The Triangulation Complexity of 3-Manifolds

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Triangulation Complexity

This work is joint with Jessica Purcell.

The triangulation complexity $\Delta(M)$ of a closed orientable 3-manifold M is the minimal number of tetrahedra in any triangulation of M.

Sometimes called just complexity.

It's a poorly understood invariant.

My goal today is to convince you that it is a natural invariant that relates to many other aspects of 3-manifold theory.

What's known?

Precise values are known only for some fairly small examples [Matveev, Martelli-Petronio].

Theorem: [Matveev] Let $M = M_1 \sharp \dots \sharp M_n$ where each M_i is prime and no M_i is S^3, $\mathbb{RP}^3$ or $L(3,1)$. Then

$$\Delta(M) = \Delta(M_1) + \dots + \Delta(M_n).$$

Theorem: [Jaco, Rubinstein, Tillmann] $\Delta(L(2n,1)) = 2n - 3$.

Theorem: [Gromov, Thurston] If M is hyperbolic then $\Delta(M) \geq \mathrm{vol}(M)/v_3$, where v_3 is the volume of a regular ideal 3-simplex.

Fibred 3-manifolds

Today, we are mostly going to focus on fibred 3-manifolds with fibre a closed orientable surface S.

These are determined by their monodromy $\phi \colon S \to S$.

Any $\phi \in MCG(S)$ acts isometrically on lots of spaces:

- the Teichmüller space of S, with either its Teichmüller metric or its Weil-Petersson metric;
- the curve complex $\mathcal{C}(S)$;
- the pants complex $\mathcal{P}(S)$;
- $MCG(S)$ itself (with some word metric).

Translation distance

Let h be an isometry of a metric space (X, d).

The translation distance of h is

$$d(h) = \inf\{d(x, h(x)) : x \in X\}.$$

The stable translation distance of h is

$$\overline{d}(h) = \inf\{d(x, h^n(x))/n : n \in \mathbb{Z}_{>0}\}.$$

Here, $x \in X$ is chosen arbitrarily.

The volume of fibred 3-manifolds

Theorem: [Brock] Let S be a compact orientable surface. Then, for any 3-manifold M that fibres over the circle with fibre S and pseudo-anosov monodromy ϕ, the following are within a bounded ratio of each other, the bound only depending on $\chi(S)$:

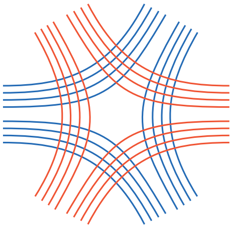
- $\mathrm{vol}(M)$ [hyperbolic volume]
- $d_{\mathrm{WP}}(\phi)$ [Weil-Petersson translation distance]
- $\overline{d}_{\mathrm{WP}}(\phi)$
- $d_{\mathcal{P}(S)}(\phi)$
- $\overline{d}_{\mathcal{P}(S)}(\phi)$

The triangulation complexity of fibred 3-manifolds

Theorem 1: [L-Purcell] Let S be a closed orientable surface. Then, for any 3-manifold M that fibres over the circle with fibre S and pseudo-anosov monodromy ϕ, the following are within a bounded ratio of each other, the bound only depending on $\chi(S)$:

- $\Delta(M)$
- $d_{MCG(S)}(\phi) = \min\{||\alpha^{-1}\phi\alpha|| : \alpha \in \mathrm{MCG}(S)\}$
- $\overline{d}_{MCG(S)}(\phi)$
- $d_{\mathrm{ThickTeich}(S)}(\phi)$ [the thick part of Teichmüller space]
- $\overline{d}_{\mathrm{ThickTeich}(S)}(\phi)$

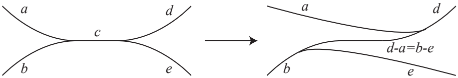
Computing translation length in MCG(S)



Any pseudo-anosov ϕ: $S \to S$ has stable and unstable laminations $\mathcal{L}_+$ and $\mathcal{L}_-$ and dilatation λ.

The stable lamination is carried by a train track τ with weights μ.

Keep splitting the branches with highest weight, giving weighted train tracks $(\tau, \mu) = (\tau_0, \mu_0), (\tau_1, \mu_1), \ldots$



Theorem:[Agol] There are positive integers m and n such that $\tau_{m+n} = \phi(\tau_m)$ and $\mu_{m+n} = \lambda^{-1}\mu_m$.

