

DETECTING LARGE GROUPS

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ABSTRACT

Let G be a finitely presented group, and let p be a prime. Then G is ‘large’ (respectively, ‘ p -large’) if some normal subgroup with finite index (respectively, index a power of p) admits a non-abelian free quotient. This paper provides a variety of new methods for detecting whether G is large or p -large. These relate to the group’s profinite and pro- p completions, to its first L_2 -Betti number and to the existence of certain finite index subgroups with ‘rapid descent’. The paper draws on new topological and geometric techniques, together with a result on error-correcting codes.

1. INTRODUCTION

A group is known as *large* if one of its finite index subgroups has a free non-abelian quotient. Large groups have many interesting properties, for example, super-exponential subgroup growth and infinite virtual first Betti number. (See also [4], [15]). It is therefore useful to be able to detect them in practice. In this paper, we will show how one may deduce that a finitely presented group is large using an array of different structures: its profinite and pro- p completions, its first L_2 -Betti number and the ‘homology growth’ of its finite index subgroups.

The detection of large groups was the aim of [5], where the author gave a characterisation of large finitely presented groups in terms of the existence of a sequence of finite index subgroups satisfying certain conditions. In this paper, we start by deducing the following consequence.

Theorem 1.1. *Let G and K be finitely presented (discrete) groups that have isomorphic profinite completions \hat{G} and \hat{K} . Then G is large if and only if K is large.*

In the above result, the term ‘isomorphic’ can be taken to mean ‘isomorphic as groups’, since any group isomorphism between profinite completions \hat{G} and \hat{K} is automatically continuous. We do not require that the isomorphism $\hat{G} \rightarrow \hat{K}$ be

induced by a homomorphism $G \rightarrow K$.

One can also define a group to be p -large, for some prime p , if it contains a normal subgroup with index a power of p that has a free non-abelian quotient. In a similar spirit to Theorem 1.1, we will prove the following.

Theorem 1.2. *Let G and K be finitely presented (discrete) groups that have isomorphic pro- p completions for some prime p . Then G is p -large if and only if K is p -large.*

A sample application of Theorem 1.2 is to weakly parafree groups, which are defined in terms of the lower central series, as follows. Denote the i th term of the lower central series of a group by $\gamma_i(\cdot)$. A group is *weakly parafree* if there is some non-trivial free group F with the ‘same’ lower central series as G . This means that there is an isomorphism $F/\gamma_i(F) \rightarrow G/\gamma_i(G)$, for each positive integer i , and that these isomorphisms are compatible with each other in the obvious way. A group is known as *parafree* if it is weakly parafree and residually nilpotent. Many interesting examples of parafree groups are given in [1] and their properties are investigated in [2]. A consequence of Theorem 1.2 is the following result.

Theorem 1.3. *Let G be a finitely presented, weakly parafree group with $b_1(G) > 1$. Then G is large.*

Here, $b_1(G)$ denotes the first Betti number of G . This has the following corollary.

Corollary 1.4. *Any finitely presented, parafree group is either large or infinite cyclic.*

There are also applications of Theorem 1.2 to low-dimensional topology, including the following.

Theorem 1.5. *If two closed 3-manifolds M_1 and M_2 are topologically \mathbb{Z}_p -cobordant, for some prime p , then $\pi_1(M_1)$ is p -large if and only if $\pi_1(M_2)$ is p -large.*

Recall that two closed 3-manifolds M_1 and M_2 are *topologically \mathbb{Z}_p -cobordant* if there is a compact topological 4-manifold X such that $\partial X = M_1 \cup M_2$, and such that the inclusion of each M_i into X induces isomorphisms of homology groups with mod p coefficients. Thus, Theorem 1.5 represents an unexpected link between two very different areas of low-dimensional topology: the theory of finite covers of 3-manifolds, and 4-dimensional topology.

One of the goals of this paper is also to relate largeness to L_2 -Betti numbers. The first L_2 -Betti number is defined in [13] for any finitely presented group G and is denoted here by $b_1^{(2)}(G)$.

Theorem 1.6. *Let G be a finitely presented (discrete) group that is virtually residually p -finite for some prime p , and such that $b_1^{(2)}(G) > 0$. Then G is large.*

This gives new examples of large groups. By applying results of Shalom from [16], we obtain the following.

Corollary 1.7. *Let G be a finitely presented, non-amenable, discrete subgroup of $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$, with $n \geq 2$ and with critical exponent strictly less than 2. Then G is large.*

This is a consequence of Theorem 1.6 because Shalom showed that such a group G has $b_1^{(2)}(G) > 0$ (Theorem 1.5 in [16]). And since it is finitely generated and linear over a field of characteristic zero, it is virtually residually p -finite, for all but finitely many primes p (Proposition 9 in Window 7 of [11]). Corollary 1.7 can be viewed as a generalisation of the following important result of Cooper, Long and Reid (Theorem 1.3 of [3]) to Lie groups other than $\mathrm{SO}(3, 1)$.

Theorem 1.8. *Let G be a finitely generated, discrete subgroup of $\mathrm{SO}(3, 1)$ that is neither virtually abelian nor cocompact. Then G is large.*

Theorem 1.6 is proved using two results. The first is due to Lück (Theorem 0.1 of [12]). It relates $b_1^{(2)}(G)$ to the ordinary first Betti number $b_1(G_i)$ of finite index normal subgroups G_i .

Theorem 1.9. (Lück [12]) *Let G be a finitely presented group, and let $\{G_i\}$ be a nested sequence of finite index normal subgroups such that $\bigcap G_i = 1$. Then*

$$\lim_{i \rightarrow \infty} \frac{b_1(G_i)}{[G : G_i]}$$

exists and equals $b_1^{(2)}(G)$.

This theorem is concerned with the growth rate of $b_1(G_i)$ for finite index subgroups G_i . Recent work of the author has instead focused on the growth rate of homology with coefficients modulo some prime. Let us fix some terminology. Let \mathbb{F}_p be the field of order a prime p . For a group G , let $d_p(G)$ be the dimension of the homology group $H_1(G; \mathbb{F}_p)$.

The second result forming the basis for Theorem 1.6 is the following, which is a consequence of the results of the author in [5].

Theorem 1.10. *Let G be a finitely presented group with a surjective homomorphism $\phi: G \rightarrow \mathbb{Z}$. Let $G_i = \phi^{-1}(i\mathbb{Z})$, and let p be a prime. Then*

1. $\lim_{i \rightarrow \infty} d_p(G_i)/[G : G_i]$ exists;
2. *this limit is positive if and only if $d_p(G_i)$ is unbounded;*
3. *if the limit is positive, then G is large.*

Thus, fast growth of $d_p(G_i)$ as a function of the index $[G : G_i]$ appears to be a strong and useful property. We say that a nested sequence of finite index subgroups $\{G_i\}$ of a group G has *linear growth of mod p homology* if $\inf_i d_p(G_i)/[G : G_i]$ is strictly positive. A notable situation where this arises is the following, which was the main result of [6].

Theorem 1.11. *Let G be a lattice in $\mathrm{PSL}(2, \mathbb{C})$ satisfying one of the following:*

1. *G contains a non-trivial torsion element, or*
2. *G is arithmetic.*

Then G contains a strictly nested sequence of finite index subgroups $\{G_i\}$ with linear growth of mod p homology, for some prime p .

It seems very likely that these lattices are large. But it remains unclear whether the conclusion of the theorem is strong enough to imply this. However, the following theorem provides an affirmative answer when $\{G_i\}$ is the derived p -series for G . Recall that this is a sequence of finite index subgroups $\{D_i^{(p)}(G)\}$ defined recursively by setting $D_0^{(p)}(G) = G$ and $D_{i+1}^{(p)}(G) = [D_i^{(p)}(G), D_i^{(p)}(G)](D_i^{(p)}(G))^p$. Thus, $D_i^{(p)}(G)/D_{i+1}^{(p)}(G)$ is simply $H_1(D_i^{(p)}(G); \mathbb{F}_p)$.

Theorem 1.12. *Let G be a finitely presented group, and let p be a prime. Suppose that the derived p -series for G has linear growth of mod p homology. Then G is p -large.*

This has implications for other series of finite index subgroups of G , for example the lower central p -series, which is defined as follows. The first term $\gamma_1^{(p)}(G)$ is G . The remaining terms are defined recursively, setting $\gamma_{i+1}^{(p)}(G) = [\gamma_i^{(p)}(G), G](\gamma_i^{(p)}(G))^p$.

Corollary 1.13. *Let G be a finitely presented group, and let p be a prime. Suppose that the lower central p -series for G has linear growth of mod p homology. Then G is p -large.*

The reason that 1.12 implies 1.13 is as follows. Each term $D_i^{(p)}(G)$ of the derived p -series contains $\gamma_j^{(p)}(G)$ for all sufficiently large j . This is because the lower central p -series of the finite p -group $G/D_i^{(p)}(G)$ terminates in the identity element, since this is true for any finite p -group. Moreover, we have the inequality

$$\frac{d_p(D_i^{(p)}(G)) - 1}{[G : D_i^{(p)}(G)]} \geq \frac{d_p(\gamma_j^{(p)}(G)) - 1}{[G : \gamma_j^{(p)}(G)]}.$$

This is an application of Lemma 3.3 in this paper, using the fact that $\gamma_j^{(p)}(G)$ is normal in $D_i^{(p)}(G)$ and has index a power of p . Thus, the assumption that the lower central p -series of G has linear growth of mod p homology implies that the same is true of the derived p -series. Theorem 1.12 then implies that G is p -large, as required. It is clear from this proof that versions of Corollary 1.13 apply to other series of subgroups, for example, the dimension subgroups modulo \mathbb{Z}_p .

