Finite covers of 3-manifolds, III

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Recall from last time:

Let G = a fin. pres. group.
Let $\phi: G \to \mathbb{Z}$ be a surj hm.
Let $G_i = \phi^{-1}(i\mathbb{Z})$.

Theorem 2.14: [L] Suppose that $d_p(G_i)$ is unbounded, for some prime p. Then G is large.

Today:

1. 3-manifolds with boundary
2. 3-orbifolds
3. arithmetic 3-manifolds
4. general 3-manifolds

We'll prove:

Thm 3.1: [Cooper-Long-Reid] If $\partial M \neq \emptyset$, then Γ is large.

Fundamental lemma: $b_1(M) \geq b_1(\partial M)/2$.

Proof of 3.1: $\partial M =$ union of tori. Let T be one of them.

Using 1.5, there is a surjective hm $\phi\colon \pi_1(M) \to PSL(2,q)$ or $PGL(2,q)$, some prime power q.

Let $f\colon \tilde{M} \to M$ corr to the kernel of ϕ.

The number of components of $f^{-1}(T) =$

$$[PSL(2,q), \phi_* \pi_1(T)] \geq 3.$$

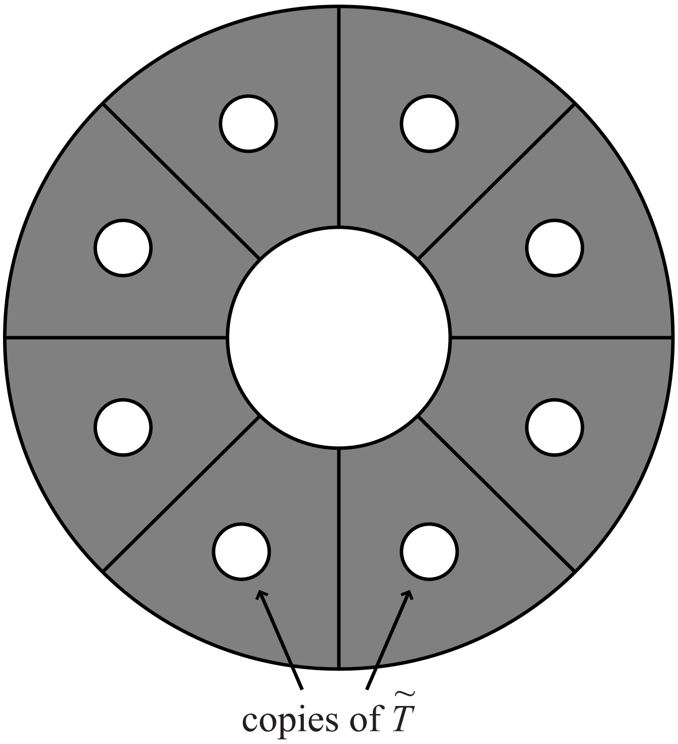
So, $|\partial \tilde{M}| \geq 3$. Let $\tilde{T}$ be one component.

By the fundamental lemma, there is a surj hm

$$\psi\colon \pi_1 \tilde{M} \to \mathbb{Z}$$

s.t. $\psi_* | \pi_1 \tilde{T} = 0$.

Let M_i be the cover corr to $\psi^{-1}(i\mathbb{Z})$.



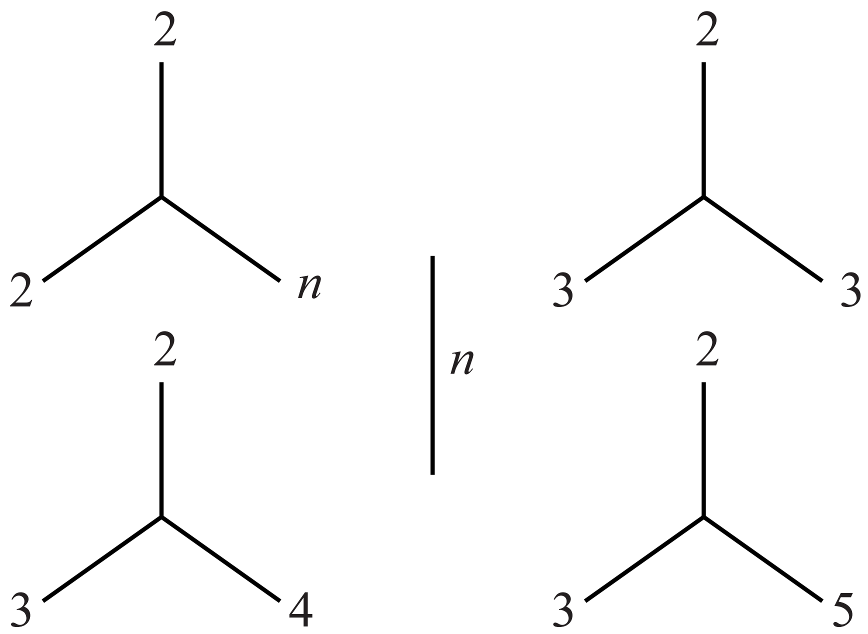
Then $|\partial M_i| \geq i$ and so $b_1(M_i)$ is unbounded.

