

ELEMENTARY KNOT THEORY

MARC LACKENBY

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1. INTRODUCTION

In the past 50 years, knot theory has become an extremely well-developed subject. But there remain several notoriously intractable problems about knots and links, many of which are surprisingly easy to state. The focus of this article is this ‘elementary’ aspect to knot theory. Our aim is to highlight what we still do not understand, as well as to provide a brief survey of what is known. The problems that we will concentrate on are the non-technical ones, although of course, their eventual solution is likely to be sophisticated. In fact, one of the attractions of knot theory is its extensive interactions with many different branches of mathematics: 3-manifold topology, hyperbolic geometry, Teichmüller theory, gauge theory, mathematical physics, geometric group theory, graph theory and many other fields.

This survey and problem list is by no means exhaustive. For example, we largely neglect the theory of braids, because there is an excellent survey by Birman and Brendle [5] on this topic. We have also avoided 4-dimensional questions, such as the slice-ribbon conjecture (Problem 1.33 in [41]). Although these do have a significant influence on elementary knot theory, via unknotting number for example, this field is so extensive that it would best be dealt with in a separate article.

In fact, this article comes with a number of health warnings. It is frequently vague, often deliberately so, and many standard terms that we use are not defined here. It is certainly far from complete, and we apologise to anyone whose work has been omitted. And it is not meant to be a historical account of the many developments in the subject. Instead, it is primarily a list of fundamental and elementary problems, together with enough of a survey of the field to put these problems into context.

2. THE EQUIVALENCE PROBLEM

The equivalence problem for knots and links asks the most fundamental question in the field: *can we decide whether two knots or links are equivalent?* Questions of this sort arise naturally in just about any branch of mathematics. In topology, there are many well-known negative results.

A central result of Markov [65] states that there is no algorithm to determine whether two closed n -manifolds are homeomorphic, when $n \geq 4$. But in dimension 3, the situation is more tractable. In particular, the equivalence problem for knots and links is soluble. In fact, there are now five known ways to solve this problem, which we describe briefly below.¹ Four of these techniques focus on the exterior $S^3 - \text{int}(N(K))$ of the link K . Now, one loses a small amount of information when passing to the link exterior, because it is possible for distinct links to have homeomorphic exteriors. (This is not the case for knots, by the famous theorem of Gordon and Luecke [23]). So, one should also keep track of a complete set of meridians for the link, in other words, a collection of simple closed curves on $\partial N(K)$, one on each component of $\partial N(K)$, each of which bounds a meridian disc in $N(K)$. Then, for two links K and K' , there is a homeomorphism between their exteriors that preserves these meridians if and only if K and K' are equivalent.

2.1. HIERARCHIES AND NORMAL SURFACES

Haken [25], Hemion [30] and Matveev [66] were the first to solve the equivalence problem for links. Their solution is lengthy and difficult. A full account of their argument was given only fairly recently by Matveev [66].

Haken used incompressible surfaces arranged into sequences called hierarchies. Recall that a surface S properly embedded in a 3-manifold M is *incompressible* if, for any embedded disc D in M with $D \cap S = \partial D$, there is a disc D' in S with $\partial D' = \partial D$. Given such a surface, it is natural to cut M along S , creating the compact manifold $\text{cl}(M - N(S))$. It turns out that one can always find another properly embedded incompressible surface in this new manifold, and thereby iterate this procedure. A *hierarchy* is a sequence of manifolds and surfaces

$$M = M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} M_n$$

where each S_i is a compact orientable incompressible surface properly embedded in M_i with no 2-sphere components, where M_{i+1} is obtained from M_i by cutting along S_i and where the final manifold M_n is a collection of 3-balls. Haken proved that the exterior of a non-split link always admits a hierarchy. He was able to solve the equivalence problem for links via the use of hierarchies with certain nice properties. An example of a hierarchy for a knot exterior is given in Figure 1.

Associated with a hierarchy, there is a 2-complex embedded in the manifold M . To build this, one extends each surface S_i in turn, by attaching a collar to ∂S_i in $N(S_1 \cup \dots \cup S_{i-1})$, so that the boundary of the new surface runs over the earlier surfaces and ∂M . The union of these extended surfaces and ∂M is the 2-complex. The 0-cells are the points where three surfaces intersect. The 1-skeleton is the set of points which lie on at least two surfaces. Since the complement of this extended hierarchy and the boundary of the manifold is a collection of open balls, we actually obtain a cell structure for M in this way. It is a trivial but important observation that this is a representation of M that depends only on the topology of the hierarchy. Haken's solution to the equivalence problem

¹ Since the first version of this article was produced, two new solutions to the related problem of determining whether two closed 3-manifolds are homeomorphic have been given by Scott and Short [91] and by Kuperberg [49]. Both rely on the solution to the Geometrisation Conjecture, as in Section 2.2.

establishes that, as long as one chooses the hierarchy correctly, there is a way of building this cell structure, starting only from an arbitrary triangulation of M .

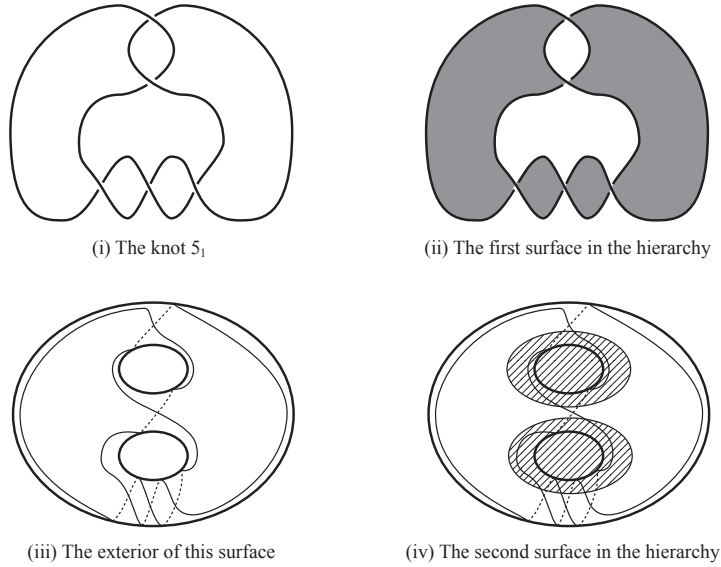


Figure 1: A hierarchy for a knot exterior

Suppose that we are given diagrams for two (non-split) links K and K' . The first step in the algorithm is to construct triangulations T and T' for their exteriors M and M' . We know that M and M' admit hierarchies with the required properties. Moreover, if M and M' are homeomorphic (by a homeomorphism preserving the given meridians), then this homeomorphism takes one such hierarchy for M to a hierarchy for M' . So, let us focus initially on the hierarchy for M . The next stage in the procedure is to place the first surface S of the hierarchy into *normal form*. By definition, this means that it intersects each tetrahedron of T in a collection of triangles and squares, as shown in Figure 2.

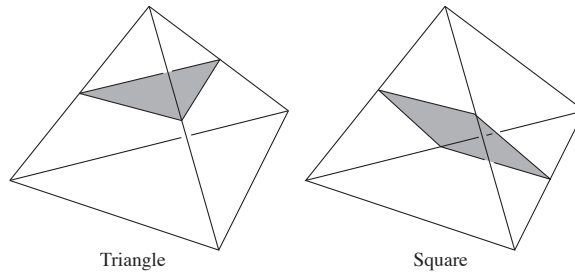


Figure 2: Normal surface

Now, any incompressible, boundary-incompressible surface S (with no components that are 2-spheres or boundary-parallel discs) in a compact orientable irreducible boundary-irreducible 3-manifold M with a given triangulation T may be ambient isotoped into normal form. It can then be encoded by a finite amount of data. The normal surface S determines a collection of $7t$ integers, where t is the number of tetrahedra in T . These integers simply record the number of triangles and squares of S in each tetrahedron. (Note from Figure 2 that there are 4 types of normal triangle and 3

types of normal square in each tetrahedron.) The list of these integers is denoted $[S]$ and is called the associated *normal surface vector*. Normal surface vectors satisfy a number of constraints. Firstly, each co-ordinate is, of course, a non-negative integer. Secondly, they satisfy various linear equations called the *matching conditions* which guarantee that, along each face of T with a tetrahedron on each side, the triangles and squares of S in the two adjacent tetrahedra patch together correctly. Finally, there are the *quadrilateral constraints* which require S to have at most one type of square in each tetrahedron. This is because two different square types cannot co-exist in the same tetrahedron without intersecting. Haken showed that there is a one-one correspondence between properly embedded normal surfaces (up to a suitable notion of ‘normal’ isotopy) and vectors satisfying these conditions. Given this strong connection, it is natural to use tools from linear algebra when studying normal surfaces. In particular, one can speak of a normal surface S as being the *sum* of two normal surfaces S_1 and S_2 if $[S] = [S_1] + [S_2]$. A normal surface S is called *fundamental* if it cannot be expressed as a sum of two other non-empty normal surfaces. Haken showed that there is a finite list of fundamental normal surfaces, and these are all algorithmically constructible. A key part of Haken’s argument is to show that some surfaces can be realised as fundamental normal surfaces with respect to *any* triangulation of the manifold. Thus, if we could find a hierarchy where each surface had this property, then we could construct the hierarchy starting with any triangulation T of M . In fact, Haken did not prove that there is a hierarchy with this property, but he did show that there is a hierarchy where each surface can be realised as a sum of a bounded number of fundamental normal surfaces. This is sufficient to make the argument work.

Thus, Haken’s algorithm essentially proceeds by starting with the triangulations T and T' of the two link exteriors. Then, one constructs all possible hierarchies for each manifold, with the property that each surface in the hierarchy is a bounded sum of fundamental surfaces. From each such hierarchy, one forms the associated cell structure for M . If two such cell structures, one from each of T and T' , are combinatorially equivalent, then the links are the same. If none of these cell structures are equivalent, then the links are distinct.