Theorem 1.12 is a consequence of a more general result, which we now describe. An *abelian p -series* for a group G is a sequence of finite index subgroups $G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \dots$ such that G_i/G_{i+1} is an elementary abelian p -group for each natural number i . We investigate finitely presented groups G having an abelian p -series $\{G_i\}$ which descends as fast as possible, in the sense that the index $[G_i : G_{i+1}]$ is (approximately) as big as it can be. Clearly, the fastest possible descent occurs for the derived p -series of a non-abelian free group F , of rank n , say. In this case,

$$[D_i^{(p)}(F) : D_{i+1}^{(p)}(F)] = p^{d_p(D_i^{(p)}(F))} = p^{[F : D_i^{(p)}(F)](n-1)+1},$$

and so

$$d_p(D_i^{(p)}(F)/D_{i+1}^{(p)}(F)) = [F : D_i^{(p)}(F)](n-1) + 1.$$

Thus, we say that an abelian p -series $\{G_i\}$ has *rapid descent* if

$$\inf_i \frac{d_p(G_i/G_{i+1})}{[G : G_i]} > 0.$$

In Sections 4-7, we will prove the following theorems.

Theorem 1.14. *Let G be a finitely presented group, and let p be a prime. Then the following are equivalent:*

1. G is large;
2. some finite index subgroup of G has an abelian p -series with rapid descent.

Theorem 1.15. *Let G be a finitely presented group, and let p be a prime. Then the following are equivalent:*

1. G is p -large;
2. G has an abelian p -series with rapid descent.

These theorems represent a significant improvement upon the results in [5], and can be viewed as the strongest theorems in this paper. They are interesting for two reasons. Firstly, (2) in each theorem does not obviously imply that G has a finite index subgroup with positive b_1 , although this is of course a consequence of the theorems. Secondly, the proof of these results involves some genuinely new techniques. As in [5] and [7], topological and geometric methods play a central role. But in this paper, some basic ideas from the theory of error-correcting codes are also used. In particular, we apply a generalisation of the so-called ‘Plotkin bound’ [14].

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2. PROFINITE COMPLETIONS AND WEAKLY PARAFREE GROUPS

Our goal in this section is to prove Theorems 1.1 - 1.5. Our starting point is the following result, which is one of the main theorems in [5].

Theorem 2.1. *Let G be a finitely presented group. Then the following are equivalent:*

1. G is large;
2. there exists a sequence $G_1 \geq G_2 \geq \dots$ of finite index subgroups of G , each normal in G_1 , such that
 - (i) G_i/G_{i+1} is abelian for all $i \geq 1$;
 - (ii) $\lim_{i \rightarrow \infty} ((\log[G_i : G_{i+1}])/[G : G_i]) = \infty$;
 - (iii) $\limsup_i (d(G_i/G_{i+1})/[G : G_i]) > 0$.

Here, $d(\)$ denotes the rank of a group, which is the minimal size of a generating

set.

Remark 2.2. In the proof of (2) \Rightarrow (1), one actually deduces that G_i admits a surjective homomorphism onto a non-abelian free group, for all sufficiently large i . (See the comments after Theorem 1.2 in [5].)

Theorem 1.1 is a rapid consequence of the above result and the following elementary facts, which follow immediately from the definition of the profinite completion of a group.

Proposition 2.3. *Let G and K be finitely generated (discrete) groups, and let $\phi: \hat{G} \rightarrow \hat{K}$ be an isomorphism between their profinite completions. Then the following hold.*

1. *There is an induced bijection (also denoted ϕ) between the set of finite index subgroups of G and the set of finite index subgroups of K .*
2. *If G_i is any finite index subgroup of G , then G_i is normal in G if and only if $\phi(G_i)$ is normal in K . In this case, there is an induced isomorphism $G/G_i \rightarrow K/\phi(G_i)$, again denoted ϕ .*
3. *If G_i and G_j are finite index subgroups of G , then $G_i \subset G_j$ if and only if $\phi(G_i) \subset \phi(G_j)$.*
4. *If $G_i \subset G_j$ are finite index subgroups of G , and G_i is normal in G , then the isomorphism $\phi: G/G_i \rightarrow K/\phi(G_i)$ sends G_j/G_i to $\phi(G_j)/\phi(G_i)$.*

We can now prove the following.

Theorem 1.1. *Let G and K be finitely presented (discrete) groups that have isomorphic profinite completions \hat{G} and \hat{K} . Then G is large if and only if K is large.*

Proof. Let $\phi: \hat{G} \rightarrow \hat{K}$ be the given isomorphism. Suppose that G is large. It therefore contains a nested sequence of finite index subgroups G_i satisfying each of the conditions in Theorem 2.1. These conditions are all detectable by the profinite completion, as follows.

Let \tilde{G}_i be the intersection of the conjugates of G_i . Proposition 2.3 (1) gives finite index subgroups $\phi(G_i)$ and $\phi(\tilde{G}_i)$ of K , which we denote by K_i and \tilde{K}_i respectively. By Proposition 2.3 (2), \tilde{K}_i is normal in K . By Proposition 2.3 (3), \tilde{K}_i is contained in K_i . By Proposition 2.3 (2) and (4), there is an isomorphism $G/\tilde{G}_i \rightarrow K/\tilde{K}_i$ which

takes G_j/\tilde{G}_i to K_j/\tilde{K}_i for any $j \leq i$. The normality of G_i in G_1 is equivalent to the normality of G_i/\tilde{G}_i in G_1/\tilde{G}_i . Thus, K_i is normal in K_1 . The isomorphism $G/\tilde{G}_{i+1} \rightarrow K/\tilde{K}_{i+1}$ takes G_i/\tilde{G}_{i+1} and G_{i+1}/\tilde{G}_{i+1} onto K_i/\tilde{K}_{i+1} and K_{i+1}/\tilde{K}_{i+1} respectively. Hence, K_i/K_{i+1} is isomorphic to G_i/G_{i+1} . Thus, the sequence K_i satisfies the conditions of Theorem 2.1. So, K is large. \square

Remark 2.4. A modified version of Proposition 2.3 holds, where the phrase ‘profinite completions’ is replaced by ‘pro- p completions for some prime p ’, and ‘finite index subgroup(s)’ is replaced throughout by ‘subnormal subgroup(s) with index a power of p ’.

We now embark upon the proof of Theorem 1.2. For this, we need the following variant of Theorem 2.1.

Theorem 2.5. *Let G be a finitely presented group and let p be a prime. Then the following are equivalent:*

1. G is p -large;
2. there exists a sequence $G_1 \geq G_2 \geq \dots$ of subgroups of G , each with index a power of p in G , such that G_1 is normal in G , and each G_i is normal in G_1 , and where the following hold:
 - (i) G_i/G_{i+1} is abelian for all $i \geq 1$;
 - (ii) $\lim_{i \rightarrow \infty} ((\log[G_i : G_{i+1}])/[G : G_i]) = \infty$;
 - (iii) $\limsup_i (d(G_i/G_{i+1})/[G : G_i]) > 0$.

Proof. (1) \Rightarrow (2): Since G is p -large, some finite index normal subgroup G_1 , with index a power of p , admits a surjective homomorphism ϕ onto a non-abelian free group F . Define the following subgroups of F recursively. Set $F_1 = F$, and let $F_{i+1} = [F_i, F_i](F_i)^{p^i}$. Set $G_i = \phi^{-1}(F_i)$. Then it is trivial to check that the conditions of (2) hold.

(2) \Rightarrow (1): By Theorem 2.1 and Remark 2.2, some G_i admits a surjective homomorphism ϕ onto a non-abelian free group F . By assumption, G_i is subnormal in G and has index a power of p . Set \tilde{G}_i to be the intersection of the conjugates of G_i . Then \tilde{G}_i is normal in G and also has index a power of p . The restriction of ϕ to \tilde{G}_i is a surjective homomorphism onto a finite index subgroup of F , which is therefore free non-abelian. Thus, G is p -large. \square

Theorem 1.2. *Let G and K be finitely presented (discrete) groups that have isomorphic pro- p completions for some prime p . Then G is p -large if and only if K is p -large.*

Proof. This is very similar to the proof of Theorem 1.1. Suppose that G is p -large. It therefore has a sequence of subgroups G_i where the conclusions of Theorem 2.5 hold. Using Remark 2.4, K also has such a sequence of subgroups. Thus, by Theorem 2.5, K is p -large. \square

Our aim now is to prove Theorem 1.3.

Theorem 1.3. *Let G be a finitely presented, weakly parafree group with $b_1(G) > 1$. Then G is large.*

As stated in the Introduction, this has the following corollary.