Thm 2.14 $\Rightarrow$ $\pi_1(M)$ is large. $\square$

3-ORBIFOLDS

Lattices Γ in $\mathrm{PSL}(2,\mathbb{C})$ with torsion $\longleftrightarrow$ Finite volume orientable hyperbolic 3-orbifolds with non-$\emptyset$ sing locus

Local models for the singular locus:



Conjecture: [L] Γ is large.

Theorem 3.3. [L] Γ has a nested sequence of finite index subgroups Γ_i with linear growth of mod p homology, for some prime p.

But, unfortunately, Γ_i is not the derived p-series of Γ_1 (cf Thm 2.13)

Corollary 3.4: [L] Γ has at least exponential subgroup growth.

Theorem 3.5: [L] Let $O = \mathbb{H}^3/\Gamma$ be a compact orientable 3-orbifold. Suppose that the singular locus is a link L, and that the singularity order of each component of L is a prime p. Then

$$d_p(\Gamma) \geq |L|.$$

Sketch proof: Let M be $O - \mathrm{int}(N(L))$. This is a compact orientable 3-manifold. So,

$$d_p(M) \geq d_p(\partial M)/2 \geq |L|.$$

Γ is obtained from $\pi_1(M)$ by quotienting p^{th} powers of certain words (the meridians of the singular locus). This does not affect d_p. $\square$

So, as far as mod p homology is concerned, these 3-orbifolds behave as though they have boundary.

Proof of 3.3

Wlog, O is closed.

Step 1: Find a finite cover $O_1 = \mathbb{H}^3/\Gamma_1$ s.t.

- $\mathrm{sing}(O_1)$ is a non-empty link
- the sing order of each component is a prime p
- $d_p(\Gamma_1) > 11$.

By Selberg's thm, there is a f.i. torsion-free normal subgroup $\tilde{\Gamma} \triangleleft \Gamma$.

We may ensure $d_p(\tilde{\Gamma}) > 10p + 1$.

(This is stronger than 1.2: to ensure that $\tilde{\Gamma}$ is both normal and has big d_p requires sophisticated techniques relating to pro-p groups.)

Let μ = some torsion element of Γ. Wlog, order of μ is a prime p. Let $\Gamma_1 = \tilde{\Gamma}\langle\mu\rangle$. By 2.8,

$$d_p(\Gamma_1) \geq \frac{d_p(\tilde{\Gamma}) - 1}{[\Gamma_1 : \tilde{\Gamma}]} + 1 > 11.$$

Let K = some component of $\mathrm{sing}(O_1)$.

Step 2: Find covers $p_i \colon O_i \to O_1$ s.t. $p_i^{-1}(K)$ is a disjoint union of copies of K.

Then 3.5 $\Rightarrow$ linear growth of mod p homology.

Let μ and λ be a meridian and longitude of K.

We must ensure $\pi_1(O_i)$ contains $\langle\langle \mu, \lambda \rangle\rangle$.

Equivalently, we must find f.i. subgroups of $G = \Gamma_1 / \langle\langle \mu, \lambda \rangle\rangle$.

Golod-Shafarevich theorem: Let $G = \langle X | R \rangle$. Suppose that

$$d_p(G)^2/4 > |R| - |X| + d_p(G).$$

Then G has an infinite nested seq of f.i. subgroups.