There were two cases that Haken could not handle using these methods. When M fibres over the circle with fibre S , then cutting M along S results in a copy of $S \times I$. In $S \times I$, there is no good choice of surface to cut along. In particular, it is not clear how to find a surface that is guaranteed to be fundamental or at least a bounded sum of fundamental surfaces. The resolution of the equivalence problem for these manifolds was not completed until work of Hemion [30], where he gave a solution to the conjugacy problem in the mapping class group of S . A similar problem arises when cutting along S yields a twisted I -bundle (in which case, S is known as a *semi-fibre*). This case was fully dealt with by Matveev [66].

In fact, there is now an alternative way of sidestepping this issue, by establishing that there is always a hierarchy for a non-split link exterior which is always constructible. This relies on the result of Culler and Shalen [13], which says that if M is a compact orientable hyperbolic 3-manifold with boundary a non-empty collection of tori, then M contains either a closed, non-separating, essential properly embedded surface or a separating, connected, essential, properly embedded surface with non-empty boundary. When M is the exterior of a hyperbolic link in the 3-sphere, this surface can be chosen to be neither a fibre nor a semi-fibre (see Section 7 in [9]). By work of Mijatovic [71], there is such a surface that can be realised as a bounded sum of fundamental surfaces.

Of course, this summary is a dramatic over-simplification. The details of the argument are very complicated. A full exposition, which lasts several hundred pages, can be found in the excellent book by Matveev [66].

2.2. GEOMETRIC STRUCTURES

One of the most striking advances in low-dimensional topology was Thurston's introduction of hyperbolic geometry, via his Geometrisation Conjecture [100]. This was proved by Perelman in 2003 [81, 82, 83], but Thurston himself proved his conjecture in the case of knot and link complements, and this can be used to provide another method of solving the equivalence problem. An approximate statement of the Geometrisation Conjecture is that every compact orientable 3-manifold admits a 'canonical decomposition' into 'geometric pieces'.

The canonical decomposition takes place in two steps. (We focus for simplicity on the case where the boundary of the 3-manifold is empty or a collection of tori.) Firstly, one decomposes the 3-manifold into its prime connected summands. Secondly, one cuts the manifold along its JSJ tori. Both processes can be achieved algorithmically using normal surface theory [34]. (See [75] and [66] for more details of JSJ theory, which was first developed by Jaco, Shalen [33] and Johannson [36].)

Once this decomposition has taken place, the resulting pieces either are Seifert fibred or admit finite-volume hyperbolic structures. The generic situation is the hyperbolic case, and it is this that we will focus on. An algorithm to find a finite-volume hyperbolic structure on a 3-manifold, provided one exists, was given by Manning [63], based on work of Casson. By combining [63] with work of Weeks [103], it is possible to compute the Epstein-Penner decomposition of the manifold [18]. This is a striking construction: it is a decomposition of a non-compact finite-volume hyperbolic 3-manifold into ideal polyhedra. The crucial thing about it is that it is canonical: it depends only on the topology of the manifold, not on any arbitrary choices. Thus, two finite-volume non-compact hyperbolic 3-manifolds are homeomorphic if and only if their Epstein-Penner decompositions are combinatorially equivalent. Moreover, one can determine whether there is a homeomorphism between the manifolds, taking one given collection of disjoint simple closed curves in the boundary to another. This therefore gives a solution to the equivalence problem for knots and links.

This is more than just a theoretical algorithm. The computer programs Knotscape [31], Snappea [102] and Snap [10] all attempt to compute the Epstein-Penner decomposition of a hyperbolic 3-manifold. Unlike the Casson-Manning algorithm, they are not guaranteed to do so. But *if* Snap or Knotscape finds what it claims is the Epstein-Penner decomposition, then this is indeed the correct decomposition, and as a result, these programs are a practical method of reliably determining whether two links are equivalent.

2.3. RELATIVELY HYPERBOLIC GROUPS

One can also use some of the machinery of geometric group theory to solve the equivalence problem, at least for hyperbolic knots.

The fundamental group of a closed hyperbolic 3-manifold is Gromov-hyperbolic. Mostow rigidity states that two such manifolds are isometric if and only if their fundamental groups are isomor-

phic. Moreover, Sela provided an algorithm to decide whether two torsion-free Gromov-hyperbolic groups are isomorphic [92]. Now, the complement of a knot is, of course, not closed. But when this complement has a hyperbolic structure, its fundamental group is a relatively hyperbolic group, relative to the fundamental group of the toral boundary. Mostow rigidity applies in this case also. Sela's result was extended to this class of relatively hyperbolic groups by Dahmani and Groves [14]. So, at least in the case of hyperbolic knots, one can solve the equivalence problem this way. This does not immediately lead to a solution for hyperbolic links, because links are not determined by their complements. But presumably, the algorithm of Dahmani and Groves could be adapted to take account of a set of meridians in the boundary tori. This method also only works in the hyperbolic case, and so to turn this into a fully-fledged solution to the equivalence problem for all links, presumably one would first need to find the 'canonical decomposition into geometric pieces' that was discussed in the previous subsection. Nevertheless, this is a genuinely different solution to the equivalence problem. But its algorithmic complexity seems to be hard to estimate.

2.4. REIDEMEISTER MOVES

There is an alternative way of interpreting the equivalence problem in terms of Reidemeister moves. Recall that a *Reidemeister move* is a local modification to a link diagram as shown in Figure 3. Reidemeister proved [85] that any two diagrams of a link differ by a sequence of Reidemeister moves. The algorithmic significance of this is that *if* two diagrams represent the same link, then one may always prove this, given enough time and computing power. This is because one can apply all possible Reidemeister moves to the first diagram, and thereby obtain a collection of diagrams of the first link. One can then apply all possible Reidemeister moves to each of these, and so on. It is a consequence of Reidemeister's theorem that if the two initial diagrams represent the same link, then this process is guaranteed to produce a sequence of Reidemeister moves taking one diagram to the other.

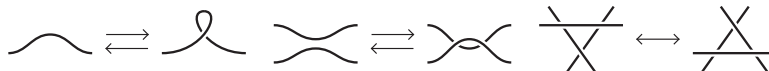


Figure 3: Reidemeister moves

Of course, this does not provide a solution to the equivalence problem, because if two diagrams represent distinct link types, then the above process does not terminate. But if one knew in advance how many Reidemeister moves were required to take one diagram to other, then one could stop the process when one had tried all possible sequences of Reidemeister moves of this length and if a sequence of Reidemeister moves taking one diagram to the other had not been found, then one could declare that the links are distinct. More specifically, a computable upper bound on the number of Reidemeister moves required to relate two diagrams of the same link provides a solution to the equivalence problem. In fact, it is not hard to show that the converse is also true: if there is a solution to the equivalence problem, then, given natural numbers n_1 and n_2 , one can compute an upper bound on the number of Reidemeister moves required to relate two diagrams of a link with n_1 and n_2 crossings. One enumerates all link diagrams with these numbers of crossings, and then one sorts them into their various link types, using the hypothesised algorithm to solve the equivalence problem. Then, for all diagrams of the same link type, one starts applying Reidemeister moves

to these diagrams. By Reidemeister's theorem, eventually a sequence of such moves will be found relating any two diagrams of the same link. Hence, one can compute an upper bound on the number of moves that are required.

Recently, Coward and the author [9] have provided an explicit, computable upper bound on Reidemeister moves, thereby giving a new and conceptually simple solution to the equivalence problem. Unfortunately, it is a huge bound, and so the resulting algorithm is very inefficient. We will provide more details in Section 4, which is devoted to Reidemeister moves.

2.5. PACHNER MOVES

It is well known that two triangulated manifolds are PL-homeomorphic if and only if they differ by a sequence of Pachner moves [80]. These are modifications that change the triangulation in a very simple way. In the case of closed 3-manifolds, the moves are shown in Figure 4. For 3-manifolds with boundary, there are some extra moves which change the triangulation on the boundary.

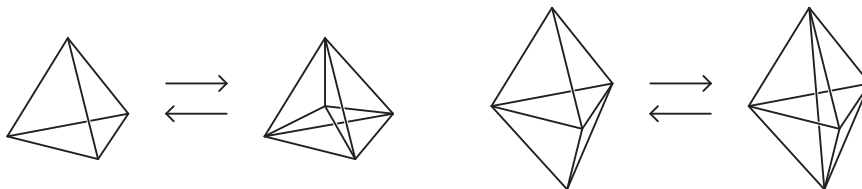


Figure 4: Pachner moves

Therefore, just as argued in the previous subsection, if two triangulated manifolds are PL-homeomorphic, then one will always be able to prove this eventually. Moreover, if one has a computable upper bound on the number of Pachner moves required to pass between two triangulations, then one can solve the PL-homeomorphism problem. In the case of knot exteriors, such a computable upper bound was found by Mijatovic [71]. Unfortunately, his bound is massive: a tower of exponentials, exponentially high. So, again, this does not lead to a practical algorithm.

Mijatovic's method built on work of Haken. As explained in Section 2.1, Haken used a hierarchy for a link exterior M to build a 'canonical' cell structure for M . This had the property that one can algorithmically build this cellulation starting from any triangulation of M . By carefully modifying the initial triangulation to one derived from this cellulation, Mijatovic was able to produce an upper bound on the number of Pachner moves required. The method is not easy, and in particular, he needed to go beyond what Haken achieved, because he also needed the Rubinstein-Thompson machinery for 3-sphere recognition [99, 70].

2.6. SOLVING THE EQUIVALENCE PROBLEM EFFICIENTLY

None of these approaches to the equivalence problem is known to be efficient. Indeed, any efficient solution seems to be out of reach at present. This leads to our first unsolved problem.