Corollary 1.4. *Any finitely presented, parafree group is either large or infinite cyclic.*

Proof. Any parafree group G with $b_1(G) \leq 1$ is infinite cyclic. \square

Theorem 1.3 is a consequence of a stronger result concerning weakly p -parafree groups, which we now define. A group G is *weakly p -parafree*, for some prime p , if there is some non-trivial free group F such that $G/\gamma_i^{(p)}(G)$ is isomorphic to $F/\gamma_i^{(p)}(F)$ for each $i \geq 1$. Recall that $\gamma_i^{(p)}(\)$ denotes the lower central p -series of a group.

Proposition 2.6. *A weakly parafree group is weakly p -parafree for each prime p .*

Proof. By assumption, there is an isomorphism between $G/\gamma_i(G)$ and $F/\gamma_i(F)$ for some non-trivial free group F . This induces an isomorphism between the lower central p -series of $G/\gamma_i(G)$ and $F/\gamma_i(F)$. Hence,

$$\frac{G/\gamma_i(G)}{\gamma_i^{(p)}(G/\gamma_i(G))} \cong \frac{F/\gamma_i(F)}{\gamma_i^{(p)}(F/\gamma_i(F))}.$$

But $\gamma_i^{(p)}(G/\gamma_i(G))$ is isomorphic to $\gamma_i^{(p)}(G)/\gamma_i(G)$, since $\gamma_i(G)$ is contained in $\gamma_i^{(p)}(G)$. So, the left-hand side is isomorphic to $G/\gamma_i^{(p)}(G)$. Similarly, the right-hand side is isomorphic to $F/\gamma_i^{(p)}(F)$. Thus, we obtain an isomorphism between $G/\gamma_i^{(p)}(G)$ and $F/\gamma_i^{(p)}(F)$. \square

Note that, when defining the weakly p -parafree group G , we do not make the assumption that the isomorphisms $G/\gamma_i^{(p)}(G) \rightarrow F/\gamma_i^{(p)}(F)$ are *compatible*. This

means that the following diagram commutes, for each $i \geq 2$:

$$\begin{array}{ccc} G/\gamma_i^{(p)}(G) & \longrightarrow & F/\gamma_i^{(p)}(F) \\ \downarrow & & \downarrow \\ G/\gamma_{i-1}^{(p)}(G) & \longrightarrow & F/\gamma_{i-1}^{(p)}(F). \end{array}$$

Here, the horizontal arrows are the given isomorphisms and the vertical maps are the obvious quotient homomorphisms. However, we can assume this, with no loss, as the following lemma implies.

Lemma 2.7. *Let G and K be finitely generated groups. Suppose that, for each integer $i \geq 1$, there is an isomorphism $\theta_i: G/\gamma_i^{(p)}(G) \rightarrow K/\gamma_i^{(p)}(K)$. Then there is a collection of such isomorphisms that are compatible.*

Proof. For each $j \leq i$, θ_i restricts to an isomorphism between $\gamma_j^{(p)}(G)/\gamma_i^{(p)}(G)$ and $\gamma_j^{(p)}(K)/\gamma_i^{(p)}(K)$, and hence, quotienting $G/\gamma_i^{(p)}(G)$ by $\gamma_j^{(p)}(G)/\gamma_i^{(p)}(G)$, we obtain an isomorphism $\theta_{i,j}: G/\gamma_j^{(p)}(G) \rightarrow K/\gamma_j^{(p)}(K)$. As $G/\gamma_2^{(p)}(G)$ is finite, some $\theta_{i,2}$ occurs infinitely often. Take this to be the given isomorphism $\phi_2: G/\gamma_2^{(p)}(G) \rightarrow K/\gamma_2^{(p)}(K)$, and only consider those θ_i for which $\theta_{i,2} = \phi_2$. Among these, some $\theta_{i,3}$ occurs infinitely often. Define this to be ϕ_3 , and so on. Then the ϕ_i form the required compatible collection of isomorphisms. \square

The above lemma is elementary and well-known, as is the following result. They are included for the sake of completeness.

Lemma 2.8. *Let G and K be finitely generated groups, and let p be a prime. Then the following are equivalent:*

1. *the pro- p completions of G and K are isomorphic;*
2. *for each $i \geq 1$, there is an isomorphism $G/\gamma_i^{(p)}(G) \rightarrow K/\gamma_i^{(p)}(K)$.*

Proof. (1) \Rightarrow (2): Let $\hat{G}_{(p)}$ denote the pro- p completion of G . Then, for each $i \geq 1$, $\hat{G}_{(p)}/\gamma_i^{(p)}(\hat{G}_{(p)})$ is isomorphic to $G/\gamma_i^{(p)}(G)$. The claim follows immediately.

(2) \Rightarrow (1): The pro- p completion $\hat{G}_{(p)}$ can be expressed as the inverse limit of $\dots \rightarrow G/\gamma_2^{(p)}(G) \rightarrow G/\gamma_1^{(p)}(G)$. This is because the kernel of any homomorphism of G onto a finite p -group contains $\gamma_i^{(p)}(G)$ for all sufficiently large i . Suppose now that, for each $i \geq 1$, there is an isomorphism $G/\gamma_i^{(p)}(G) \rightarrow K/\gamma_i^{(p)}(K)$. According to Lemma 2.7, these isomorphisms can be chosen compatibly. This implies there is an

isomorphism between the inverse limits:

$$\varprojlim G/\gamma_i^{(p)}(G) \cong \varprojlim K/\gamma_i^{(p)}(K).$$

Thus, $\hat{G}_{(p)}$ and $\hat{K}_{(p)}$ are isomorphic. \square

Setting K to be a free group in Lemma 2.8 gives the following characterisation of weakly p -parafree groups in terms of pro- p completions.

Corollary 2.9. *Let G be a finitely generated (discrete) group and let p be a prime. Then G is weakly p -parafree if and only if its pro- p completion is isomorphic to the pro- p completion of a non-trivial free group.*

Thus, Theorem 1.3 is a consequence of the following.

Theorem 2.10. *Let G be a finitely presented group that is weakly p -parafree for some prime p . Suppose that $d_p(G) > 1$. Then G is p -large.*

Proof. The assumption that G is weakly p -parafree implies that the pro- p completion of G is isomorphic to the pro- p completion of a free group F , by Corollary 2.9. Since $d_p(G) > 1$, F is a non-abelian free group. In particular, it is p -large. Thus, by Theorem 1.2, G is p -large. \square

We close this section with a topological application of Theorem 1.2.

Theorem 1.5. *If two closed 3-manifolds M_1 and M_2 are topologically \mathbb{Z}_p -cobordant, for some prime p , then $\pi_1(M_1)$ is p -large if and only if $\pi_1(M_2)$ is p -large.*

This is an immediate consequence of Theorem 1.2 and the following result.

Theorem 2.11. *If two closed 3-manifolds M_1 and M_2 are topologically \mathbb{Z}_p -cobordant, for some prime p , then the pro- p completions of their fundamental groups are isomorphic.*

This is, in turn, a consequence of the following theorem, applied to the homomorphisms $\pi_1(M_1) \rightarrow \pi_1(X)$ and $\pi_1(M_2) \rightarrow \pi_1(X)$ induced by the inclusion maps of M_1 and M_2 into X , where X is the 4-manifold in the definition of \mathbb{Z}_p -cobordant.

Theorem 2.12. *Let $\phi: G \rightarrow K$ be a homomorphism between finitely presented groups that induces an isomorphism $H_1(G; \mathbb{F}_p) \rightarrow H_1(K; \mathbb{F}_p)$ and a surjection $H_2(G; \mathbb{F}_p) \rightarrow H_2(K; \mathbb{F}_p)$, for some prime p . Then, the pro- p completions of G and K are isomorphic.*

Proof. For brevity, let $G_i = \gamma_i^{(p)}(G)$ and $K_i = \gamma_i^{(p)}(K)$. We will show that, for each $i \geq 1$, ϕ induces an isomorphism $G/G_i \rightarrow K/K_i$. The theorem then follows from Lemma 2.8. We will prove this by induction on i . For $i = 1$, it is trivial, and for $i = 2$, it follows from the natural isomorphism between G/G_2 and $H_1(G; \mathbb{F}_p)$. So, let us suppose that it is known for a given integer $i \geq 2$.

In the following commutative diagram, the rows are the Stallings exact sequence [17] and the vertical homomorphisms are induced by ϕ . All homology groups are with respect to mod p coefficients.

$$\begin{array}{ccccccccc}
H_2(G) & \longrightarrow & H_2(G/G_i) & \longrightarrow & G_i/G_{i+1} & \longrightarrow & H_1(G) & \longrightarrow & H_1(G/G_i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(K) & \longrightarrow & H_2(K/K_i) & \longrightarrow & K_i/K_{i+1} & \longrightarrow & H_1(K) & \longrightarrow & H_1(K/K_i)
\end{array}$$

The first vertical homomorphism is surjective by hypothesis. The second, fourth and fifth are isomorphisms by induction and by hypothesis. So, by the 5-lemma, the third homomorphism is an isomorphism. This is the first vertical map in the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_i/G_{i+1} & \longrightarrow & G/G_{i+1} & \longrightarrow & G/G_i & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K_i/K_{i+1} & \longrightarrow & K/K_{i+1} & \longrightarrow & K/K_i & \longrightarrow & 1
\end{array}$$

Here, the rows are exact and the vertical homomorphisms are induced by ϕ . The third vertical map is also an isomorphism, by induction, and hence so is the middle vertical map. This proves the theorem by induction. \square

3. HOMOLOGY GROWTH IN CYCLIC COVERS

The goal of this section is to prove the following result and then Theorem 1.6.

