf_n \frac{\log s_{[[n]]}(G)}{k[[n]]} \geq \frac{\lambda \log p}{kp} > 0.$$

Thus, G has at least exponential subgroup growth. In the case where each G_i is subnormal in G, the subgroups we are counting here are also subnormal, and so G then has exponential subnormal subgroup growth. □

Remark. One might be tempted to think that Theorem 1.3 may be strengthened, because its proof appears to be quite wasteful in the way it counts subgroups. In particular, it uses the existence of only one tower of finite covers $O_i \to O''$, but in fact many such towers are known to exists, that have the property that the number of components of the singular locus of O_i grows linearly in the degree of the cover. This is because these covers were constructed from a nested sequence of finite index subgroups of the group Γ. Such a sequence was proved to exist by Theorem 4.4, using the fact that Γ fails the Golod-Shafarevich condition. However, it is known that if a group fails this condition, then it has many finite index subgroups (see Theorem 4.6.4 in [16] for example). However, this does not lead to any significant improvement in the subgroup growth of $\pi_1(O)$, since the known lower bounds for the subgroup growth of Γ are swamped by the exponential terms arising from the linear growth of mod p homology of $\{\pi_1(O_i)\}$.

We would now like to establish the following result, which is a stronger version of Theorem 1.3. We will first prove it in the case where the orbifold admits a finite volume hyperbolic structure. The remaining seven geometries will be dealt with in the final chapter.

Theorem 5.2. *Let O be a compact orientable geometric 3-orbifold with non-empty singular locus. Then, the subnormal subgroup growth of $\pi_1(O)$ is*

$$\begin{cases} \text{polynomial}, & \text{if } O \text{ admits an } S^3, \mathbb{E}^3, S^2 \times \mathbb{E}, \text{ Nil or Sol geometry;} \\ \text{exponential}, & \text{otherwise.} \end{cases}$$

We will need the following lemma.

Lemma 5.3. *Let $\{G_i\}$ be a sequence of finite index subgroups of a finitely generated group G, and let H be a finite index subnormal subgroup of G. If $\{G_i\}$*

has linear growth of mod p homology for some prime p, then $\{G_i \cap H\}$ does also, after possibly discarding finitely many subgroups $G_i \cap H$.

Proof. An obvious induction allows us to reduce to the case where H is normal in G. Now, $G_i/(G_i \cap H)$ is isomorphic to G_iH/H, which is a subgroup of G/H. This places a uniform upper bound, independent of i, on $[G_i : (G_i \cap H)]$ and, since there are only finitely many subgroups of G/H, a uniform upper bound on $d_p(G_i/(G_i \cap H))$. Hence, by Proposition 3.6, there is a uniform upper bound on $d_p(G_i) - d_p(G_i \cap H)$. Thus, $\liminf_i d_p(G_i \cap H)/[G : G_i \cap H]$ is positive. Hence the infimum of $d_p(G_i \cap H)/[G : G_i \cap H]$ is also positive, once some initial terms of the sequence have been deleted. The lemma now follows. □

Proof of Theorem 5.2 (finite volume hyperbolic case). When O has boundary, the covering spaces $\{O_i\}$ we constructed in the proof of Theorem 1.1 had fundamental groups that were subnormal in $\pi_1(O)$, and successive covers $O_{i+1} \to O_i$ had degree p. Hence by Proposition 5.1, $\pi_1(O)$ has exponential subnormal subgroup growth.

Suppose now that O is closed. In the proof of Theorem 1.1, we considered a tower of covers $O_i \to O'' \to O' \to O$. But we do not necessarily know that $O'' \to O'$ and $O' \to O$ are regular, and so we do not know that $\pi_1(O_i)$ is a subnormal subgroup of $\pi_1(O)$. Thus Proposition 5.1 cannot be applied directly to deduce that $\pi_1(O)$ has exponential subnormal subgroup growth.