PROBLEM 1. *What is the complexity class of the equivalence problem for knots and links?*

It is quite striking that, of the five above approaches to the equivalence problem, only the final two give an *a priori* estimate for its running time. However, the disadvantage of the approaches using Reidemeister moves or Pachner moves is that they are almost inevitably lengthy.

One of the main difficulties in Haken's approach to the equivalence problem is that, in order to apply normal surface theory, one needs to build a triangulation T_i for each 3-manifold M_i in the hierarchy. However, the surface S_i may be exponentially complicated in T_i , in the sense of having exponentially many triangles and squares, as a function of the number of tetrahedra in T_i . So, it seems inevitable that the next triangulation T_{i+1} should be exponentially more complicated than that of T_i . So, the number of tetrahedra in T_n is massive. As a function of the crossing number of a given diagram of the link, it is a tower of exponentials. The height of this tower is n , the length of the hierarchy, and unfortunately, due to the technical requirements of Haken's hierarchies, n can be quite large too.

On the other hand, the approach to the equivalence using geometric structures is of unknown complexity. However, it appears to be computationally efficient in practice, even if one cannot yet prove this. So, there remains some hope that a provably efficient solution to the equivalence problem may be found in the future.

One might ask for a polynomial-time algorithm in Problem 1. But this seems rather unlikely, and there certainly seems no prospect of a proof in sight. Even an NP algorithm seems beyond reach at the moment. (For a very brief explanation of NP and other complexity classes, see Section 3.1.) Alternatively, one might try to find lower bounds on the complexity of the problem, conditional upon well-known conjectures in theoretical computer science. For example, is the equivalence problem NP-hard? In other words, is it at least as hard as any other NP problem? In particular, if there was a polynomial-time solution to the equivalence problem for knots and links, would this imply that $P = NP$? This seems plausible. Indeed, a seemingly simpler algorithm in 3-manifold theory is known to be NP-complete, by work of Agol, Hass and Thurston [3]. This is the problem of determining whether a given knot in a given 3-manifold bounds a compact orientable embedded surface with genus at most a given integer.

3. THE RECOGNITION PROBLEM

An algorithmic problem that is closely related to the equivalence problem is the *recognition problem* for a fixed link type. Here, one fixes a link type K , and one asks whether a given diagram represents this link type. Fairly obviously, a solution to the equivalence problem implies a solution to the recognition problem for each link type. However, by fixing the link type, we simplify the problem, and so there may be some hope of coming up with a more efficient solution. The case of the unknot is particularly intriguing. The following remains unknown.

PROBLEM 2. *Is there a polynomial-time algorithm to recognise the unknot?*

Currently, the unknot recognition problem is known to lie in the following complexity classes: NP, co-NP and E. We briefly remind the reader of the definitions of these and various other classes.

3.1. SOME BASIC COMPLEXITY CLASSES

The class P is possibly simplest to describe. It consists of those problems that be solved in time n^k , where n is the size of the input and k is a fixed constant. Similarly, a problem is in the class E if it can be solved in time k^n , where n is the size of the input and k is a fixed constant. Somewhat confusingly, this is not the same as the class EXP of exponential-time algorithms, since, by definition, these can be solved in time k^{n^c} for constants k and c .

A problem is in NP if it admits a polynomial-time certificate. This is an extra piece of information that is not provided by the algorithm, rather it is given by some external source. The point is that if the answer to the problem is ‘yes’, then the certificate proves that it is ‘yes’, and this can be verified in polynomial time. There are many examples of NP problems; one is the problem of deciding whether a given positive integer is composite. In this case, a very simple certificate is two integers (greater than 1) whose product is the given integer. It can be verified in polynomial time (as a function of the number of digits of the input) that these two integers do indeed multiply together to produce the given number. One might legitimately wonder why NP problems are solvable at all, given that they require information from an external source. But the point is that this information must have polynomially bounded size, because a computer has time enough only to check this many bits of information. So, a deterministic algorithm proceeds by checking all possible certificates of at most this size. Thus, problems in NP are also in EXP.

A computational problem is in co-NP if its negation is in NP. Thus, for example, the problem of deciding whether a positive integer is composite is in co-NP, because there is an NP algorithm to determine whether a positive integer is prime. This is not at all obvious however. It was first proved by Pratt [84], using the Lucas primality test.

There is also the well-known class of NP-complete problems. A problem is NP-complete if it is in NP and any other NP problem may be (efficiently) reduced to it. In particular, if an NP-complete problem has a polynomial-time solution, then this would imply that $P = NP$. It is widely believed that $P \neq NP$, and so NP-complete problems are, in some conditional sense, provably hard. However, it is very unlikely that any problem that is both in NP and co-NP is NP-complete, because this would imply that $NP = co-NP$, which also is viewed as rather unlikely.

3.2. HAKEN’S UNKNOT RECOGNITION ALGORITHM

The first person to provide an unknot recognition algorithm was Haken [24]. A knot K is the unknot if and only if its exterior M has compressible boundary. Haken showed how one may use normal surface theory to determine this. As in Section 2.1, one builds a triangulation of M using the given diagram of K . Haken showed that if M contains an essential properly embedded disc, then it contains one that is a fundamental normal surface. He also showed that all fundamental normal surfaces may be algorithmically constructed. Thus, his unknot recognition algorithm proceeds by constructing all possible fundamental normal surfaces, and determining whether any of these is an essential disc.

3.3. THE NP ALGORITHM OF HASS, LAGARIAS AND PIPPENGER

Hass, Lagarias and Pippenger [27] provided an NP algorithm, by building on Haken's work. Instead of using fundamental surfaces, they used vertex surfaces. By definition, a normal surface S is a *vertex surface* if it is connected and whenever $n[S] = [S_1] + [S_2]$ for a positive integer n and normal surfaces S_1 and S_2 , then each of $[S_1]$ and $[S_2]$ is a multiple of $[S]$. The reason for the vertex terminology is that the set of all normal surface vectors can be viewed as integer points within a larger subset of \mathbb{R}^{7t} , where t is the number of tetrahedra of the 3-manifold M . This subset is the set of vectors satisfying the matching equations and quadrilateral constraints, but where the vectors are required only to be non-negative *real* numbers. It is not hard to see that this subset is a union of convex polytopes, each of which is the cone on a compact polytope, where the cone point is the origin. Vertex surfaces are precisely those connected normal surfaces that are a multiple of a vertex of one of these compact polytopes. Hass, Lagarias and Pippenger showed that if M contains an essential properly embedded disc, then it has one that is a vertex normal surface. Moreover, for each such vertex surface that is an essential disc, one can certify that it is an essential disc in polynomial time. Thus, unknot recognition lies in NP. It is also not hard to prove that the number of vertex surfaces is at most 2^{7t} , because each one is obtained as the unique solution (up to scaling) of the matching equations plus some extra constraints that force certain co-ordinates to be zero. So, if one checks each such vertex surface in turn, one can determine whether the given knot is the unknot in at most k^n steps where n is the crossing number of the diagram and k is a fixed constant. This implies that unknot recognition lies in E.

3.4. THE CO-NP ALGORITHM OF KUPERBERG

Kuperberg has recently proved that unknot recognition is in co-NP, assuming the Generalised Riemann Hypothesis [48]. Here, one wants an efficient way of certifying that a non-trivial knot K is indeed knotted. Kuperberg achieves this by establishing the existence of a representation from $\pi_1(S^3 - K)$ to some finite group with non-abelian image. The unknot clearly admits no such representation, because the fundamental group of its complement is \mathbb{Z} . Thus, this does establish that the knot is non-trivial.

The existence of such a representation is not at all clear. Moreover, Kuperberg requires a representation with small enough image, so that the representation can be verified as a homomorphism with non-abelian image in polynomial time. The starting point is the major theorem of Kronheimer and Mrowka [45] that establishes that, for a non-trivial knot K , there is a homomorphism $\pi_1(S^3 - K) \rightarrow SU(2)$ with non-abelian image. This is established using a highly sophisticated argument that uses taut foliations, contact structures, symplectic fillings, and the instanton equations. (In the case where the knot K is hyperbolic, one can instead use the associated representation $\pi_1(S^3 - K) \rightarrow PSL(2, \mathbb{C})$ in Kuperberg's argument.)

Once one has a linear representation of $\Gamma = \pi_1(S^3 - K)$, there is a standard method of obtaining finite representations, that goes back to the result of Malcev [62] which states that finitely generated linear groups are residually finite. One considers a finite generating set for Γ , and for each generator, one considers the entries of the corresponding matrix (in $SU(2)$). The ring R generated by these entries is a finitely generated subring of \mathbb{C} . Thus, the image of Γ lies in $SL(2, R)$. Such a ring R

contains many finite index ideals I . The desired finite quotients are obtained by mapping $SL(2, R)$ onto $SL(2, R/I)$ for some such I . The Generalised Riemann Hypothesis is used to prove the existence of an ideal I with small index, and hence one obtains a finite quotient of Γ with small size. The actual tools that are used here are results of Korain [43], Lagarias-Odlyzko [59] and Weinberger [104] which imply that, for any integer polynomial, one may reduce its coefficients modulo some small prime and obtain a polynomial with a root, assuming the Generalised Riemann Hypothesis.

3.5. THE CO-NP ALGORITHM OF AGOL

Agol has announced the existence of another certificate for establishing that a non-trivial knot is knotted [1]. This has the advantage that it is unconditional on any conjectures, and it also provides more information about the knot, by determining its genus. Recall that the *genus* of a knot K is the minimal genus of a Seifert surface for K . It is a fundamental quantity, but for the purposes of this algorithm, all that one needs is that the genus is zero if and only if the knot is trivial.