Theorem 1.10. *Let G be a finitely presented group with a surjective homomorphism $\phi: G \rightarrow \mathbb{Z}$. Let $G_i = \phi^{-1}(i\mathbb{Z})$, and let p be a prime. Then*

1. $\lim_{i \rightarrow \infty} d_p(G_i)/[G : G_i]$ exists;
2. this limit is positive if and only if $d_p(G_i)$ is unbounded;
3. if the limit is positive, then G is large.

We will need the following lemma.

Lemma 3.1. *Let k be a non-negative integer and let $f: \mathbb{N}_{>0} \rightarrow \mathbb{R}$ be a function satisfying*

$$|f(i+j) - f(i) - f(j)| \leq k, \quad (*)$$

for any $i, j \in \mathbb{N}$. Then

1. $\lim_{i \rightarrow \infty} f_i/i$ exists;
2. this limit is non-zero if and only if f_i is unbounded.

Proof. Note first that a trivial induction establishes that $f(i) \leq i(f(1) + k)$ for each positive $i \in \mathbb{N}$. Hence $\limsup_i f(i)/i$ is finite, M say.

Claim. Suppose that $f(m) > 2k$, for some positive $m \in \mathbb{N}$. Then $\liminf_i f(i)/i > 0$.

We will show that $f(nm) > (n+1)k$, for each positive $n \in \mathbb{N}$, by induction on n . The induction starts trivially. For the inductive step, note that $f((n+1)m) \geq f(m) + f(nm) - k > 2k + (n+1)k - k = (n+2)k$. This establishes the inequality. The claim now follows by noting that if $i = nm + r$, for $0 \leq r < m$, then

$$|f(i) - f(nm)| \leq k + \max_{1 \leq r < m} |f(r)|.$$

Now let $g(i) = Mi - f(i)$. By the definition of M , $\liminf_i g(i)/i = 0$. Now, g satisfies $(*)$ and so by the claim, g is bounded above by $2k$. Hence, $f(i) \geq Mi - 2k$. Thus, $\liminf_i f(i)/i = M$. This proves (1). To prove (2), note that if f satisfies $(*)$, then so does $-f$. Thus, applying the claim, we deduce that either $|f(i)| \leq 2k$ for all positive i or $\lim_{i \rightarrow \infty} f(i)/i$ is non-zero. \square

Proof of Theorem 1.10. We claim that there is a non-negative integer k such that, for all $i, j \geq 1$,

$$|d_p(G_{i+j}) - d_p(G_i) - d_p(G_j)| \leq k.$$

This and Lemma 3.1 will then imply (1) and (2).

Let K be a finite connected 2-complex with fundamental group G . We may find a map $F: K \rightarrow S^1$ so that $F_*: \pi_1(K) \rightarrow \pi_1(S^1)$ is $\phi: G \rightarrow \mathbb{Z}$. Let b be a point in S^1 . Then, after a small homotopy, we may assume that $F^{-1}(b)$ is a finite graph Γ , that a regular neighbourhood of Γ is a copy of $\Gamma \times I$ and that the restriction of F to this neighbourhood is projection onto the I factor, followed by inclusion of I into S^1 . Let \overline{K} be the result of cutting K along Γ . Let K_i be the i -fold cover of K corresponding to G_i . Then K_i can be obtained from i copies of \overline{K} glued together in a circular

fashion. Cut K_i along one of the copies of Γ in K_i , and let \overline{K}_i be the result. The Mayer-Vietoris sequence (applied to the decomposition of K_i into \overline{K}_i and $\Gamma \times I$) gives the following inequalities:

$$-d_p(\Gamma) \leq d_p(K_i) - d_p(\overline{K}_i) \leq |\Gamma|.$$

Similarly, since the disjoint union of \overline{K}_i and \overline{K}_j is obtained by cutting \overline{K}_{i+j} along a copy of Γ , we have

$$-d_p(\Gamma) \leq d_p(\overline{K}_{i+j}) - d_p(\overline{K}_i) - d_p(\overline{K}_j) \leq |\Gamma|.$$

Since $d_p(K_i) = d_p(G_i)$, the claim now follows, letting $k = 2d_p(\Gamma) + 2|\Gamma|$.

Conclusion (3) is a consequence of the following result (Theorem 1.2 of [5]), setting $H_i = G$, $J_i = G_i$ and $K_i = [G_i, G_i](G_i)^p$. \square

Theorem 3.2. *Let G be a finitely presented group and suppose that, for each natural number i , there is a triple $H_i \geq J_i \geq K_i$ of finite index normal subgroups such that*

- (i) H_i/J_i is abelian for all i ;
- (ii) $\lim_{i \rightarrow \infty} ((\log[H_i : J_i])/[G : H_i]) = \infty$;
- (iii) $\inf_i (d(J_i/K_i)/[G : J_i]) > 0$.

Then, K_i admits a surjective homomorphism onto a non-abelian free group, for all sufficiently large i .

To prove Theorem 1.6, we need one more fact, which is well known. It appears as Proposition 3.7 in [6], for example.

Lemma 3.3. *Let G be a finitely generated group and let K be a normal subgroup with index a power of a prime p . Then*

$$d_p(K) \leq (d_p(G) - 1)[G : K] + 1.$$

We can now prove Theorem 1.6.

Theorem 1.6. *Let G be a finitely presented (discrete) group, that is virtually residually p -finite, for some prime p , and such that $b_1^{(2)}(G) > 0$. Then G is large.*

Proof. Since G is virtually residually p -finite, it has a finite index normal subgroup G_1 that is residually p -finite. Thus, G_1 contains a nested sequence of normal subgroups G_i , each with index a power of p , such that $\bigcap_i G_i = 1$. By the multiplicativity of $b_1^{(2)}$ (Theorem 1.7 (1) of [13]), $b_1^{(2)}(G_1) = b_1^{(2)}(G)[G : G_1] > 0$. By Lück's theorem (Theorem 1.9), $\lim_{i \rightarrow \infty} b_1(G_i)/[G_1 : G_i]$ exists and equals $b_1^{(2)}(G_1)$, which is positive. Hence, by relabelling the G_i , we may assume that $b_1(G_1) > 0$. Let $\phi: G_1 \rightarrow \mathbb{Z}$ be a surjective homomorphism, and let $K_i = \phi^{-1}(p^i \mathbb{Z})$. We claim that $\liminf_i d_p(K_i)/[G : K_i]$ is positive. Consider the subgroups $G_i \cap K_i$. Each is a finite index normal subgroup of G_1 and their intersection is the identity. Hence, by Theorem 1.9, $\lim_{i \rightarrow \infty} b_1(G_i \cap K_i)/[G_1 : G_i \cap K_i]$ exists and is positive. In particular, $\liminf_i d_p(G_i \cap K_i)/[G : G_i \cap K_i]$ is positive. Now, the quotient $K_i/(G_i \cap K_i)$ is isomorphic to $K_i G_i / G_i$, which is a subgroup of G_1 / G_i , and so has order a power of p . Hence, by Lemma 3.3,

$$\frac{d_p(G_i \cap K_i)}{[G : G_i \cap K_i]} \leq \frac{(d_p(K_i) - 1)[K_i : G_i \cap K_i] + 1}{[G : G_i \cap K_i]}.$$

Therefore, $\liminf_i d_p(K_i)/[G : K_i]$ is positive, as claimed.

In particular, $d_p(K_i)$ is unbounded. Thus, by Theorem 1.10, G is large. \square

4. COCYCLE SIZE AND PROPERTY (τ)

Most of the remainder of the paper is devoted to the proof of Theorems 1.14 and 1.15.

Theorem 1.14. *Let G be a finitely presented group and let p be a prime. Then the following are equivalent:*

1. G is large;
2. some finite index subgroup of G has an abelian p -series with rapid descent.

Theorem 1.15. *Let G be a finitely presented group, and let p be a prime. Then the following are equivalent:*

1. G is p -large;
2. G has an abelian p -series with rapid descent.

The difficult direction in each of these theorems is (2) \Rightarrow (1). For this, one needs a method for proving that a finitely presented group is large or p -large. Various

techniques have been developed with this aim. The one we will use deals with the ‘relative size’ of cocycles on 2-complexes.

Let K be a connected finite 2-complex with fundamental group G . Let K_i be the covering space corresponding to a finite index subgroup G_i . The key to our approach is to consider cellular 1-cocycles on K_i representing non-trivial elements of $H^1(K_i; \mathbb{F}_p)$.

For a cellular 1-dimensional cocycle c on K_i , let its *support* $\text{supp}(c)$ be those 1-cells with non-zero evaluation under c . For an element $\alpha \in H^1(K_i; \mathbb{F}_p)$, consider the following quantity, which was defined in [7]. The *relative size* of α is

$$\text{relsize}(\alpha) = \frac{\min\{|\text{supp}(c)| : c \text{ is a cellular cocycle representing } \alpha\}}{\text{Number of 1-cells of } K_i}.$$

The following result was proved in [7] and is central to our approach.