Γ_1 has a presentation $\langle X_1 | R_1 \rangle$ where

$$|R_1| - |X_1| = |\mathrm{sing}(O_1)| \leq d_p(\Gamma_1)$$

So, $G = \Gamma_1/\langle\langle \mu, \lambda \rangle\rangle$ has a presentation $\langle X | R \rangle$ s.t.

$$|R| - |X| \leq d_p(\Gamma_1) + 2 \leq d_p(G) + 4.$$

So

$$\begin{aligned} &d_p(G)^2/4 - |R| + |X| - d_p(G) \\ &\quad \geq d_p(G)^2/4 - 2d_p(G) - 4 \end{aligned}$$

$d_p(\Gamma_1) > 11 \Rightarrow d_p(G) > 9 \Rightarrow$ GS inequality holds.

$\square$ Thm 3.3

Thm 3.6: [L-Long-Reid] Let O be a closed orientable 3-orbifold and let K be a s.c.c component of $\mathrm{sing}(O)$. Let $\phi\colon \pi_1(O) \to \mathbb{Z}$ be a surj hm s.t. $\phi|K = 0$. Then $\pi_1(O)$ is large.

Proof: Apply 3.5 and 2.14. $\square$

Given a s.c.c. component K of $\mathrm{sing}(O)$, such a ϕ can always be found if $b_1(O) \geq 2$.

Thm 3.7: [L-Long-Reid] Let $O = \mathbb{H}^3/\Gamma$ be a closed orientable 3-orbifold where Γ contains $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. If $vb_1(O) \geq 4$, then Γ is large.

Proof:

Let $\tilde{\Gamma}$ be a torsion-free f.i. normal subgroup of Γ. Let $\tilde{M} = \mathbb{H}^3/\tilde{\Gamma}$.

We may assume $b_1(\tilde{M}) \geq 4$.

Let x_1, x_2 and x_3 be the 3 non-trivial elements of $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Let $\Gamma_i = \tilde{\Gamma}\langle x_i \rangle$.
Let $O_i = \mathbb{H}^3/\Gamma_i$.

Then $\mathrm{sing}(O_i)$ is a non-empty collection of simple closed curves.

Claim: $b_1(\Gamma_i) \geq 2$ for some i.

x_i acts as an involution on $\tilde{M}$ and so on $H_1(\tilde{M}; \mathbb{R})$.

So, $H_1(\tilde{M}; \mathbb{R})$ decomposes into a $+1$ and -1 eigenspace.

$b_1(\Gamma_i) \geq$ dimension of $+1$ e-space.

Since $b_1(\tilde{M}) \geq 4$, one of these $+1$ e-spaces has dim ≥ 2.

$\square$ Thm 3.7

Thm 3.8: [L-Long-Reid] Any arithmetic 3-mfld M is commensurable with an orbifold O s.t. $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \leq \pi_1(O)$.

Proof: omitted

Corollary 3.9: If M is arithmetic and $vb_1(M) \geq 4$, then Γ is large.

This is known to hold in many circumstances. 'Arithmetic methods' [Clozel] often give $b_1(\Gamma_1) > 0$ for some congruence subgroup Γ_1.

Thm 3.10: [Borel] If some congruence subgroup has $b_1 > 0$, then $vb_1 = \infty$. Hence, $\pi_1(M)$ is large.

e.g. The Weeks manifold has large π_1.

Any arithmetic Kleinian group containing A_4, S_4 or A_5 is large.

Using vertices

Let $O = \mathbb{H}^3/\Gamma$ be closed.
Suppose $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \leq \Gamma$.

Let $\tilde{\Gamma}$ = torsion-free f.i. normal subgroup.
Let $\Gamma_1 = (\tilde{\Gamma})((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}))$.
Let $O_1 = \mathbb{H}^3/\Gamma_1$

Then:

- every singular arc and curve of O_1 has order 2;
- $\mathrm{sing}(O_1)$ contains vertices.

Claim: Under these hypotheses,

$$d_2(\Gamma_1) \geq b_1(\mathrm{sing}(O_1)).$$

Proof: Same as for 3.5.

Let $|O_i| \to |O_1|$ be a cover between manifolds and let $O_i \to O_1$ be the induced orbifold covers.

Then $b_1(\mathrm{sing}(O_i))$ grows linearly in the covering degree.

So, $\pi_1(O_i)$ has linear growth of mod 2 homology.