It is a consequence of Haken's work that the genus of a knot is algorithmically computable, because a minimal genus Seifert surface can be arranged to be a fundamental normal surface. However, Thurston and Gabai found another method for determining the genus of knots, by using the theory of taut foliations [101, 21]. In fact, this method naturally measures not the genus of a compact surface S but its *Thurston complexity* $\chi_-(S)$. By definition, when S is connected, then $\chi_-(S) = \max\{-\chi(S), 0\}$. When S is disconnected with components S_1, \dots, S_n , then $\chi_-(S) = \sum_{i=1}^n \chi_-(S_i)$. Thus, when S is orientable, $\chi_-(S)$ is roughly the negative of the Euler characteristic of the surface, but one first discards disc and sphere components. So, for knots, a Seifert surface has minimal possible χ_- if and only if it has minimal genus. But for links, this need not be the case, because the link may have disconnected Seifert surfaces as well as connected ones.

The theory of Thurston and Gabai is concerned with foliations on a 3-manifold, where the leaves of the foliation have codimension one and where there is a consistent transverse orientation on these leaves. Such a foliation is *taut* if there is a simple closed curve in the 3-manifold that is transverse to the foliation and that intersects every leaf. This may seem a slightly strange definition, but it turns out that the existence of such a curve rules out various trivial examples of foliations, and in fact the following was proved by Thurston [101]: *If S is a compact leaf of a taut foliation on a compact orientable 3-manifold M , then S has minimal χ_- in its homology class in $H_2(M, \partial S)$.* Hence, if one starts with a Seifert surface S for an oriented link L , and one can find a taut foliation on the exterior of L in which S is a compact leaf, then S has minimal χ_- among all Seifert surfaces for L . Gabai [21] introduced a method for constructing these foliations and in fact proved the converse statement: *If S is a Seifert surface for an oriented non-split link L with minimal χ_- , then there is some taut foliation on the exterior of L in which S is a leaf.*

One can view this taut foliation as a way of certifying the genus of a knot. However, it is not a certificate in the strict algorithmic sense, because in principle, an infinite amount of information is required to specify the foliation. But the key features of Gabai's foliations are encoded by a finite amount of information as follows. Gabai utilised hierarchies to construct his foliations, and it was first observed by Scharlemann [88] that many of the useful topological consequences of Gabai's theory can be extracted solely from the hierarchy, without referring to the foliation. The hierarchies

of Gabai and Scharlemann naturally occur in the setting of 3-manifolds with the following extra structure.

A *sutured manifold* is a compact oriented 3-manifold M with its boundary decomposed into two subsurfaces R_- and R_+ . A transverse orientation is imposed on these subsurfaces, so that R_+ points out of M and R_- points into it. These surfaces intersect along a collection of simple closed curves γ , called *sutures*. It is usual to denote a sutured manifold by (M, γ) , but the surfaces R_+ and R_- are also an essential part of the structure. A sutured manifold (M, γ) is *taut* if M is irreducible, R_+ and R_- are incompressible and they minimise χ_- in their homology classes in $H_2(M, \gamma)$.

When a sutured manifold (M, γ) is decomposed along a properly embedded, transversely oriented surface S that intersects γ transversely, the resulting 3-manifold M' inherits a sutured manifold structure. This decomposition is *taut* if M and M' are both taut as sutured manifolds and S has minimal χ_- in its homology class in $H_2(M, \partial S)$. A sequence of such decompositions terminating in a collection of 3-balls is a *sutured manifold hierarchy*. A key feature of sutured manifold theory is: *If a sequence of sutured manifold decompositions ends in a collection of taut sutured 3-balls, then (provided some simple conditions hold) every decomposition in this sequence is taut.*

This sequence of decompositions can therefore be viewed as a certificate for knot genus. However, it is not at all clear that one can produce such a certificate that can be verified in polynomial time. The principal difficulty is the one explained in Section 2.6 in reference to Haken's use of hierarchies. It seems, at first sight, that one needs to keep track of a triangulation of each manifold in this hierarchy, and the complexity of these triangulations seems to grow rapidly as one proceeds along the hierarchy. Agol's alternative approach is to make the entire sutured manifold hierarchy normal (in some suitable sense) with the respect to the initial triangulation of M . The idea is that the sutured manifold hierarchy is closely related to a taut foliation and so normalising such a hierarchy is somewhat analogous to normalising a taut foliation. Techniques exist for placing taut foliations in 'normal' form, due to Brittenham [8] and Gabai [22], and versions of these are used by Agol. Unfortunately, full details of this proof have not yet been written down. An alternative approach may be to use the techniques of the author in [50], where sutured manifold hierarchies were normalised in some weak sense.²

3.6. KHOVANOV HOMOLOGY

In the mid 1980s, knot theory underwent a dramatic revolution, with the introduction of the Jones polynomial [37], and then Witten's interpretation of this using Chern-Simons theory [105]. Right from the outset of this work, it was asked: *Does the Jones polynomial detect the unknot?* In other words, must a non-trivial knot necessarily have non-trivial Jones polynomial? This question remains unanswered. Indeed, it may be viewed as a central open problem in knot theory, but it is not 'elementary' and so does not make it onto our problem list.

It was shown recently by Kronheimer and Mrowka [47] that a more refined invariant, Khovanov homology, does detect the unknot. This followed from a slightly indirect relationship between

² This approach has been successfully completed by the author [56], giving the first full unconditional proof that unknot recognition is in co-NP.

Khovanov homology and 4-dimensional gauge-theoretic invariants.

It seems unlikely that this result will lead to an efficient way of recognising the unknot. This is because determining Khovanov homology is a computationally intensive process. In fact, the less refined Jones polynomial is, in one sense, provably hard to calculate. For example, it was shown by Jaeger, Vertigana and Welsh [35] that computation of the Jones polynomial of a link is ‘ \sharp P-hard’. (This is somewhat similar to being NP-hard.)

3.7. HEEGAARD FLOER HOMOLOGY

Another major advance in knot theory has been the introduction of Heegaard Floer homology by Ozsváth and Szabó [77]. This was based on earlier instanton invariants, as pioneered by Donaldson [16] in dimension 4, and then Floer [19] in dimension 3, and monopole invariants, as developed by Seiberg and Witten [106] in dimension 4 and then Kronheimer and Mrowka [46] in dimension 3. It was clear from the outset that these gauge-theoretic invariants are particularly powerful, but they were undoubtedly hard to calculate. One of the most attractive features of Heegaard Floer homology is that, right from its inception, it has been possible to calculate it in many important examples. Indeed, there are now algorithms to calculate the various different versions of Heegaard Floer homology, due to Sarkar and Wang [86] and Manolescu, Ozsváth and Sarkar [64]. One of the earliest results about Heegaard Floer homology was that it detects the unknot [78]. So, coupled with the fact that it is computable, this provides another unknot recognition algorithm. Again, however, it seems unlikely that this algorithm is in any way efficient.

3.8. RECTANGULAR DIAGRAMS AND ARC PRESENTATIONS

In 2003, Dynnikov introduced a new and striking solution to the unknot recognition problem [17]. This used rectangular diagrams and arc presentations. A *rectangular diagram* (also called *grid diagram*) is a planar link diagram that consists of vertical and horizontal arcs, no two of which are colinear, and with the key property that whenever a horizontal and a vertical arc cross, it is the vertical arc that is the over-arc at the resulting crossing. The number of vertical arcs is equal to the number of horizontal arcs; this is known as the *arc index* of the rectangular diagram.

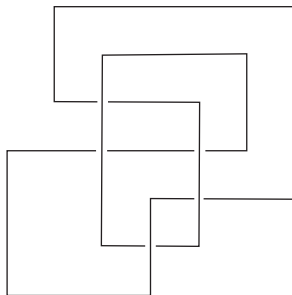


Figure 5: A rectangular diagram

Rectangular diagrams were studied in detail by Cromwell [12]. He introduced a set of moves which change a rectangular diagram, without changing the link type, much like Reidemeister moves.

He showed that any two rectangular diagrams for a link differ by a sequence of such moves. (See Figure 6.) Two of these types of moves (*cyclic permutations* and *exchange moves*) leave the arc index unchanged. The other type of move is a *stabilisation/destabilisation* which increases/decreases the arc index by one.

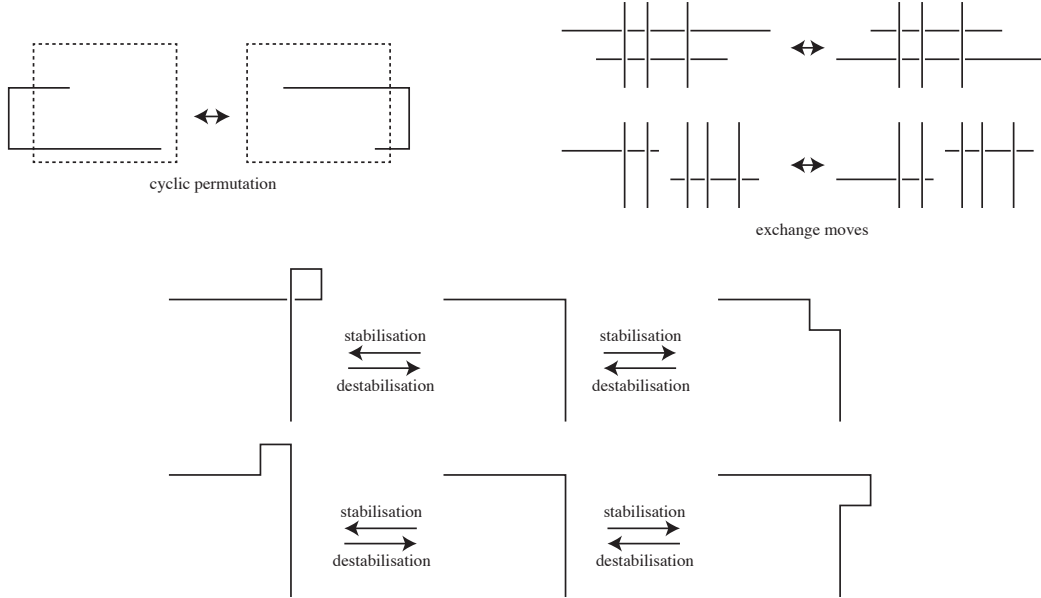


Figure 6: Modifications to a rectangular diagram

Dynnikov proved that any rectangular diagram of the unknot can be reduced to the trivial diagram using these moves, but crucially, *no stabilisations are required*. This leads to the following unknot recognition algorithm. Start with the given diagram, and using an isotopy of the plane, make it rectangular. Then keep applying all possible cyclic permutations, exchange moves and destabilisations. Only finitely many moves are possible at each stage. The key point is that the number of rectangular diagrams with bounded arc index is finite. So, eventually, one can explore the entire set of rectangular diagrams that are reachable from the initial one by sequences of these moves. The given knot is the unknot if and only if the trivial diagram is in this collection.