Theorem 4.1. *Let K be a finite connected 2-complex, and let $\{K_i \rightarrow K\}$ be a collection of finite-sheeted covering spaces. Suppose that $\{\pi_1(K_i)\}$ has linear growth of mod p homology for some prime p . Then one of the following must hold:*

- (i) $\pi_1(K_i)$ is p -large for infinitely many i , or
- (ii) there is some $\epsilon > 0$ such that the relative size of any non-trivial class in $H^1(K_i; \mathbb{F}_p)$ is at least ϵ , for all i .

This is a slightly modified version of Theorem 6.1 of [7]. In that result, it is not explicitly stated that $\pi_1(K_i)$ is p -large for infinitely many i , merely that $\pi_1(K)$ is large. But the proof does indeed give a normal subgroup of $\pi_1(K_i)$ with index a power of p that has a free non-abelian quotient.

In this section, we will relate the relative size of cocycles to Property (τ) . While not directly needed in the remainder of the paper, this is a potentially important link.

We now recall the definition of Property (τ) . Let G be a group with a finite generating set S . Let $\{G_i\}$ be a collection of finite index subgroups of G . Let $X_i = X(G/G_i; S)$ be the Schreier coset graph of G/G_i with respect to the generating set S .

The *Cheeger constant* $h(X_i)$ is defined to be

$$h(X_i) = \min \left\{ \frac{|\partial A|}{|A|} : A \subset V(X_i), 0 < |A| \leq |V(X_i)|/2 \right\}.$$

Here, $V(X_i)$ denotes the vertex set of X_i , and for a subset A of $V(X_i)$, ∂A denotes the set of edges with one endpoint in A and one not in A .

Then G has *Property* (τ) with respect to $\{G_i\}$ if $\inf_i h(X(G/G_i; S)) > 0$. This turns out not to depend on the choice of finite generating set S .

When a group G has *Property* (τ) with respect to an infinite collection of finite index subgroups $\{G_i\}$, there are lots of nice consequences. For example, the resulting Schreier coset graphs have applications in theoretical computer science and coding theory. But when a group does not have *Property* (τ) , there are other useful conclusions one can often draw, which we now briefly describe. This is particularly the case in low-dimensional topology, where the following important conjecture remains unresolved.

Conjecture. [10] (Lubotzky-Sarnak) *Let G be the fundamental group of a finite-volume hyperbolic 3-manifold. Then G does not have *Property* (τ) with respect to some collection $\{G_i\}$ of finite index subgroups.*

To appreciate the significance of this conjecture, note the following theorem, which appears as Corollary 7.4 in [9].

Theorem. [9] *The Lubotzky-Sarnak conjecture and the Geometrisation Conjecture together imply that any arithmetic lattice in $\mathrm{PSL}(2, \mathbb{C})$ is large.*

Thus, it is important to develop new methods for showing that a group does not have *Property* (τ) . The following result, which is the main theorem in this section, may be a useful tool.

Theorem 4.2. *Let K be a finite connected 2-complex with fundamental group G . Let $\{G_i\}$ be a collection of finite index subgroups of G , and let $\{K_i\}$ be the corresponding covering spaces of K . Suppose that there is a prime p and, for each i , there is a non-trivial class α_i in $H^1(K_i; \mathbb{F}_p)$, such that $\mathrm{relsize}(\alpha_i) \rightarrow 0$ as $i \rightarrow \infty$. Let \tilde{G}_i be the kernel of the homomorphism $G_i \rightarrow \mathbb{Z}/p\mathbb{Z}$ induced by α_i . Then G does not have *Property* (τ) with respect to $\{\tilde{G}_i\}$.*

In the proof of this theorem, we will need the following construction, which will also be important later in the paper. Let K be a finite connected 2-complex with some 0-cell as a basepoint b . Let c be a cocycle on K representing a non-trivial element of $H^1(K; \mathbb{F}_p)$ and let (\tilde{K}, \tilde{b}) be a (based) covering space of (K, b) . Suppose

that $\pi_1(\tilde{K}, \tilde{b})$ lies in the kernel of the homomorphism $\pi_1(K, b) \rightarrow \mathbb{Z}/p\mathbb{Z}$ determined by c . Then one can define, for any 0-cell v of \tilde{K} , its c -value $c(v)$, which is an integer mod p . It is defined as follows. Pick a path β from \tilde{b} to v in the 1-skeleton of \tilde{K} , project it to a path in K and define $c(v)$ to be the evaluation of c on this path. To see that this is independent of the choice of β , let β' be any other path in \tilde{K} from \tilde{b} to v in the 1-skeleton of \tilde{K} . Then $\beta' \cdot \beta^{-1}$ is a closed loop in \tilde{K} . This projects to a closed loop in K . By our hypothesis on the covering space \tilde{K} , the evaluation of c on any such closed loop is trivial. This immediately implies that $c(v)$ is indeed well-defined.

An example is useful. Let K be the wedge of 3 circles, labelled x , y and z . Let c be the mod 2 cocycle supported on the x labelled edge. Let \tilde{K} be the covering space corresponding to the second term of the derived 2-series of K . This is shown in Figure 1. (Note that the dotted edges join up with each other.) The c -value of its vertices is shown.

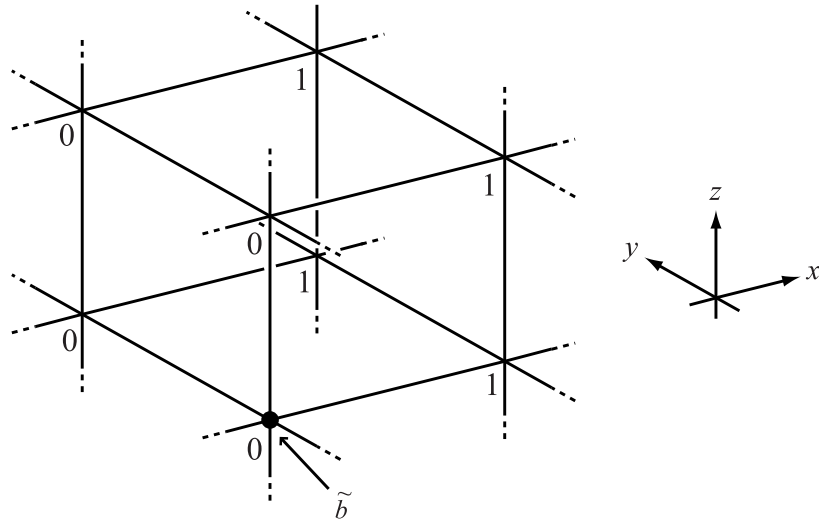


Figure 1.

Lemma 4.3. *Let K be a finite connected 2-complex, and let α be a non-trivial element of $H^1(K; \mathbb{F}_p)$. Let \tilde{K} be a finite-sheeted covering space of K , such that $\pi_1(\tilde{K})$ lies in the kernel of the homomorphism $\pi_1(K) \rightarrow \mathbb{Z}/p\mathbb{Z}$ determined by α . Let \tilde{X} be the 1-skeleton of \tilde{K} . Let $V(K)$ and $E(K)$ be the 0-cells and 1-cells of K respectively. Then*

$$h(\tilde{X}) \leq \frac{|E(K)|}{|V(K)|/p} \text{resize}(\alpha).$$

Proof. Let c be a cocycle on K representing α and for which $|\text{supp}(c)|$ is minimal. Let A be the vertices in \tilde{K} with zero c -value. We claim that $|A| = |V(\tilde{K})|/p$. Let ℓ be any loop in K based at the basepoint b such that $c(\ell) = 1$. Then $[\ell]$ represents an element of the covering group $\pi_1(K)/\pi_1(\tilde{K})$, which increases the c -value of every vertex by 1 modulo p . Hence, the number of vertices with given c -value is $|V(\tilde{K})|/p$, which proves the claim. As a consequence, $|A| = d|V(K)|/p$, where d is the degree of the cover $\tilde{K} \rightarrow K$. Any edge in ∂A must lie in the inverse image of the support of c . Thus, $|\partial A| \leq d|\text{supp}(c)| = d|E(K)|\text{relsize}(\alpha)$. So,

$$h(\tilde{X}) \leq \frac{|\partial A|}{|A|} \leq \frac{d|E(K)|}{d|V(K)|/p}\text{relsize}(\alpha),$$

and the required formula now follows. \square

Proof of Theorem 4.2. Let \tilde{X}_i be the 1-skeleton of the covering space of K corresponding to \tilde{G}_i . By Lemma 4.3,

$$h(\tilde{X}_i) \leq \frac{|E(K_i)|}{|V(K_i)|/p}\text{relsize}(\alpha_i) = \frac{|E(K)|}{|V(K)|/p}\text{relsize}(\alpha_i).$$

Since we are assuming that the relative size of α_i tends to zero, and since the other terms on the right-hand side of the above formula depend only on K , we deduce that $h(\tilde{X}_i) \rightarrow 0$. Hence, G does not have Property (τ) with respect to $\{\tilde{G}_i\}$. \square

5. COCYCLES IN COVERING SPACES

Our aim over the next few chapters is to prove (2) \Rightarrow (1) of Theorems 1.14 and 1.15, and thereby establish that the group G in these theorems is large or p -large as appropriate. The proof will be topological, and so we consider a finite connected 2-complex K with fundamental group G . We are assuming that G has a finite index subgroup G_1 with a rapidly descending p -series G_i . (In the proof of Theorem 1.15, take G_1 to be G .) Let K_i be the corresponding covering spaces of K . Theorem 4.1 gives a criterion for establishing that G_1 is p -large, in terms of the existence of 1-cocycles on K_i with relative size tending to zero. We therefore, in this section, investigate how 1-cocycles on a 2-complex can be used to construct 1-cocycles in covering spaces (with potentially smaller relative size). If U is a set of 1-cocycles on a cell complex K , we define its *support* $\text{supp}(U)$ to be the union of the supports of the cocycles in U . Our main result is the following.