Theorem: [Masur-Mosher-Schleimer] The translation distance of ϕ in $MCG(S)$ is equal to n (up to a bounded factor).

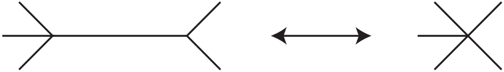
The spine graph of a surface

Let S be a closed orientable surface.

A spine for S is an embedded graph Γ that has no vertices with valence 1 or 2, and where $S \backslash\backslash \Gamma$ is a disc.

The spine graph $\mathrm{Sp}(S)$ has:

- a vertex for each spine of S up to isotopy;
- two vertices are joined by an edge if the corresponding spines differ by an edge contraction or expansion.

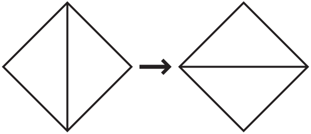


Švarc-Milnor lemma $\Rightarrow$ $\mathrm{MCG}(S)$ and $\mathrm{Sp}(S)$ are quasi-isometric.

The triangulation graph

The triangulation graph $\mathrm{Tr}(S)$ has:

- a vertex for 1-vertex triangulation of S up to isotopy;
- two vertices are joined by an edge if the corresponding triangulations differ by a 2-2 Pachner move.



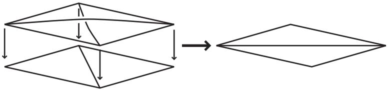
Again, $\mathrm{MCG}(S)$ and $\mathrm{Tr}(S)$ are quasi-isometric.

So it suffices to show that

$$d_{\mathrm{Sp}(S)}(\phi) \preceq \Delta((S \times [0,1])/\phi) \preceq d_{\mathrm{Tr}(S)}(\phi).$$

One direction of the proof

- Let $\mathcal{T}_0, \ldots, \mathcal{T}_n = \phi(\mathcal{T}_0)$ be a path in $\mathrm{Tr}(S)$ realising the translation distance of ϕ.
- Start with a triangulation of $S \times [0,1]$ where $S \times \{0\}$ and $S \times \{1\}$ have the same triangulation $\mathcal{T}_0$.
- Each time we perform a 2-2 Pachner move, attach a tetrahedron onto $S \times \{1\}$ to perform this move.



- Once we reach $\mathcal{T}_n$, glue bottom to top using ϕ.
- The result is a triangulation of $(S \times [0,1])/\phi$ where the number of tetrahedra is $\mathrm{const} + d_{\mathrm{Tr}(S)}(\phi)$.

So,

$$\Delta((S \times [0,1])/\phi) \leq \mathrm{const} + d_{\mathrm{Tr}(S)}(\phi).$$

Brock's theorem for products

Brock also considered geometrically finite hyperbolic structures on $S \times [0,1]$. He related

the volume of its convex core

to

the Weil-Petersson distance

between the points in Teichmüller space

given by $S \times \{0\}$ and $S \times \{1\}$

Triangulations of products

We prove an analogous statement:

Theorem 2: [L-Purcell] Let S be a closed orientable surface. Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be 1-vertex triangulations of $S \times \{0\}$ and $S \times \{1\}$. Then the following are within a bounded ratio of each other, the bound only depending on the genus of S:

- the minimal number of tetrahedra in any triangulation of $S \times [0, 1]$ that equals $\mathcal{T}_0$ and $\mathcal{T}_1$ on $S \times \{0\}$ and $S \times \{1\}$;
- the minimal number of 2-2 Pachner moves relating $\mathcal{T}_0$ to $\mathcal{T}_1$.

Let's assume this theorem for the moment.

In fact a version of Theorem 2 is used to prove Theorem 1.

Lens spaces

Theorem 3: [L-Purcell] Let $L(p,q)$ be a lens space where $0 < q < p$. Let $[a_1, \dots, a_n]$ be the continued fraction expansion of p/q, where each $a_i > 0$. Then there is an absoute constant $c > 0$ such that

$$c \sum a_i \leq \Delta(L(p,q)) \leq \sum a_i.$$

The lower bound depends on Theorem 2 and on:
Theorem: [L-Schleimer] In any triangulation $\mathcal{T}$ of a lens space other than $\mathbb{RP}^3$, there is a simplicial curve in $\mathcal{T}^{(5)}$ that has exterior either a solid torus or a twisted I-bundle over a Klein bottle.