This does not immediately lead to an efficient solution. This is because the number of rectangular diagrams with a given arc index is a super-exponential function, and so the search space might be quite large. But it is a beautiful and striking result that is perhaps the closest in spirit to elementary knot theory among all the known unknot recognition algorithms. It also has the following consequence, as noted by Dynnikov himself: *If a diagram of the unknot has n crossings, then there is a sequence of Reidemeister moves taking it to the trivial diagram, where each diagram in this sequence has crossing number at most $2(n+1)^2$* . This is simply because a rectangular diagram with arc index m , say, has crossing number at most $(m-1)^2/2$.

Dynnikov's proof builds on previous work of Cromwell [12], Birman, Menasco [7] and Bennequin [4]. Cromwell observed a correspondence between rectangular diagrams and arc presentations, which are defined as follows. One fixes a round unknot in the 3-sphere, which is a reference circle, known

as the *binding circle*. The complement of the binding circle is an open solid torus, which is foliated by open discs called *pages*. A link is in an *arc presentation* if it intersects the binding circle in finitely many points called *vertices*, and it intersects each page in either the empty set or a single open arc joining distinct vertices. Cromwell proved that there is a one-one correspondence between arc presentations and rectangular diagrams up to cyclic permutation.

Thus, Dynnikov's proof mainly concentrates on arc presentations. Given an arc presentation of the unknot, its spanning disc can be arranged so that its intersection with the pages induces a singular foliation on the spanning disc. The singularities occur when the disc slices through the binding circle, and also at Morse-type singularities in the complement of the binding circle. Dynnikov's measure of complexity for the disc is the number of singularities of this foliation. Using an argument that relies on the fact that the disc has positive Euler characteristic, Dynnikov showed that this singular foliation must have, at some point, a certain configuration. He then showed that this implies the existence of some exchange moves, cyclic permutations and possibly a destabilisation, after which the arc index has been reduced or the number of singularities of the disc has been reduced. Thus, this process is guaranteed to terminate. In particular, it is possible to simplify the disc without using stabilisations. Once the disc is as simple as possible, so too is the rectangular diagram.

This style of argument was very much based on earlier work of Cromwell, Birman, Menasco and Bennequin. For example, Birman and Menasco studied representations of the unknot as a closed braid, and proved that it may be reduced to the trivial braid (of braid index one) by a sequence of moves, also known as exchange moves and destabilisations. Thus, the braid index does not increase during this sequence of moves. However, this does not immediately lead to an unknot recognition algorithm, because in the case of braids, there are infinitely many possible exchange moves that can be applied at each stage, and the number of braids with a given braid index is infinite. Nevertheless, Birman and Hirsch [6] did use braids to produce an unknot recognition algorithm, by combining these singular foliation techniques with Hass and Lagarias's work on normal surfaces [26].

4. REIDEMEISTER MOVES

Reidemeister proved that any two diagrams of a link differ by a sequence of Reidemeister moves [85]. This result has many applications. For example, it is often used to show that an invariant of link diagrams actually leads to an invariant of knots and links, by showing that the invariant is unchanged under each Reidemeister move. It also leads to many natural and interesting questions, including the following.

PROBLEM 3. *Find good upper and lower bounds on the number of Reidemeister moves required to relate two diagrams of a knot or link.*

In addition to being natural and attractive, this question has algorithmic implications. As explained in Section 2.4, there exists a computable upper bound on Reidemeister moves if and only if there exists a solution to the equivalence problem. Moreover, if one just considers diagrams of a fixed link type, then the existence of a computable upper bound on Reidemeister moves is equivalent to the existence of a solution to the recognition problem for that link type. Obviously, one would

like upper bounds that are as small as possible, because this leads to more efficient algorithms.

We now give a summary of what is known here. The gap between the known upper and lower bounds are quite startling.

4.1. LOWER BOUNDS

There is, of course, a linear lower bound on the number of Reidemeister moves required to relate two diagrams of a link. More precisely, if two diagrams have n_1 and n_2 crossings, then the number of moves relating them is at least $|n_1 - n_2|/2$, simply because each Reidemeister move changes the crossing number by at most 2.

Hass and Nowik [28] proved that, in general, at least quadratically many moves may be needed. For each link type, they produced a sequence of diagrams D_n , where the crossing number of D_n grows linearly with n , but where the minimal number of Reidemeister moves relating D_n to D_0 is a quadratic function of n . They did this by finding an invariant of knot diagrams that does not lead to a knot invariant, but that changes in a controlled way when a Reidemeister move is performed. Hence, a ‘large’ difference in the invariants of two diagrams implies that they necessarily differ by a long sequence of Reidemeister moves.

4.2. UPPER BOUNDS FOR GENERAL KNOTS AND LINKS

In contrast to these lower bounds, the known upper bounds on Reidemeister moves are vast. The best known bound in general is due to Coward and the author [9]: *If two diagrams of a link have n and n' crossings respectively, then these diagrams differ by a sequence of at most*

$$2^{2^{\dots 2^{(n+n')}}} \} \text{height } c^{(n+n')}$$

Reidemeister moves, where $c = 10^{1000000}$. This is huge, but it was the first known upper bound that is primitive recursive.

The proof relied on Mijatovic’s upper bound on Pachner moves between triangulations of a link exterior, as described in Section 2.5, which in turn built on Haken’s work. Mijatovic’s upper bound was a tower of exponentials, much like the formula above. However, it is not at all clear how one can go from a bound on Pachner moves to a bound on Reidemeister moves. The idea is to use the sequence of Pachner moves to build an explicit homeomorphism from the 3-sphere to itself taking the link arising from the first diagram to the link arising from the second diagram. This homeomorphism is isotopic to the identity (via the ‘Alexander trick’), and so one obtains a one-parameter family of links interpolating between the two given ones. Projecting these links gives a one-parameter family of diagrams. The control that we have over the homeomorphism of the 3-sphere can be used to control the complexity of these link projections, and so one obtains an upper bound on the crossing number of all the intermediate diagrams. This in turn can be used to bound the number of Reidemeister moves (as explained in more detail in Section 4.5).

4.3. UPPER BOUNDS FOR THE UNKNOT

A recent theorem of the author [55] provides a polynomial upper bound on Reidemeister moves for the unknot: *If a diagram of the unknot has n crossings, then there is a sequence of at most $(236n)^{11}$ Reidemeister moves taking it to the trivial diagram.*

Previously, the best known upper bound was an exponential function of n . This was due to Hass and Lagarias [26]. Their method of proof relied heavily on Haken's solution of the recognition problem for the unknot. One is given a diagram of the unknot with n crossings, and from this one builds a triangulation of the unknot's exterior. It is not hard to arrange that each tetrahedron of this triangulation is straight in \mathbb{R}^3 (apart from the tetrahedron enclosing the point at infinity) and that the number of tetrahedra is bounded above by a linear function of n . Now a spanning disc for the unknot can be placed in normal form, and can in fact be realised as a vertex surface. Crucially, there is an upper bound on the number of triangles and squares in a vertex normal surface, which is an exponential function of the number of tetrahedra in the triangulation. One then simply slides the unknot along this disc. Each time one slides across a triangle or square, this induces a controlled number of Reidemeister moves in the projection to the plane of the diagram.

So, it is the exponential upper bound on the size of the normal spanning disc that leads to the Hass-Lagarias upper bound on the number of Reidemeister moves. It is possible to prove that the exponential upper bound on the size of the spanning disc cannot be improved upon. Hass, Snoeyink and Thurston [29] found a sequence of unknot diagrams D_n , where the crossing number is a linear function of n , but where any piecewise-linear spanning disc for the unknot must consist of at least exponentially many triangles.

To prove that there is a polynomial upper bound on Reidemeister moves, the author had to use both normal surfaces and arc presentations. As in Dynnikov's work (which is described in Section 3.8), one starts with an arc presentation of the unknot, and the aim is to apply exchange moves, cyclic permutations and destabilisations to reduce it to the trivial arc presentation. Dynnikov examined a spanning disc and its induced singular foliation, and showed that there is always a move that can be performed which results in a reduction in the number of singularities in the foliation or a reduction in the arc index. Using normal surface theory, it is not hard to show that initially, the disc can be arranged to have at most exponentially many singularities. So, although this gives an upper bound on the number of moves, this bound is exponential. To obtain an improved estimate, one needs to go deeper into normal surface theory. It is possible to speak of two bits of normal surface as being *normally parallel*. The aim is to find many parallel pieces of the disc which can be moved all at the same time. In this way, one obtains, using a controlled number of exchange moves cyclic permutations and destabilisations, a substantial decrease in the complexity of the disc. This decrease is large enough that only polynomially many moves are required before the disc has been completely simplified, which implies that the arc presentation has become trivial.