Theorem 5.1. *Let K be a finite connected 2-complex with r 2-cells. Let U be a collection of cocycles on K that represent linearly independent elements of $H^1(K; \mathbb{F}_p)$.*

Let $u = |U|$. Let $q: \tilde{K} \rightarrow K$ be a finite regular cover such that $\pi_1(K)/\pi_1(\tilde{K})$ is an elementary abelian p -group with rank n . Then there is a collection \tilde{U} of cocycles on \tilde{K} representing linearly independent elements of $H^1(\tilde{K}; \mathbb{F}_p)$ such that

1. $\text{supp}(\tilde{U}) \subset q^{-1}(\text{supp}(U))$;
2. $|\tilde{U}| \geq (n - u)u - r$.

The point behind Theorem 5.1 is that it provides not just a lower bound on the dimension of $H^1(\tilde{K}; \mathbb{F}_p)$ but also gives information about certain cocycles on \tilde{K} representing this cohomology.

We now embark on the proof of Theorem 5.1. Consider the following subspaces of $H^1(K; \mathbb{F}_p)$:

1. the space spanned by the elements of U ;
2. the classes that have trivial evaluation on all elements of $\pi_1(\tilde{K})$.

Let V_1 and V_2 be these two subspaces. Then the dimensions of V_1 and V_2 are u and n respectively.

Pick a complementary subspace for $V_1 \cap V_2$ in V_2 , and let C be a set of cocycles on K that represents a basis for this subspace. Note that $|C| \geq n - u$. Note also that, by construction, $C \cup U$ forms a linearly independent set of elements in $H^1(K; \mathbb{F}_p)$.

For each $c_1 \in U$ and $c_2 \in C$, we will show how to construct a cochain on \tilde{K} , which we denote $c_1 \wedge c_2$. These cochains will play a vital role in the proof of Theorem 5.1.

Pick an orientation on each of the 1-cells of K . This pulls back to give an orientation on each 1-cell e of \tilde{K} . Let $i(e)$ denote the initial vertex of e .

Let \tilde{c}_1 be the inverse image of c_1 in \tilde{K} . This is a cocycle on \tilde{K} . Since each 1-cell e is oriented, $\tilde{c}_1(e)$ is a well-defined element of \mathbb{F}_p .

Fix a basepoint b in the 0-skeleton of K , and let \tilde{b} be a basepoint for \tilde{K} in the inverse image of b . Recall from Section 4 that each vertex v of \tilde{K} then has a well-defined c_2 -value, denoted by $c_2(v)$.

We now define $c_1 \wedge c_2$. Since the edges of \tilde{K} are oriented, it suffices to assign an

integer $(c_1 \wedge c_2)(e)$ modulo p to each edge e . We define this to be

$$(c_1 \wedge c_2)(e) = \tilde{c}_1(e) \cdot c_2(i(e)),$$

where the product is multiplication in \mathbb{F}_p .

Note that $\text{supp}(c_1 \wedge c_2) \subset \text{supp}(\tilde{c}_1) \subset q^{-1}(\text{supp}(U))$.

It may be helpful to consider the case where K is the wedge of 3 circles labelled x , y and z , where $p = 2$ and where $\pi_1(\tilde{K})$ is the second term in the derived 2-series for $\pi_1(K)$. Let c_1 and c_2 be the cochains supported on the x -labelled and y -labelled edges of K , respectively. Then the following is a diagram of \tilde{K} , and the edges in the support of $c_1 \wedge c_2$ are shown in bold.

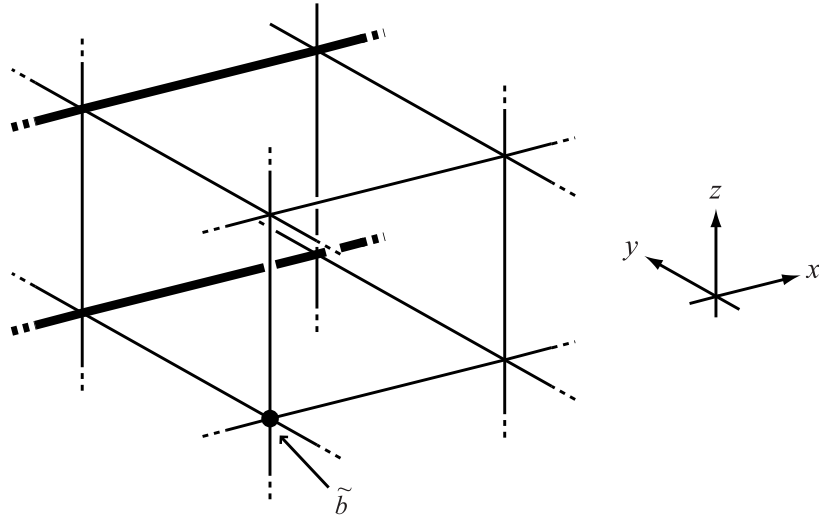


Figure 2.

We denote by $\langle U \wedge C \rangle$ the space of cochains on \tilde{K} spanned by elements $c_1 \wedge c_2$, where $c_1 \in U$ and $c_2 \in C$. Let $Z^1(\tilde{K})$ denote the space of 1-cocycles on \tilde{K} with mod p coefficients. We will establish the following.

Claim 5.2. *The dimension of $\langle U \wedge C \rangle$ is $|U| |C|$, which is at least $u(n - u)$.*

Claim 5.3. *The subspace $Z^1(\tilde{K}) \cap \langle U \wedge C \rangle$ of cocycles in $\langle U \wedge C \rangle$ has codimension at most r (the number of 2-cells of K).*

Claim 5.4. *The map from $Z^1(\tilde{K}) \cap \langle U \wedge C \rangle$ into $H^1(\tilde{K}; \mathbb{F}_p)$ is an injection.*

Thus, setting \tilde{U} to be a basis for $Z^1(\tilde{K}) \cap \langle U \wedge C \rangle$ will establish Theorem 5.1.

Lemma 5.5. *Let g and h be loops in K based at the same point. Let $[g, h]$ denote any lift of $ghg^{-1}h^{-1}$ to \tilde{K} . Then*

$$(c_1 \wedge c_2)([g, h]) = c_1(h)c_2(g) - c_1(g)c_2(h),$$

where equality is in \mathbb{F}_p .

Proof. Each letter in the word g corresponds to an edge e , say, in the first part of the loop $[g, h]$. The inverse of this letter appears in g^{-1} , where the loop runs over the edge e' . Between the vertices $i(e)$ and $i(e')$ is a word w conjugate to h . We claim that $c_2(i(e')) - c_2(i(e)) = c_2(h)$. Pick a path from the basepoint \tilde{b} of \tilde{K} to $i(e)$. Then $c_2(i(e))$ is the evaluation under c_2 of the projection of this path to K . If we extend this path using the word w , we obtain a path from \tilde{b} to $i(e')$. Thus, the difference between $c_2(i(e'))$ and $c_2(i(e))$ is the evaluation of c_2 on the projection of w . The parts of g and g^{-1} in w project to the same edges in K , but with reverse orientations. Hence, $c_2(i(e')) - c_2(i(e))$ equals $c_2(h)$, as required. Therefore, the evaluation of the loop $[g, h]$ along the edges in g and g^{-1} is $-c_1(g)c_2(h)$ in total. Similarly, along the edges in h and h^{-1} , it is $c_1(h)c_2(g)$. So, the total evaluation is $c_1(h)c_2(g) - c_1(g)c_2(h)$, as required. \square

Let $C^1(\tilde{K})$ and $B^1(\tilde{K})$ be the space of 1-cochains on \tilde{K} , with mod p coefficients, and the subspace of coboundaries.