In turn this relies on:
Theorem: [L] In any triangulation $\mathcal{T}$ of a solid torus, there is a simplicial curve in $\mathcal{T}^{(5)}$ that is a core curve.

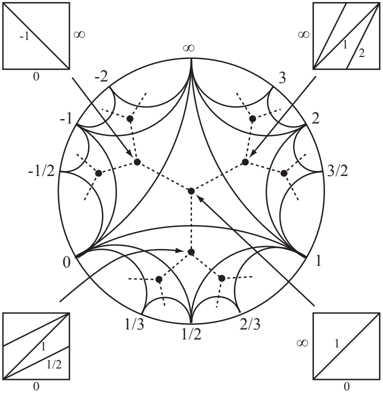
Triangulations of the torus

$\mathrm{Tr}(T^2)$ is the Farey tree.

Corresponding to the points ∞ and q/p, there are infinite lines A_∞ and $A_{q/p}$ in the tree.

These consist of the edges in the tree that are dual to the edges in Farey graph emanating from ∞ and from q/p.

The distance between these lines is more-or-less $\sum a_i$.



Lens space proof

- The upper bound is easy.
- Let $\mathcal{T}$ be a triangulation of $L(p,q)$ with $\Delta(L(p,q))$ tetrahedra.
- Suppose $L(p,q)$ does not contain an embedded Klein bottle.
- Using the above theorems, there are core curves C and C' of the solid tori making up $L(p,q)$ that are simplicial in $\mathcal{T}^{(12)}$.
- Drill these out and obtain a triangulation of $T^2 \times [0,1]$.
- In $\partial N(C)$, the meridian μ_C of $N(C)$ is 'short', and in $\partial N(C')$, the meridian $\mu_{C'}$ of $N(C')$ is 'short'.
- So we can modify the triangulation of $T^2 \times \{0,1\}$ so that they are 1-vertex and so that μ_C and $\mu_{C'}$ are edges.
- Theorem 2 gives a sequence of 2-2 Pachner moves relating the triangulation of $\partial N(C)$ to the triangulation of $\partial N(C')$.
- The number of moves is $\preceq \Delta(L(p,q))$.
- This a path in $\mathrm{Tr}(T^2)$ joining A_∞ and $A_{q/p}$.
- So $\Delta(L(p,q)) \succeq \sum a_i$.

Triangulations of products

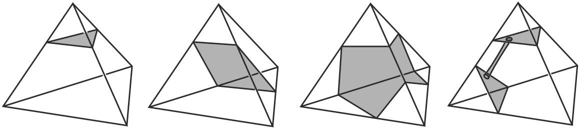
Theorem 2: [L-Purcell] Let S be a closed orientable surface. Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be 1-vertex triangulations of $S \times \{0\}$ and $S \times \{1\}$. Then the following are within a bounded ratio of each other, the bound only depending on the genus of S:

- the minimal number of tetrahedra in any triangulation of $S \times [0, 1]$ that equals $\mathcal{T}_0$ and $\mathcal{T}_1$ on $S \times \{0\}$ and $S \times \{1\}$;
- the minimal number of 2-2 Pachner moves relating $\mathcal{T}_0$ to $\mathcal{T}_1$.

This is proved using almost normal surfaces.

Almost normal surfaces

A surface properly embedded in a triangulated 3-manifold is almost normal if it intersects each tetrahedron in a union of triangles and squares, except in precisely one tetrahedron where it is a union of triangles and squares and exactly one almost normal piece:



Theorem: [Rubinstein, Stocking] Let M be a triangulated 3-manifold. Let S_0 and S_1 be closed normal surfaces that are topologically parallel but not normally parallel. Then between them there is an almost normal surface that is topologically parallel to each of them.

Overview of Theorem 2

Let $\mathcal{T}$ be a triangulation of $S \times [0, 1]$.
Let $\Delta(\mathcal{T})$ be its number of tetrahedra.
Suppose that the triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ of $S \times \{0\}$ and $S \times \{1\}$ are 1-vertex.

We start with the spine Γ_0 in $S \times \{0\}$ dual to $\mathcal{T}_0$.

We want to convert this to the spine that is dual to $\mathcal{T}_1$ using edge contractions and expansions.