4.4. AN UPPER BOUND FOR EACH LINK TYPE

The polynomial upper bound on Reidemeister moves described above has recently been generalised by the author, as follows [58]. *For each link type K , there is a polynomial p_K such that any*

two diagrams of K with crossing number n and n' differ by a sequence of at most $p_K(n) + p_K(n')$ Reidemeister moves. As a consequence, for each link type K , the K -recognition problem is in NP. The certificate is simply a sequence of Reidemeister moves taking the given diagram of K to some fixed diagram.

The proof uses and extends the techniques that were developed in the case of the unknot. In particular, both normal surfaces and arc presentations play a central role. In the case of the unknot, the spanning disc was simplified at each stage of the procedure. For a general knot K , one uses instead an entire hierarchy for the exterior of K . The spanning disc for the unknot could be simplified until its singular foliation contained just two singularities. This is not possible for the surfaces in a hierarchy. Instead, the goal is to simplify these surfaces until their number of singularities is bounded by a polynomial function of the initial arc index. Once this has been achieved, one considers a regular neighbourhood of this hierarchy together with a regular neighbourhood of the knot. This is a ball B , and the complexity of B is, in some suitable sense, controlled. The ball inherits a handle structure by thickening the cell structure of the hierarchy, and K lies inside this handle structure in a way that depends only on the topology of the hierarchy. The proof proceeds by isotoping K through B until it lies within a single 0-handle and projects to some fixed diagram for K . Since B has controlled complexity, so too does the projection of K throughout this isotopy. Thus, one obtains a bound on the crossing number of the intermediate diagrams, and with a bit more work, a polynomial bound on the number of Reidemeister moves used in this process.

4.5. BOUNDS ON THE CROSSING NUMBER OF INTERMEDIATE DIAGRAMS

In addition to bounding the number of Reidemeister moves required to pass between two link diagrams, one can also seek to bound the complexity of the diagrams in this sequence.

PROBLEM 4. *Given two diagrams of a link, find a sequence of Reidemeister moves relating them where all diagrams in the sequence have small crossing number.*

This is, of course, related to Problem 3. Firstly, an upper bound on the number of moves immediately gives an upper bound on the intermediate crossing numbers. This is because each Reidemeister move changes the crossing number by at most 2. Secondly, if one has an upper bound on the crossing number of each intermediate diagram, then one obtains an upper bound on the number of moves. This is because there are at most $(24)^{n+1}$ connected diagrams with crossing number at most n (see Theorem 6.5 in [9] for example). Moreover, in a shortest sequence of Reidemeister moves joining two diagrams, one never visits the same diagram twice. So, if each diagram in the sequence has crossing number at most n , say, then the length of the sequence is at most $(24)^{n+1}$.

However, one can sometimes establish a much better bound than this. As explained in Section 3.8, Dynnikov found, in the case of the unknot, a quadratic upper bound on the crossing number of intermediate diagrams.

5. CROSSING NUMBER

The *crossing number* $c(K)$ of a knot or link K is defined to be the minimal number of crossings in any diagram for K .

5.1. COMPUTING THE CROSSING NUMBER

Crossing number is algorithmically computable. Indeed, this fact is closely related to the solubility of the equivalence problem for links. For example, the problem of determining whether a knot's crossing number is zero is clearly equivalent to determining whether it is the unknot. One can use a solution to the equivalence problem to determine a knot's crossing number in the following naive way. Starting with a knot diagram having n crossings, enumerate all diagrams with less than n crossings, and for each, determine whether it is the same knot as the original one. The one with the smallest number of crossings clearly realises the knot's crossing number. This approach is obviously not very efficient but it seems unlikely that there is any quicker way of determining a link's crossing number in general. In fact, this method has been successfully used in practice to compile knot tables, by Hoste, Thistlethwaite and Weeks for example [32].

5.2. DETERMINING THE CROSSING NUMBER FOR CERTAIN LINK CLASSES

Although there seems little hope of anything more than using a brute-force search to determine the crossing number of an arbitrary link, one can often determine the crossing number more efficiently in various interesting cases. For example, the crossing numbers of alternating links are completely understood, by work of Kauffman [40], Murasugi [72] and Thistlethwaite [97]. Their work, and a broader analysis of alternating links, is discussed in Section 7.1.

Torus links form another interesting collection, although they are of course more restricted. Murasugi [73] proved that the crossing number of the (p, q) torus link is $pq - \max\{p, q\}$.

5.3. THE NUMBER OF KNOT TYPES

Although crossing number is computable for any given knot, its behaviour is not well understood for infinite collections of knots. In particular, the following problem remains unresolved.

PROBLEM 5. *If $N(c)$ is the number of knot types with crossing number c , determine the asymptotic behaviour of $N(c)$ as c tends to infinity.*

There is a more-or-less obvious exponential upper bound to $N(c)$. As explained in Section 4.5, the number of connected diagrams with crossing number c is at most 24^{c+1} , and so $N(c) \leq 24^{c+1}$.

One can also obtain an exponential lower bound on $N(c)$, by considering alternating knots. As described in the previous subsection, the crossing numbers of alternating knots are completely understood: a 'reduced' alternating diagram has minimal crossing number. (For an explanation of this terminology, see Section 7.1.) Moreover, the resolution of the Tait Flyping conjecture by Menasco and Thistlethwaite [69] implies that one can determine exactly when two alternating diagrams represent the same knot type. So, enumerating the number of alternating knot types with crossing

number c is nearly the same as enumerating the number of alternating diagrams. Very accurate asymptotics were established by Sundberg and Thistlethwaite [95] who showed that, if $N_{\text{alt}}(c)$ is the number of prime alternating link types with crossing number c , then

$$\lim_{c \rightarrow \infty} (N_{\text{alt}}(c))^{1/c} = \frac{101 + \sqrt{21001}}{40}.$$

Using similar methods, Thistlethwaite [98] was also able to show that alternating links are increasingly scarce among all link types, as the crossing number tends to infinity. In particular, $N_{\text{alt}}(c)/N(c) \rightarrow 0$ as $c \rightarrow \infty$. But more precise asymptotics for $N(c)$ seem a long way from resolution.

5.4. THE CROSSING NUMBER OF COMPOSITE KNOTS

One of the most basic operations in knot theory is connected sum. Here, one starts with two oriented knots K_1 and K_2 in the 3-sphere. For each knot, one finds a 3-ball that intersects the knot in a properly embedded unknotted arc. One removes the interior of this ball. The result is, for each knot, a knotted arc in the complementary 3-ball. If one glues these two balls together, the result is the 3-sphere. One performs this gluing so that the two arcs join together to form a knot, and so that the orientations on the arcs patch together correctly. The resulting knot is the *connected sum* of K_1 and K_2 , denoted $K_1 \# K_2$.

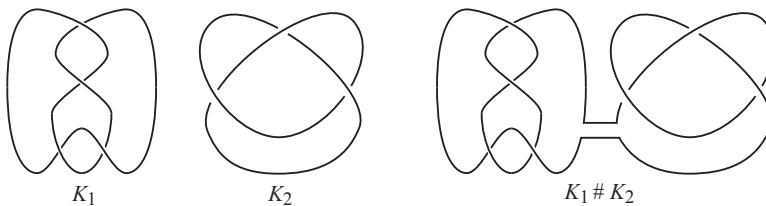


Figure 7: Connected sum

As is evident from Figure 7, one can create a diagram D for $K_1 \# K_2$ starting from diagrams of K_1 and K_2 . If we use diagrams of K_1 and K_2 with minimal crossing number, we obtain a diagram D with crossing number $c(D)$ equal to $c(K_1) + c(K_2)$. Hence, $c(K_1 \# K_2) \leq c(K_1) + c(K_2)$. It seems ‘obvious’ that this should be an equality, but actually, it is far from clear, and this is in fact one of the most notorious unsolved problems in elementary knot theory.

PROBLEM 6. *Must $c(K_1 \# K_2) = c(K_1) + c(K_2)$?*

This has been verified for certain knot classes, for example when K_1 and K_2 are both alternating [40, 72, 97] or when K_1 and K_2 are both torus knots [15]. The only general result in this direction is due to the author [52]: *For oriented knots K_1 and K_2 ,*

$$\frac{c(K_1) + c(K_2)}{152} \leq c(K_1 \# K_2) \leq c(K_1) + c(K_2).$$

We will give an overview of the proof, because it uses some unexpected methods which may be applicable to other problems in elementary knot theory.

There is a simple operation on knots and links that is closely related to connected sum. This is *disjoint union*. Here one starts with two links L_1 and L_2 , each lying in the 3-sphere. One encloses each link in the interior of a ball, and then one identifies the boundaries of these balls, thereby forming a new link in the 3-sphere, which is denoted $L_1 \sqcup L_2$.

Now, crossing number *does* behave well with respect to disjoint union. It is easy to prove that $c(L_1 \sqcup L_2) = c(L_1) + c(L_2)$. Again we have the upper bound $c(L_1 \sqcup L_2) \leq c(L_1) + c(L_2)$, because one can start with diagrams for L_1 and L_2 and use these to create a diagram for $L_1 \sqcup L_2$. To prove the inequality in the other direction, one must start with a diagram of $L_1 \sqcup L_2$ (which we may take to have minimal crossing number) and use this to create diagrams for L_1 and L_2 . But this is straightforward: to form the diagram for L_1 , say, one simply throws away the parts of the link projection that come from L_2 . Thus, one obtains diagrams for L_1 and L_2 , with the property that the sum of the number of crossings in these diagrams is at most the crossing number of the original diagram of $L_1 \sqcup L_2$. Thus, $c(L_1) + c(L_2) \leq c(L_1 \sqcup L_2)$.