Lemma 5.6. *The cochains $\{c_1 \wedge c_2 : c_1 \in U, c_2 \in C\}$ map to linearly independent elements in $C^1(\tilde{K})/B^1(\tilde{K})$.*

Proof. Since $U \cup C$ forms a linearly independent set of classes in $H^1(K; \mathbb{F}_p)$, there are loops ℓ_i in K , based at the basepoint of K , where $i \in U \cup C$, such that, for all $c \in U \cup C$,

$$c(\ell_i) = \begin{cases} 1 & \text{if } i = c \\ 0 & \text{otherwise.} \end{cases}$$

Let $i \in U$ and $j \in C$. Then, by Lemma 5.5, for any $c_1 \in U$ and $c_2 \in C$,

$$(c_1 \wedge c_2)([\ell_j, \ell_i]) = c_1(\ell_i)c_2(\ell_j) - c_1(\ell_j)c_2(\ell_i) = \begin{cases} 1 & \text{if } i = c_1 \text{ and } j = c_2; \\ 0 & \text{otherwise.} \end{cases}$$

Since every element of $B^1(\tilde{K})$ has trivial evaluation on any loop in \tilde{K} , we deduce the lemma. \square

Lemma 5.6 implies Claim 5.2. It also implies that the restriction of the quotient homomorphism $C^1(\tilde{K}) \rightarrow C^1(\tilde{K})/B^1(\tilde{K})$ to $\langle U \wedge C \rangle$ is an injection. Thus, it is an

injection on any subspace of $\langle U \wedge C \rangle$. This gives Claim 5.4. We now verify Claim 5.3.

Lemma 5.7. *Let ℓ and ℓ' be the boundary loops of two 2-cells of \tilde{K} that differ by a covering transformation of \tilde{K} . Then,*

$$(c_1 \wedge c_2)(\ell') = (c_1 \wedge c_2)(\ell).$$

Proof. Let g be a path in the 1-skeleton of \tilde{K} from the basepoint of ℓ to the basepoint of ℓ' . Thus, the loop $g\ell g^{-1}$ runs from the basepoint of ℓ to the basepoint of ℓ' , then goes around ℓ' and then returns to the basepoint of ℓ . Since g and ℓ project to loops in K based at the same point, Lemma 5.5 gives that

$$(c_1 \wedge c_2)([g, \ell]) = c_1(\ell)c_2(g) - c_1(g)c_2(\ell).$$

This is zero because ℓ is the boundary of a 2-cell and so has zero evaluation under the cocycles c_1 and c_2 . So,

$$(c_1 \wedge c_2)(\ell') = (c_1 \wedge c_2)([g, \ell]) + (c_1 \wedge c_2)(\ell) = (c_1 \wedge c_2)(\ell).$$

□

The cocycles in $\langle U \wedge C \rangle$ are precisely those cochains in $\langle U \wedge C \rangle$ that have zero evaluation on the boundary of any 2-cell in \tilde{K} . But Lemma 5.7 states that if two 2-cells differ by a covering transformation, then they have the same evaluation. Thus, one need only check the evaluation of the boundary of just one 2-cell in each orbit of the covering action. There are precisely r such orbits, where r is the number of 2-cells in K . Thus, the codimension of $Z^1(\tilde{K}) \cap \langle U \wedge C \rangle$ in $\langle U \wedge C \rangle$ is at most r .

This proves Claim 5.3 and hence Theorem 5.1. □

Remark 5.8. Although Theorem 5.1 suffices for the purposes of this paper, it is possible to strengthen it a little. One can in fact find a set \tilde{U} satisfying the requirements of Theorem 5.1, but with the stronger inequality

$$|\tilde{U}| \geq un - \frac{d(d+1)}{2} - r,$$

where $d = \dim(V_1 \cap V_2)$. This is proved as follows. Pick a basis for $V_1 + V_2$ so that it contains a basis for $V_1 \cap V_2$, a basis for V_1 and a basis for V_2 . Pick a total order

on the basis for $V_1 \cap V_2$. Then consider all cochains $c_1 \wedge c_2$, where c_1 lies in the basis for V_1 , c_2 lies in the basis for V_2 , and $c_1 < c_2$ if c_1 and c_2 both lie in $V_1 \cap V_2$. The number of such cochains is $un - d(d+1)/2$. It is possible to prove the corresponding versions of 5.2, 5.3 and 5.4 for these cochains. Hence, the required inequality follows.

Remark 5.9. The cochains $c_1 \wedge c_2$ we have considered in this section are, in fact, special cases of a much more general construction. In [8], a more general class of cochain was used to provide new lower bounds on the homology growth and subgroup growth of certain groups. These more general cochains had a certain integer ℓ , known as their level, assigned to them. The cochains $c_1 \wedge c_2$ are those with level one.

6. THE SUBSPACE REDUCTION THEOREM

Let E be a finite set, and let \mathbb{F}_p^E be the vector space over \mathbb{F}_p consisting of functions $E \rightarrow \mathbb{F}_p$. The *support* of an element ϕ of \mathbb{F}_p^E is

$$\text{supp}(\phi) = \{e \in E : \phi(e) \neq 0\}.$$

The *support* of a subspace W of \mathbb{F}_p^E is

$$\text{supp}(W) = \bigcup_{\phi \in W} \text{supp}(\phi).$$

The main example we will consider is where E is the set of 1-cells in a finite 2-complex K (with some given orientations). Then \mathbb{F}_p^E is just $C^1(K)$, the space of 1-cochains on K . Recall that our goal is to find cocycles representing non-trivial elements of $H^1(K; \mathbb{F}_p)$ and with small relative size. The following result, which is the main theorem of this chapter, will be the tool we use.

Theorem 6.1. *Let V be a subspace of \mathbb{F}_p^E with dimension v , and let w be a positive integer strictly less v . Then, V contains a subspace W with dimension w such that*

$$|\text{supp}(W)| \leq \frac{p^v - p^{v-w}}{p^v - 1} |\text{supp}(V)| \leq \frac{p^{w+1} - p}{p^{w+1} - 1} |\text{supp}(V)|.$$

In our case, V will be a subspace of $C^1(K)$ spanned by v cocycles, representing linearly independent elements of $H^1(K; \mathbb{F}_p)$. We will use Theorem 6.1 to pass to a set of w cocycles (where w is a fixed integer less than v) spanning a subspace W with

support which is smaller than the support of V by a definite factor, independent of v .

We now embark on the proof of Theorem 6.1. The following lemma gives a formula relating the support of a subspace to the support of each of its elements.

Lemma 6.2. *For a non-zero subspace W of \mathbb{F}_p^E ,*

$$|\text{supp}(W)| = \frac{1}{(p-1)p^{\dim(W)-1}} \sum_{\phi \in W} |\text{supp}(\phi)|.$$

Proof. Focus on an element $e \in E$ in the support of W . Then, there is a ψ in W such that $\psi(e) \neq 0$. Decompose W as a direct sum $\langle \psi \rangle \oplus W'$. Then we may express W as a union of translates of W' , as follows:

$$W = \bigcup_{i=0}^{p-1} (i\psi + W').$$

Now, for any element $\phi' \in W'$, $i\psi(e) + \phi'(e) = 0$ for exactly one value of i between 0 and $p-1$. Denote the indicator function of an element ϕ in \mathbb{F}_p^E by $I_\phi: E \rightarrow \{0, 1\}$. This is defined as follows:

$$I_\phi(e) = \begin{cases} 0 & \text{if } \phi(e) = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Then,

$$\left(\sum_{\phi \in W} I_\phi \right) (e) = \left(\sum_{\phi' \in W'} \sum_{i=0}^{p-1} I_{\phi'+i\psi} \right) (e) = (p-1)p^{\dim(W)-1}.$$

Summing this over all e in the support of W gives

$$\sum_{\phi \in W} |\text{supp}(\phi)| = (p-1)p^{\dim(W)-1} |\text{supp}(W)|,$$

as required. \square

Theorem 6.3. *Let V be a non-zero subspace of \mathbb{F}_p^E with dimension v . Then there is a codimension one subspace W of V such that*

$$|\text{supp}(W)| \leq \frac{p^v - p}{p^v - 1} |\text{supp}(V)|.$$

Proof. Note that the theorem holds trivially if $v = 1$. We therefore assume $v \geq 2$. There are $(p^v - 1)/(p - 1)$ codimension one subspaces W of V . Summing the formula of Lemma 6.2 over each of these gives:

$$\sum_W |\text{supp}(W)| = \frac{1}{(p-1)p^{v-2}} \sum_W \sum_{\phi \in W} |\text{supp}(\phi)|.$$

The number of times a non-zero element ϕ of V appears in the sum $\sum_W \sum_{\phi \in W}$ is independent of the element ϕ . Since there are $(p^v - 1)/(p - 1)$ codimension one subspaces W , each containing $p^{v-1} - 1$ non-zero elements, and there are $p^v - 1$ non-zero elements of V , the number of times a non-zero element ϕ of V appears in the sum $\sum_W \sum_{\phi \in W}$ is therefore

$$\frac{(p^v - 1)(p^{v-1} - 1)}{(p-1)(p^v - 1)} = \frac{p^{v-1} - 1}{p-1}.$$

Hence,

$$\sum_W |\text{supp}(W)| = \frac{1}{(p-1)p^{v-2}} \frac{(p^{v-1} - 1)}{(p-1)} \sum_{\phi \in V} |\text{supp}(\phi)|.$$