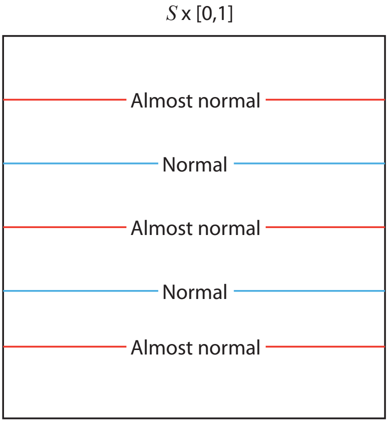
The number of these must be $\preceq \Delta(\mathcal{T})$.

Normal and almost normal surfaces

Pick a maximal collection of disjoint normal fibres in $S \times [0,1]$, no two of which are normally parallel.

Between them, we find almost normal fibres.

[Actually, we don't exactly do this, but never mind.]

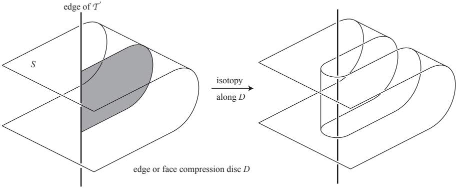


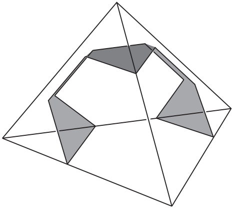
Isotoping almost normal surfaces

There is a natural way of isotoping an almost normal surface to a normal one.

One can do this so each component of intersection with each tetrahedron is one of finitely many types.

The resulting surfaces are nearly normal.





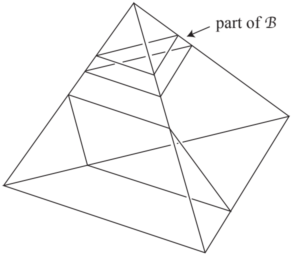
Transferring a spine from $S \times \{0\}$ to $S \times \{1\}$

- Let $\mathcal{T}$ be a triangulation of $S \times [0,1]$.
- Using nearly normal surfaces, we may interpolate between $S \times \{0\}$ and $S \times \{1\}$.
- Each one is obtained from its neighbours using a simple move.
- Suppose that the number of these surfaces is $\preceq \Delta(\mathcal{T})$.
- Then one could keep track of spines in each nearly normal surface, each spine respecting the cell structure on the surface, to get a sequence of spines relating Γ_0 in $S \times \{0\}$ to a simplicial spine in $S \times \{1\}$.
- Unfortunately, there may be many more than $\Delta(\mathcal{T})$ nearly normal surfaces in our collection.
- But 'parallelity bundles' save the day.

Parallelity bundles

Let M be a compact orientable 3-manifold.
Let $\mathcal{T}$ be a triangulation of M with $\Delta(\mathcal{T})$ tetrahedra.
Let S be a normal or almost normal surface embedded in M.

Let $\mathcal{B}$ be the union of the handles lying between parallel normal discs of S.



This is the parallelity bundle for S.

It is an I-bundle over a surface.

All but at most $\preceq \Delta(\mathcal{T})$ handles of $M \backslash\backslash S$ lie in $\mathcal{B}$.

Enlarging the parallelity bundle

The vertical boundary of $\mathcal{B}$ forms annuli properly embedded in $M \backslash\backslash S$.

[L] $\Rightarrow$ $\mathcal{B}$ may be enlarged to an I-bundle $\mathcal{B}'$ so that each component

- either is an I-bundle over a disc;
- or has incompressible vertical boundary.

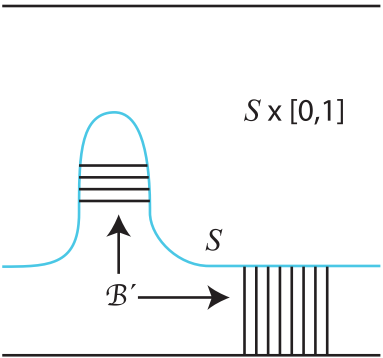
Linearly many isotopies

In our case, S is union of normal and almost normal fibres in $S \times [0,1]$.

So, $\mathcal{B}'$ lies in a union of copies of $S \times [0,1]$.

Each component of $\mathcal{B}'$ is either

- an I-bundle over a disc,
- vertical in $S \times [0,1]$, or
- of the form $(\text{annulus}) \times I$, where the vertical boundary is boundary parallel.



Key idea: Treat an isotopy across a component of $\mathcal{B}'$ as a single move.

So we only need to consider $\preceq \Delta(\mathcal{T})$ nearly normal surfaces.