Hence:

Thm 3.11: [L-Long-Reid] Suppose $vb_1(M) > 0$ for any compact orientable 3-manifold M with infinite $\pi_1(M)$. Then any arithmetic Kleinian group is large.

Sketch proof. By 3.8, the arithmetic mfld is commensurable with $O = \mathbb{H}^3/\Gamma$ where $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \leq \Gamma$.

Let O_1 be as above. Find a cover $\tilde{O}_1$ of O_1 s.t.

- every singular arc and curve of $\tilde{O}_1$ has order 2;
- $\mathrm{sing}(\tilde{O}_1)$ contains vertices;
- $|\tilde{O}_1|$ has infinite π_1 (ensure this using the GS inequality).

By hypothesis, $|\tilde{O}_1|$ has a finite cover $|O_1'|$ with $b_1 > 0$.

Let O_1' be the induced orbifold cover.

Then $\pi_1(O_1') \to \mathbb{Z}$.

Then O_i' be the cyclic covers.

These have linear growth of mod 2 homology.

2.14 $\Rightarrow \pi_1(O_1)$ is large.

$\square$ Thm 3.11

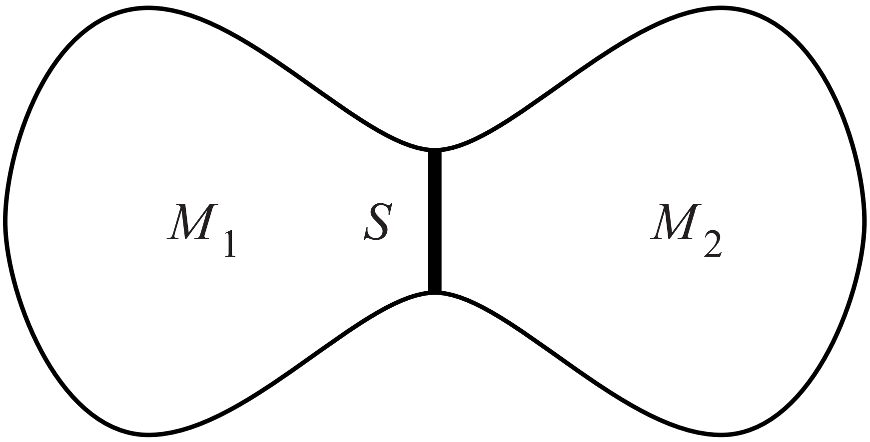
Growth rates of other quantities

Given a sequence of finite covers $M_i \to M$, how do the following grow?

- mod p homology: $d_p(M_i)$
- rank of π_1: $d(\pi_1(M_i))$
- Heegaard genus: $g(M_i)$
- the Cheeger constant: $h(M_i)$
- 1^{st} eigenvalue of Laplacian: $\lambda_1(M_i)$

The Cheeger constant $h(M)$ is defined as follows. One considers all decompositions of M along codimension 1 submanifolds S dividing M into M_1 and M_2. Then

$$h(M) = \inf \left\{ \frac{\text{Area}(S)}{\min\{\text{Vol}(M_1), \text{Vol}(M_2)\}} \right\}$$



Conjecture: [Lubotzky-Sarnak] M has a tower of finite covers M_i where $h(M_i) \to 0$.

This is a group-theoretic property:

Let G = fin. pres. group.
Let $\{G_i\}$ = a collection of f.i. subgroups.
Let M = a closed Riem mfld with $\pi_1(M) = G$.
Let M_i = covering space of M corr. to G_i.

G has Property (τ) w.r.t. $\{G_i\}$ if $\inf_i h(M_i) > 0$.

Conjecture: [Lubotzky-Sarnak] Γ does not have Property (τ) w.r.t. some nested seq of f.i. subgroups.

Note: positive vb_1 conjecture $\Rightarrow$ Lub-Sarnak.

A consequence of Lubotzky-Sarnak

Thm 3.12: [L-Long-Reid] The Lubotzky-Sarnak conjecture and the geometrisation conjecture together imply that any arithmetic Kleinian group is large.

Theorem 3.13: [L] Let G be a finitely presented group, and let $\{G_i\}$ be a collection of normal subgroups each with index a power of p. Suppose that

1. G does not have Property (τ);
2. G_i has linear growth of mod p homology.

Then G is large.

Proof: Follows same lines as 2.14.