One can immediately see from this the main difficulty in proving that $c(K_1) + c(K_2) \leq c(K_1 \# K_2)$. If one starts with a diagram D for $K_1 \# K_2$ with minimal crossing number, then there is no obvious way of forming diagrams for K_1 and K_2 from this. Although one might view a part of the curve $K_1 \# K_2$ as coming from K_1 , say, this part is only an arc and so one needs to add in something extra to form a closed curve representing K_1 . This may dramatically increase the number of crossings. This is evident when one considers the 2-sphere that is used to form the connected sum. If one performs an isotopy on $K_1 \# K_2$ so that it projects to give the diagram D , the 2-sphere may be extremely distorted in 3-space. It is precisely an arc on this sphere that needs to be inserted to form a copy of K_1 . The proof of the lower bound on $c(K_1 \# K_2)$ proceeds by controlling the complexity of this 2-sphere. In fact, one cuts $K_1 \# K_2$ at its two points of intersection with the sphere, and then one adds two arcs, one on either side of this sphere, to form a copy of $K_1 \sqcup K_2$. By controlling the sphere, one can control the number of new crossings this introduces. One thereby finds a diagram for $K_1 \sqcup K_2$ with crossing number at most $152c(D)$, which proves the theorem.

The way that the 2-sphere is controlled is via normal surface theory. One starts with a diagram D for $K_1 \# K_2$ with minimal crossing number. From this one can build a triangulation of the exterior M of $K_1 \# K_2$ in a reasonably natural way. The restriction of the 2-sphere to M is a properly embedded annulus A . Since A is essential, it can be isotoped into normal form. Hence, A intersects each tetrahedron in a collection of triangles and squares, as shown in Figure 2. Within a tetrahedron, there are four types of triangle and three types of square. Between adjacent triangles or squares of the same type, there lies a product region. The union of these product regions forms an I -bundle \mathcal{B} which is embedded in the exterior of A . This is known as the *parallelity bundle* for the exterior of A . The ∂I -bundle is the *horizontal boundary* of \mathcal{B} , denoted $\partial_h \mathcal{B}$. Note that $\partial_h \mathcal{B}$ lies in the two copies of A in the exterior of A . The key part of the argument establishes that one can find properly embedded arcs in these two copies of A , each joining the two boundary components of the annulus, and which avoid $\partial_h \mathcal{B}$. The significance of this is that these arcs then can be made to intersect each tetrahedron of the triangulation in a restricted way. To see this, note that in each tetrahedron, all but at most 6 pieces in the exterior of a normal surface lie in \mathcal{B} . Since these arcs intersect each tetrahedron in a controlled way, one can bound from above the crossing number of their diagrammatic projection. Thus, we obtain a diagram for $K_1 \# K_2$, and with some work, one can

show that it has at most $152c(D)$ crossings.

5.5. THE CROSSING NUMBER OF SATELLITE KNOTS

A natural generalisation of connected sum is the satellite knot construction. A knot K is said to be a *satellite* of a non-trivial knot L if K lies in a regular neighbourhood $N(L)$ of L , but it does not lie within a 3-ball inside $N(L)$ and is not a core curve of $N(L)$. The knot L is called the *companion*.

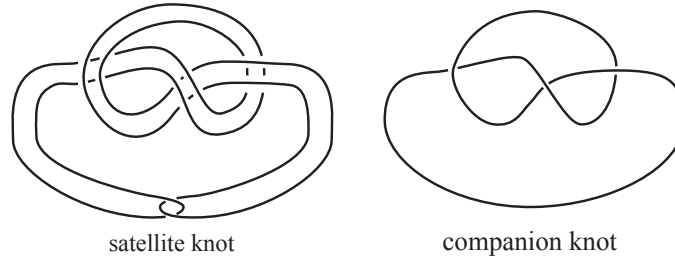


Figure 8: A satellite knot

The *wrapping number* of the satellite is the minimal number of intersections between K and a meridian disc for $N(L)$. It is always positive. Note that when K_1 and K_2 are non-trivial knots, then $K_1 \# K_2$ is a satellite of K_1 (and of K_2) with wrapping number 1.

The following remains famously unsolved.

PROBLEM 7. *If K is a satellite of L , must $c(K)$ be at least $c(L)$? More generally, must $c(K)$ be at least $w^2c(L)$, where w is the wrapping number of the satellite?*

The motivation here is that there is a way of building a diagram for the satellite K starting from a diagram for L . Each crossing of L becomes w^2 crossings for K . There may be other crossings, and so unlike in Problem 6, this is a conjectured inequality rather than equality. One might be tempted to speculate that there is an equality relating $c(K)$ to the crossing number of L , the wrapping number and the crossing number of the knot in $N(L)$, once this has been suitably defined. We do not do so here, although it is possible that something along these lines may be true.

Unsurprisingly, the methods used to analyse the crossing number of composite knots extend to the case of satellite knots, giving the following result of the author [54]: *If K is a satellite of a knot L , then $c(K) \geq 10^{-13}c(L)$.*

In the proof, one starts with a diagram D of K with minimal crossing number, and the goal is to create a diagram of L with crossing number at most $10^{13}c(D)$. One uses D to construct a handle structure for the exterior of K . The satellite torus T can be placed in ‘normal’ form with respect to this handle structure. Cutting along T gives two pieces, one a copy of the exterior of L , the other $N(L) - \text{int}(N(K))$. Each piece inherits a handle structure. We then modify this handle structure, primarily by replacing parts of the parallelity bundle which are I -bundles over discs by 2-handles. These 2-handles may end up lying in 3-space in a complicated way, but the remainder of the handle structure is controlled. We then re-attach $N(K)$ to $N(L) - \text{int}(N(K))$ to form a handle

structure of the solid torus $N(L)$. A theorem of the author [53] gives that in any handle structure of the solid torus, there is a core curve which lies entirely within the 0-handles and 1-handles, and which intersects each such handle in a restricted way. In this case, the core curve is a copy of L , which then lies inside 3-space in a controlled way. Projecting this gives the required diagram of L .

6. CROSSING CHANGES AND UNKNOTTING NUMBER

An elementary operation that one can perform on a knot is to change a crossing in one of its diagrams, as in Figure 9. This is strikingly difficult to analyse, and it leads to some of the most challenging questions in the subject. An excellent survey of this topic can also be found in [90].

6.1. COMPUTING UNKNOTTING NUMBER

The *unknotting number* $u(K)$ of a knot K is the minimal number of crossing changes that one can make to some diagram that turns it into the unknot. The difficulty of course is that one quantifies over all possible diagrams, and so although any diagram for K easily gives an upper bound on $u(K)$, it is rarely clear that this is the best possible. In fact, unknotting number is probably best understood in a manner that avoids diagrams: it is the minimal number of times that a knot must pass through itself in a one-parameter family of curves that starts with K and ends with the unknot.

PROBLEM 8. *Is the unknotting number of a knot algorithmically computable? Is there even an algorithm to decide whether a given knot has unknotting number one?*

The first of these questions seems a long way out reach at present. It is more than just a theoretical question: there exist many explicit knots with unknown unknotting number (see [41] for example). As a result, many intriguing and clever methods have been developed that provide lower bounds on unknotting number [38, 39, 42, 60, 74, 76, 79, 94, 96]. These utilise a wide variety of techniques, including the Alexander module [74], the linking form on the double branched cover [60, 94], 4-dimensional gauge theory [96] and more recently, Heegaard Floer homology [76, 79]. A major piece of research in this area was the solution by Kronheimer and Mrowka [44] of Milnor's conjecture: *the unknotting number of the (p, q) torus knot is $(p - 1)(q - 1)/2$.*

However, with each new lower bound on unknotting number, there remain knots with unknotting number that cannot be determined. This will continue to be the case until Problem 8 is resolved.

A useful way of understanding and approaching this problem is via the use of crossing circles. Given some crossing in a diagram of a knot K , the associated *crossing circle* is a simple closed curve in the complement of K that encircles the crossing, as shown in Figure 9. There are two crossing circles associated with each crossing (that are obtained from one another by rotating the diagram through 90 degrees). One usually focuses on the crossing circle that has zero linking number with K . This bounds a disc which intersects K in two points of opposite sign.

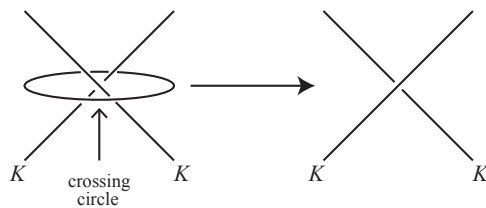


Figure 9: A crossing change

Crossing circles are useful for several reasons. Firstly, the crossing change is achieved by ± 1 surgery along the crossing circle, and so one can draw on the well-developed theory of Dehn surgery. Secondly, it also permits crossings in different diagrams to be compared: the crossings are *equivalent* if their associated crossing circles are ambient isotopic in the complement of K . Clearly, changing equivalent crossings results in equivalent knots, and so this is a helpful concept. Some interesting problems can be phrased in terms of crossing circles.

Although it seems unlikely that we will have an algorithm to determine unknotting number in the near future, the question of whether unknotting number one knots can be detected is a tantalising one. Here, the situation seems considerably more promising. To answer this question, it would suffice to produce a finite list of potential crossing circles with the property that if K has unknotting number 1, then changing the crossing at one of these crossing circles would unknot the knot. For one could then check each of these crossing changes systematically to determine whether they did unknot the knot. (This would use one of the known solutions to the unknot recognition problem.) Thus, the following arises naturally.