By Lemma 6.2,

$$\sum_{\phi \in V} |\text{supp}(\phi)| = (p-1)p^{v-1} |\text{supp}(V)|.$$

Hence,

$$\sum_W |\text{supp}(W)| = \frac{(p^v - p)}{(p-1)} |\text{supp}(V)|.$$

The average, over all W , of $|\text{supp}(W)|$ is therefore

$$\frac{p^v - p}{p^v - 1} |\text{supp}(V)|.$$

Hence, there is a codimension one subspace W with support at most this size. \square

Proof of Theorem 6.1. We prove the first inequality by induction on the codimension $v - w$. When this quantity is 1, this is Theorem 6.3. For the inductive step, suppose that we have a subspace W' of V with dimension $w + 1$ such that

$$|\text{supp}(W')| \leq \frac{p^v - p^{v-w-1}}{p^v - 1} |\text{supp}(V)|.$$

By Theorem 6.3, W' has a subspace W with dimension w such that

$$\begin{aligned} |\text{supp}(W)| &\leq \frac{p^{w+1} - p}{p^{w+1} - 1} |\text{supp}(W')| \\ &\leq \left(\frac{p^{w+1} - p}{p^{w+1} - 1} \right) \left(\frac{p^v - p^{v-w-1}}{p^v - 1} \right) |\text{supp}(V)| \\ &= \frac{p^v - p^{v-w}}{p^v - 1} |\text{supp}(V)|, \end{aligned}$$

as required. The second inequality is trivial, because

$$\frac{p^v - p^{v-w}}{p^v - 1} \bigg/ \frac{p^{w+1} - p}{p^{w+1} - 1} = \frac{p^{v-w-1}(p^{w+1} - 1)}{p^v - 1} = \frac{p^v - p^{v-w-1}}{p^v - 1} \leq 1.$$

□

It is instructive to consider the case $w = 1$ in Theorem 6.1. This states that in any subspace V of \mathbb{F}_p^E with dimension $v > 1$, there is an element with at most

$$\frac{p^v - p^{v-1}}{p^v - 1} |E| \leq \frac{p^2 - p}{p^2 - 1} |E|$$

non-zero co-ordinates. This is a theorem in the theory of error-correcting codes, known as the ‘Plotkin bound’ [14]. For, a *linear code* is just a subspace of \mathbb{F}_p^E , and the *Hamming distance* of such a code is the minimal number of non-zero co-ordinates in any non-zero element of the subspace. Thus, Theorem 6.1 can be viewed as a generalisation of the Plotkin bound, giving information not just about elements of V but whole subspaces. It is probably well-known to experts on error-correcting codes.

7. PROOF OF THEOREMS 1.14 AND 1.15

One direction of Theorems 1.14 and 1.15 is easy: the implication (1) \Rightarrow (2). The proof is as follows. Suppose that $\phi: G_1 \rightarrow F$ is a surjective homomorphism from a finite index subgroup of G onto a non-abelian free group F . For the proof of Theorem 1.15, assume in addition that G_1 is normal in G and has index a power of p . Let $\{F_i\}$ be the derived p -series of F , and let $G_i = \phi^{-1}(F_i)$. Since $\{F_i\}$ is an abelian p -series for F with rapid descent, $\{G_i\}$ is therefore an abelian p -series with rapid descent, as required.

The difficult part of Theorems 1.14 and 1.15 is the implication (2) \Rightarrow (1). So, suppose that some finite index subgroup G_1 of G has an abelian p -series $\{G_i\}$ with rapid descent. In the proof of 1.15, take G_1 to be G . We will show that G_1 is p -large, which will establish the theorems. Since $d_p(G_i/G_{i+1}) \leq d_p(G_i)$, the rapid descent of $\{G_i\}$ implies that it has linear growth of mod p homology.

Let K be a connected finite 2-complex with fundamental group G . Let K_i be the finite-sheeted covering space corresponding to the subgroup G_i . Recall from Section 4 the definition of the relative size of an element of $H^1(K_i; \mathbb{F}_p)$, and the following result.

Theorem 4.1. *Let K be a finite connected 2-complex, and let $\{K_i \rightarrow K\}$ be a collection of finite-sheeted covering spaces. Suppose that $\{\pi_1(K_i)\}$ has linear growth of mod p homology for some prime p . Then one of the following must hold:*

- (i) $\pi_1(K_i)$ is p -large for infinitely many i , or
- (ii) there is some $\epsilon > 0$ such that the relative size of any non-trivial class in $H^1(K_i; \mathbb{F}_p)$ is at least ϵ , for all i .

Thus, our plan is to prove that (ii) of Theorem 5.1 does not hold, and therefore deduce that G_1 is p -large. We will keep track of a set U_i of cellular 1-dimensional cocycles on K_i that represent linearly independent elements of $H^1(K_i; \mathbb{F}_p)$. The cardinality $|U_i|$ will be some fixed positive integer u independent of i . (The precise size of u will depend on data from the group G and the series $\{G_i\}$.) Our aim is to ensure that

$$\frac{|\text{supp}(U_i)|}{\text{Number of 1-cells of } K_i} \rightarrow 0. \quad (\dagger)$$

In particular, the relative size of any element of U_i tends to zero, which means that (ii) does not hold.

We establish (\dagger) using the following method. Let $q_i: K_{i+1} \rightarrow K_i$ be the covering map. We will find a set of cocycles U_{i+1}^+ on K_{i+1} representing linearly independent elements of $H^1(K_{i+1}; \mathbb{F}_p)$, with the following properties:

- I. $\text{supp}(U_{i+1}^+) \subset q_i^{-1}(\text{supp}(U_i))$;
- II. $|U_{i+1}^+| > u$.

Note that the inequality in (II) is strict.

Let E denote the set of 1-cells of K_{i+1} with given orientation. Then $C^1(K_{i+1})$ is isomorphic to \mathbb{F}_p^E , the vector space of functions $E \rightarrow \mathbb{F}_p$. Let V be the subspace of $C^1(K_{i+1})$ spanned by U_{i+1}^+ , and let $w = u$. We apply Theorem 6.1 to V .

Theorem 6.1. *Let V be a subspace of \mathbb{F}_p^E with dimension v , and let w be a positive integer strictly less v . Then, V contains a subspace W with dimension w such that*

$$|\text{supp}(W)| \leq \frac{p^v - p^{v-w}}{p^v - 1} |\text{supp}(V)| \leq \frac{p^{w+1} - p}{p^{w+1} - 1} |\text{supp}(V)|.$$

Let U_{i+1} be a basis for the subspace W given by Theorem 6.1. Note that the factor $(p^{w+1} - p)/(p^{w+1} - 1)$ is strictly less than 1, and is dependent only on the

fixed integers u and p . Now, $|q_i^{-1}(\text{supp}(U_i))|$ is obtained from $|\text{supp}(U_i)|$ by scaling by the degree of the cover q_i . The same relation holds between the number of 1-cells in K_{i+1} and the number of 1-cells in K_i . Thus, (†) follows.

The key, then, is to construct the cocycles U_{i+1}^+ with properties (I) and (II). For this, we use Theorem 5.1 (setting $K = K_i$, $U = U_i$ and $\tilde{K} = K_{i+1}$)

Theorem 5.1. *Let K be a finite connected 2-complex with r 2-cells. Let U be a collection of cocycles on K that represent linearly independent elements of $H^1(K; \mathbb{F}_p)$. Let $u = |U|$. Let $q: \tilde{K} \rightarrow K$ be a finite regular cover such that $\pi_1(K)/\pi_1(\tilde{K})$ is an elementary abelian p -group with rank n . Then there is a collection \tilde{U} of cocycles on \tilde{K} representing linearly independent elements of $H^1(\tilde{K}; \mathbb{F}_p)$ such that*

1. $\text{supp}(\tilde{U}) \subset q^{-1}(\text{supp}(U))$;
2. $|\tilde{U}| \geq (n - u)u - r$.

We define U_{i+1}^+ to be the set \tilde{U} provided by this theorem. Condition (1) of the theorem is just (I) above. We need to ensure that Condition (II) holds. Thus, we require

$$(n - u)u - r > u.$$

We will ensure that this holds by using the hypothesis that G_i has rapid descent and by a suitable choice of u .

Let $\langle X|R \rangle$ be a finite presentation for G . Let λ be

$$\liminf_i \frac{d_p(G_i/G_{i+1}) - 2}{[G : G_i]}.$$

Since $\{G_i\}$ is rapidly descending, λ is positive. Let u be

$$\left\lceil \frac{4|R|}{\lambda} \right\rceil,$$

which is a positive integer. Now pick a sufficiently large integer j , such that, for all $i \geq j$,

$$\frac{d_p(G_i/G_{i+1}) - 2}{[G : G_i]} > \frac{\lambda}{2}$$

and

$$\lambda[G : G_i] > 4u.$$

Hence,

$$\lambda[G : G_i]/2 - u > \lambda[G : G_i]/4.$$

Now,

$$n = d_p(G_i/G_{i+1}) \geq \lambda[G : G_i]/2 + 2.$$

So,

$$(n - u)u - r \geq (\lambda[G : G_i]/2 + 2 - u)u - r \geq \lambda[G : G_i]u/4 + 2u - r \geq 2u,$$

since

$$\lambda u \geq 4|R| = 4r/[G : G_i].$$

This proves (2) \Rightarrow (1) of Theorems 1.14 and 1.15. \square

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