Proof of 3.12: Exactly as for 3.11 $\square$

Thm 3.14: [L]

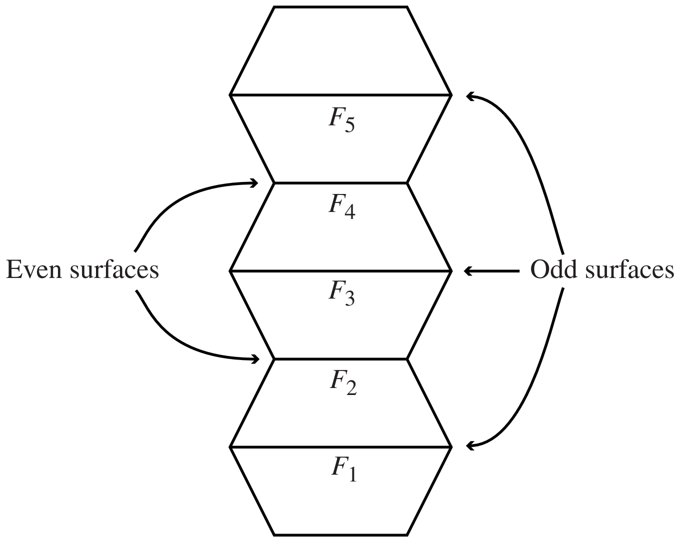
$$h(M) \leq \frac{8\pi(g(M)-1)}{\mathrm{Vol}(M)}$$

Sketch proof: Let F be a minimal genus Heegaard surface: $|\chi(F)| = 2(g(M)-1)$.

Untelescope to a generalised Heegaard splitting $\{F_1, \ldots, F_n\}$ s.t.

- F_{even} is incompressible
- F_{odd} is strongly irreducible
- $|\chi(F_i)| \leq |\chi(F)|\ \forall i$.

For i odd, let M_i be the manifold between F_{i-1} and F_{i+1}.



[Freedman-Hass-Scott] $\Rightarrow$ can isotope F_{even} to a minimal surface $\overline{F}_{\text{even}}$ (not quite true)

[Pitts-Rubinstein] $\Rightarrow$ can isotope F_{odd} to a minimal surface $\overline{F}_{\text{odd}}$ (not quite true)

Gauss-Bonnet $\Rightarrow$ $\text{Area}(\overline{F}_i) \leq 2\pi|\chi(F_i)|$.

For each i odd and $\epsilon > 0$, [Pitts-Rubinstein] $\Rightarrow$ there is a sweepout of M_i by F_i with max area at most $\text{Area}(\overline{F}_i) + \epsilon$.

Some F_i divides M into two pieces of equal volume. $\square$

Let $\{M_i \to M\}$ be a collection of finite covers with degree d_i.

The inverse image of any Heegaard surface for M is a Heegaard surface for M_i. So

$$g(M_i) - 1 \leq (g(M) - 1)d_i$$

The Heegaard gradient of $\{M_i \to M\}$ is

$$\inf_i \frac{2(g(M_i) - 1)}{\mathrm{degree}(M_i \to M)}$$

Corollary 3.15: Heegaard gradient zero $\Rightarrow$ $\inf_i h(M_i) = 0$.

Heegaard gradient conjecture: [L] M has zero Heegaard gradient if and only if M has a finite cover that fibres over S^1.

Evidence: The only known M with zero Heegaard gradient are virtually fibred.

Thm 3.16: [L] Let $\{M_i \to M\}$ be a collection of finite reg covers with degree d_i. If $g(M_i) = o(\sqrt[4]{d_i})$, then M_i fibres over S^1 for all $i >> 0$.

An approach to the virtually Haken conjecture

Thm 3.17: [L] The Heegaard gradient conjecture and the Lub-Sarnak conjecture $\Rightarrow$ the virtually Haken conjecture

Conclusions

- Mod p homology provides a systematic method for studying finite covers of 3-manifolds

- 3-manifolds have fast homology growth

- Orbifolds and arithmetic 3-manifolds have the fastest possible homology growth

- This gives fast subgroup growth

- In conjuction with $vb_1 > 0$ or $h \to 0$, this gives largeness for certain arithmetic 3-manifolds

- Heegaard gradient conj and Lubotzky-Sarnak $\Rightarrow$ virtually Haken conjecture

- Fast homology growth $\Rightarrow$ Lubotzky-Sarnak ??