Now, by Proposition 2.1, there is a finite regular cover $M \to O'$ such that the composite cover $M \to O' \to O$ is regular. Hence, $\pi_1(M)$ is normal in $\pi_1(O')$ and $\pi_1(O)$. According to Theorem 3.1, there is a finite regular cover $\tilde{O}' \to O''$ such that the composite cover $\tilde{O}' \to O'' \to O'$ is regular. Hence, $\pi_1(\tilde{O}')$ is normal in $\pi_1(O')$ and $\pi_1(O'')$. Therefore, $\pi_1(M) \cap \pi_1(\tilde{O}')$ is normal in $\pi_1(O')$, $\pi_1(\tilde{O}')$ and $\pi_1(M)$. Now, $\pi_1(O_i)$ is subnormal in $\pi_1(O'')$ and so $\pi_1(\tilde{O}') \cap \pi_1(O_i)$ is subnormal in $\pi_1(\tilde{O}')$. This implies that $\pi_1(M) \cap \pi_1(\tilde{O}') \cap \pi_1(O_i)$ is subnormal in $\pi_1(M) \cap \pi_1(\tilde{O}')$. We therefore have the chain of subgroups $\pi_1(M) \cap \pi_1(\tilde{O}') \cap \pi_1(O_i) \lhd \lhd \pi_1(M) \cap \pi_1(\tilde{O}') \lhd \pi_1(M) \lhd \pi_1(O)$. So each element of the sequence $\{\pi_1(M) \cap \pi_1(\tilde{O}') \cap \pi_1(O_i)\}$ is subnormal in $\pi_1(O)$.

We claim that this sequence has linear growth of mod p homology, after one has possibly discarded some initial terms in the sequence. Now, by Theorem 3.1, $\pi_1(\tilde{O}')$ is normal in $\pi_1(O'')$ and has finite index. By Proposition 2.1, $\pi_1(M)$ is normal in $\pi_1(O')$ and has finite index. Hence, $\pi_1(M) \cap \pi_1(\tilde{O}')$ is normal in $\pi_1(\tilde{O}')$ and has finite index. Therefore, $\pi_1(M) \cap \pi_1(\tilde{O}')$ is subnormal in $\pi_1(O'')$ and has finite index. Applying Lemma 5.3 (with $G = \pi_1(O'')$ and $G_i = \pi_1(O_i)$ and $H = \pi_1(M) \cap \pi_1(\tilde{O}')$), we deduce that this sequence has linear growth of mod p homology, after possibly discarding some initial terms of the sequence. The theorem now follows by Proposition 5.1. □

6. Property (τ), large groups and linear growth of homology

In this section, we investigate to what extent the following recent result [8] can be used to establish that the fundamental group of a closed hyperbolic 3-orbifold O with non-empty singular locus is large.

Theorem 1.7. *Let G be a finitely presented group, let p be a prime and suppose that $G \geq G_1 \triangleright G_2 \triangleright \dots$ is a nested sequence of finite index subgroups, such that G_{i+1} is normal in G_i and has index a power of p, for each i. Suppose that $\{G_i\}$ has linear growth of mod p homology. Then, at least one of the following must hold:q*

(i) G is large;

(ii) G has Property (τ) with respect to $\{G_i\}$.

Let O'' be as in the proof of Theorem 1.1, and let G be $\pi_1(O'')$. As in Theorem 4.1, let C be a component of $\mathrm{sing}_p^c(O'')$, and let $\langle\langle \pi_1(\partial N(C)) \rangle\rangle$ be the subgroup of $\pi_1(O'')$ normally generated by $\pi_1(\partial N(C))$. Let Γ be $\pi_1(O'')/\langle\langle \pi_1(\partial N(C)) \rangle\rangle$. It is shown in the proof of Theorem 4.1 that Γ has a presentation where the inequality of the GS-τ Conjecture is satisfied. Let us suppose that this conjecture is true. It would imply that Γ does not have Property (τ) with respect to some nested sequence $\{\Gamma_i\}$ of normal subgroups, each with index a power of p. Let G_i be the inverse image of Γ_i in G. Then, G does not have Property (τ) with respect to $\{G_i\}$. As in the proof of Theorem 4.1, let O_i be the covering space of O'' corresponding to G_i. It is shown there that $|\mathrm{sing}_p^c(O_i)|$ is at least the degree of the cover $O_i \to O''$. By Proposition 4.3, $\{G_i\}$ therefore has linear growth of mod p homology. So, Theorem 1.7 implies that G is large, which implies that $\pi_1(O)$ is large. Thus, we have proved the following.

Theorem 1.9. *The GS-τ Conjecture implies that the fundamental group of every closed hyperbolic 3-orbifold with non-empty singular locus is large.*

7. Manifolds commensurable with hyperbolic 3-orbifolds

In this section, we consider manifolds M that are commensurable with a hyperbolic 3-orbifold O with non-empty singular locus. The aim is prove that many of the properties we have deduced for O also hold for M. In particular, our goal is to prove Theorem 1.14.

Lemma 7.1. *Let G be a finitely generated group with exponential subnormal subgroup growth. Then any finite index normal subgroup H of G also has exponential subnormal subgroup growth.*

Proof. If G_i is a finite index subnormal subgroup of G, then $G_i \cap H$ is subnormal in H. The index $[H : G_i \cap H]$ is at most $[G : G_i]$. For any given subgroup of H with index n, the number of ways of writing it as $G_i \cap H$ for some subgroup G_i of G is at most $(mn)^{\log m}$, where $m = [G : H]$ (see the proof of Corollary 1.2.4 in [16]). Hence, the fact that G has exponential subnormal subgroup growth implies that H does also. □

Theorem 1.14. *Let M be a 3-manifold that is commensurable with a compact orientable finite-volume hyperbolic 3-orbifold O with non-empty singular locus. Let p be a prime that divides the order of a torsion element of $\pi_1(O)$. Then $\pi_1(M)$ has exponential subnormal subgroup growth, and M has a nested sequence of finite-sheeted covers that have linear growth of mod p homology and have Property (τ).*

Proof. Let M' be the common finite cover of O and M. We may find a finite cover M'' of M' such that $M'' \to M'$ and $M'' \to O$ are both regular covers. We may find a further finite cover M''' of M'' such that $M''' \to M''$ and $M''' \to M$ are regular.

By Theorem 5.2, $\pi_1(O)$ has exponential subnormal subgroup growth. Lemma 7.1, applied twice, implies that $\pi_1(M''')$ does also, and hence so does $\pi_1(M)$.

By Theorem 1.1, O has a nested sequence of finite-sheeted covers $\{O_i\}$ such that $\{\pi_1(O_i)\}$ has linear growth of mod p homology, and where each $\pi_1(O_i)$ is normal in $\pi_1(O_{i-1})$ and has index p. Let M_i be the covering space of O corresponding to the subgroup $\pi_1(M''') \cap \pi_1(O_i)$. These cover M''' which covers M. By Lemma 5.3 (with $G = \pi_1(O)$, $G_i = \pi_1(O_i)$ and $H = \pi_1(M'''))$, these covers have linear growth of mod p homology, after possibly discarding some initial terms in the sequence.

Now, each $\pi_1(M_i)$ is subnormal in $\pi_1(M_1)$ and has index a power of p. Hence, by Theorem 1.12 (with $G = \pi_1(M_1)$), we may arrange that, in addition, this sequence of subgroups has Property (τ). $\square$

8. Geometric non-hyperbolic 3-orbifolds

In this section, we study compact orientable 3-orbifolds O that admit a geometric structure other than hyperbolic. Our goal is to prove Theorem 5.2 (and hence Theorem 1.3) in this case.

Note first that any compact geometric 3-orbifold is *very good*: it admits a finite-sheeted manifold cover (Corollary 2.27 of [3]). We start by considering orbifolds that admit a geometry based on $\mathbb{H}^2 \times \mathbb{E}$ or $\widetilde{PSL_2}(\mathbb{R})$, but which do not admit any of the remaining 6 geometries. Pass to a finite-sheeted manifold cover M. Then, M is Seifert fibred, and the base orbifold is hyperbolic. The base orbifold therefore admits a finite-sheeted cover that is an orientable surface S with negative Euler characteristic. This pulls back to give a finite covering $\tilde{M} \to M$. Now, the Seifert fibration induces a surjective homomorphism $\pi_1(\tilde{M}) \to \pi_1(S)$, and $\pi_1(S)$ admits a surjective homomorphism onto $\mathbb{Z} * \mathbb{Z}$. Therefore, $\pi_1(O)$ is large. In particular, its subgroup growth and subnormal subgroup growth are (at least) exponential.

Suppose now that the orbifold admits a geometry based on S^3, $\mathbb{E}^3$, $S^2 \times \mathbb{E}$, Nil or Sol geometry. Any 3-orbifold modelled on spherical geometry clearly has finite fundamental group, and hence its subgroup growth is trivially polynomial. When the model geometry is $\mathbb{E}^3$, Nil or Sol, the orbifold is finitely covered by a torus bundle over the circle. Hence, its fundamental group G has a chain of subgroups

$$G \triangleright G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = \{e\} \tag{1}$$

where $|G/G_0|$ is finite and $G_i/G_{i+1} \cong \mathbb{Z}$. It is a well-known fact, which is easy to prove (see Corollary 1.4.3 of [16] for example), that this implies that G has polynomial subgroup growth. The same argument allows us to deal with orbifolds modelled on $S^2 \times \mathbb{E}$ geometry. Thus, this proves Theorem 5.2 for compact orientable 3-orbifolds that admit a non-hyperbolic geometry.

In the cases above where the subgroup growth of the fundamental group G is polynomial, it is natural to enquire about the *degree* of this growth. This is defined to be

$$\alpha(G) = \limsup_n \frac{\log s_n(G)}{\log n}.$$

The determination of this quantity is likely to be a tractable problem, but it is by no means trivial. We merely mention here a few remarks about it.

It can be shown (Proposition 5.6.5 of [16]) that $\alpha(G) \leq h(G)+1$, where $h(G)$ is the Hirsch length of G. This is defined to be the integer n in the sequence (1) above. In all the cases G we considered, $h(G)$ is at most 3, and therefore $\alpha(G)$ is at most 4.

Our second note is that $\alpha(G)$ is not necessarily unchanged on passing to a finite index subgroup H. It can be shown (Proposition 5.6.4 of [16]) that $\alpha(H)$ lies between $\alpha(G)$ and $\alpha(G)+1$. It is slightly surprising to note that the upper bound is sometimes realised. For example, when G is the infinite dihedral group, $\alpha(G) = 2$, but G contains $\mathbb{Z}$ as an index two normal subgroup, and $\alpha(\mathbb{Z}) = 1$.

It is clear that the degree of polynomial subgroup growth for these 3-orbifold groups is worthy of further investigation.

References

1. M. Boileau, B. Leeb, J. Porti, *Geometrisation of 3-dimensional orbifolds*, Ann. Math. 162 (2005) 195–290.

2. J. Button, *Strong Tits alternatives for compact 3-manifolds with boundary,* J. Pure Appl. Algebra 191 (2004) 89–98.

3. D. Cooper, C. Hodgson, S. Kerckhoff, *Three-dimensional Orbifolds and Cone-Manifolds*, MSJ Memoirs, Vol. 5 (2000)

4. D. Cooper, D. Long, A. Reid, *Essential closed surfaces in bounded 3-manifolds,* J. Amer. Math. Soc. 10 (1997) 553–563.

5. D. Gabai, G.R. Meyerhoff, N. Thurston, *Homotopy hyperbolic 3-manifolds are hyperbolic*, Ann. Math. 157 (2003) 335–431.

6. M. Lackenby, *Heegaard splittings, the virtually Haken conjecture and Property (τ)*, Invent. Math. 164 (2006) 317–359.

7. M. Lackenby, *Some 3-manifolds and 3-orbifolds with large fundamental group*, Preprint.

8. M. Lackenby, *Large groups, Property (τ) and the homology growth of subgroups*, Preprint.

9. M. Lackenby, D. Long, A. Reid, *Covering spaces of arithmetic 3-orbifolds,* Preprint.

10. A. Lubotzky, *Group presentations, p-adic analytic groups and lattices in SL (2,C),* Ann. Math. 118 (1983) 115–130.

11. A. Lubotzky, *Dimension function for discrete groups,* Proceedings of groups – St. Andrews 1985, 254–262, London Math. Soc. Lecture Note Ser., 121, Cambridge Univ. Press, Cambridge (1986).

12. A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Progress in Math. **125** (1994)

13. A. Lubotzky, *Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem*, Ann. Math. **144** (1996) 441–452.

14. A. Lubotzky, Private communication.

15. A. Lubotzky, A. Mann, *Powerful p-groups II. p-adic analytic groups,* J. Algebra, 105 (1987) 506–515.

16. A. Lubotzky, D. Segal, *Subgroup growth.* Progress in Mathematics, 212. Birkhäuser Verlag (2003)

17. J. Ratcliffe, *Euler characteristics of 3-manifold groups and discrete subgroups of* $\mathrm{SL}(2,\mathbb{C})$, J. Pure Appl. Algebra 44 (1987) 303–314.

18. P. Scott, *The geometries of 3-manifolds*. Bull. London Math. Soc. 15 (1983) 401–487.

19. W. Thurston, *The geometry and topology of 3-manifolds*, Lecture Notes, Princeton, 1978.

Mathematical Institute, University of Oxford,
24-29 St Giles', Oxford OX1 3LB, UK.