PROBLEM 9. *If a knot has unknotting number one, are there only finitely many crossing circles, up to ambient isotopy, that can be used to unknot the knot?*

Much of what is known about knots with unknotting number one comes from sutured manifold theory. (See Section 3.5 for a very brief summary of some of this theory.) It is possible to show that if a knot K has unknotting number 1, then its exterior admits a taut sutured manifold hierarchy, where the penultimate manifold in this decomposition is a solid torus neighbourhood of the crossing circle plus possibly some 3-balls. In particular, this crossing circle is disjoint from some minimal genus Seifert surface S for K . In fact, by pushing this argument further, Scharlemann and Thompson [89] were able to show that this crossing circle can be isotoped to lie in S . This suggests a route to determining whether a knot has unknotting number 1: find all minimal genus Seifert surfaces for K , and then search on each such surface for all possible crossing circles which unknot the knot. The first of these steps is possible, using normal surface theory, as long as K is not a satellite knot. However, there are infinitely many isotopy classes of curves on a surface with positive genus, and it is hard to see how to reduce this to a finite list of crossing circles on which to focus.

6.2. THE UNLINKING AND SPLITTING NUMBERS OF A LINK

There is a natural analogue of unknotting number for links. The *unlinking number* $u(L)$ of a link L is the minimal number of crossing changes required to turn it into the unlink. Many of the techniques that can be used to analyse unknotting number apply equally well to unlinking number. But they also often work in the following more general setting. The *splitting number* $s(L)$ of a link

L is the minimal number of crossing changes required to turn it into a split link.

Unlike the case of unknotting number one, links with splitting number one or unlinking number one are known to be detectable, under fairly mild hypotheses. It is a recent theorem of the author [57] that *there is an algorithm to determine whether a hyperbolic link of at least 3 components has splitting number one.*

This is proved as follows. Suppose that C is a crossing circle, such that changing this crossing turns a link L into a split link. Just as in Section 6.1, there is a sutured manifold hierarchy for the exterior of L where the penultimate manifold is a solid torus neighbourhood of C , plus some balls. The key part of the proof is to place this hierarchy into ‘normal’ form with respect to some initial triangulation of the exterior of L as in [50]. Thus, one obtains a finite computable list of possibilities for C . In particular, the version of Problem 9 in this setting is answered affirmatively. Moreover, each of these crossing changes can be analysed to determine whether they result in the unlink. So, under the above hypotheses, *there is also an algorithm to determine whether the link has unlinking number one.*

6.3. ADDITIVITY OF UNKNOTTING NUMBER

One can also ask how unknotting number behaves with respect to simple knot-theoretic operations. For example, the following is very far from resolution.

PROBLEM 10. *Does $u(K_1 \# K_2) = u(K_1) + u(K_2)$ always hold? Less ambitiously, what about the inequality $u(K_1 \# K_2) \geq \max\{u(K_1), u(K_2)\}$?*

Scharlemann [87] proved that composite knots have unknotting number at least 2. But we cannot rule out the following bizarre possibility: two knots might each have unknotting number 100 but their connected sum has unknotting number 2!

6.4. COSMETIC CROSSING CHANGES

A crossing circle is *nugatory* if it bounds a disc in the complement of the knot. Clearly changing a nugatory crossing does not alter the knot. Is this the only way?

PROBLEM 11. *Suppose that a crossing change to a knot K does not alter its knot type. Let C be the associated crossing circle with zero linking number. Must C be nugatory?*

This is known when the knot K is the unknot, by work of Scharlemann and Thompson [89], which relied on Gabai’s sutured manifold theory [21]. However, the general case seems to remain just out of reach.

7. SPECIAL CLASSES OF KNOTS

Knots are usually specified by means of a diagram. Typically, this diagram reveals little immediate information about the knot. For example, a superficially complicated diagram may actually represent the unknot. However, there are certain types of diagram which force a knot to have much more structure. It is these that we discuss in this section.

7.1. ALTERNATING KNOTS

A diagram for a link is *alternating* if, as one travels along each component of the link, one meets ‘over’ and ‘under’ crossings alternately. A link is *alternating* if it has an alternating diagram.

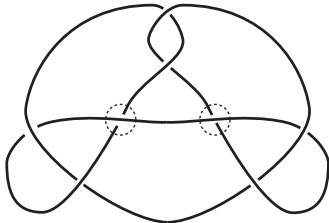


Figure 10: A non-alternating diagram (highlighted are successive ‘over’ crossings)

Some alternating diagrams can be immediately simplified as follows. We say that a diagram is *reducible* if, at some crossing, two of the incident regions are actually the same region of the diagram. When a diagram is reducible, there is an obvious modification that can be made to it, which reduces the crossing number by 1. When the original diagram is alternating, so too is the new diagram. So, when dealing with alternating links, there is no loss of generality in assuming that its alternating diagram is not reducible. It is then known as *reduced*.

A great deal of information about a link is evident from an alternating diagram. Possibly the most striking illustration of this is the following result of Kauffman [40], Murasugi [72], and Thistlethwaite [97]: *If D is a reduced alternating diagram of a link L , then $c(D) = c(L)$.*

This was proved by means of the Jones polynomial. This link invariant is defined in terms of some diagram. It is therefore unsurprising that the ‘complexity’ of the polynomial should be bounded above by the ‘complexity’ of some diagram. In fact, the *width* of the polynomial, which is the difference in degree between the highest order term and lowest order term, is always at most the crossing number of any diagram. But, Kauffman, Murasugi and Thistlethwaite proved that this inequality is an equality for a reduced alternating diagram D . This implies the above result, because the width of the Jones polynomial is then the crossing number of D , and the link cannot have a diagram with fewer crossings, because this would force the width of the polynomial to be at most this number.

A consequence of this theorem is that there is a very rapid and simple algorithm to decide whether an alternating diagram represents the unknot: *an alternating diagram of the unknot either has no crossings or is reducible*. In fact, this was already known, due to earlier techniques of Menasco [68], which relied on a very precise analysis of surfaces in the exterior of an alternating link. This works particularly well when the surface has non-negative Euler characteristic. So, by examining spheres in the complement of the link, one can also detect whether an alternating link is split. Menasco showed that *an alternating link is split if and only if its alternating diagram is disconnected*. Similarly, *an alternating knot is prime if and only if a reduced alternating diagram is prime*. Here, a connected diagram is *prime* if and only if any simple closed in the plane which intersects the link projection transversely in two points away from the crossings divides the plane into two components, one of which contains no crossings.

In Menasco's proof, he constructs, using the diagram, a special position for the link in \mathbb{R}^3 . The diagram is viewed as lying in the horizontal plane $\mathbb{R}^2 \times \{0\}$. The link mostly lies in this plane, except near each crossing, where two sub-arcs of the link take detours above and below the plane of the diagram. More precisely, one places a small 3-ball at each crossing with centre lying in the plane of the diagram, and the two arcs involved in the crossing follow the boundary of this ball. (See Figure 11.) Let \mathbb{R}_+^3 denote the set of points lying above the union of the plane and the balls, and define \mathbb{R}_-^3 similarly. Menasco showed that a closed incompressible surface in the link exterior may be arranged so that it intersects each of these balls in a collection of parallel 'saddles', and intersects \mathbb{R}_+^3 and \mathbb{R}_-^3 in a collection of discs. The boundary of these discs in \mathbb{R}_+^3 is a collection of simple closed curves, which can be viewed as lying within the diagram. A careful analysis of these curves (particularly, an innermost one in the plane), together with the condition that the diagram is connected and alternating, implies that the surface S has a *meridional compression disc*. This is a disc D in \mathbb{R}^3 such that $D \cap S = \partial D$ and $D \cap K$ is a single point in the interior of D . The existence of such a disc is impossible for a splitting sphere, for example, and this can be used to establish Menasco's theorem about split alternating links.

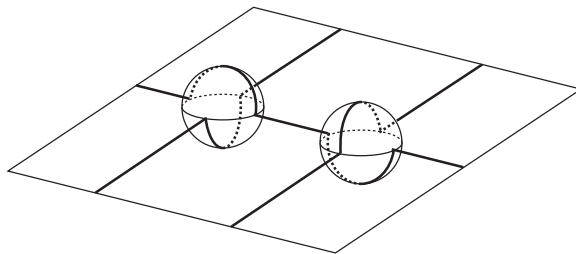


Figure 11: Balls near crossings

A significantly deeper version of this analysis was used to prove the following theorem of Menasco and Thistlethwaite [69]: *any two reduced alternating diagrams of a link differ by a sequence of flypes* (as shown in Figure 12.) This was Tait's flyping conjecture. The proof of this is very complex, and so we can only give a very brief overview. Each alternating diagram determines two checkerboard surfaces. In Menasco and Thistlethwaite's proof, they examine the surfaces arising from one diagram and how they interact with other diagram. A careful analysis, together with a surprising intervention of the Jones polynomial at one point, establishes the existence of a flype which, in some sense, simplifies these surfaces, and so the theorem is proved by induction.

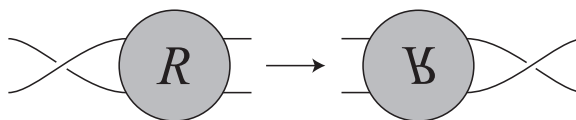


Figure 12: Flype

Although alternating links are now very well-understood, there are some invariants and properties of these links that are worthy of further analysis. Some of these are given in the following problem.

PROBLEM 12. Can one detect (or possibly just estimate) the following invariants of a knot, given an alternating diagram: its bridge number; its tunnel number; its unknotting number?³

One may also ask for estimates for more sophisticated invariants. Indeed, in [51], the author has given a way of reading off the hyperbolic volume of an alternating link's complement, up to a bounded multiplicative error. While such problems are interesting, they fall outside the remit of this survey, which is focused on much more elementary questions.

7.2. POSITIVE KNOTS AND POSITIVE BRAIDS

Given the manifest success in understanding the topological properties of alternating knots, it is natural to consider other possible classes of knot diagrams. A reasonable way of assessing whether it is a good class of diagrams is whether it satisfies the minimal requirement that a diagram specifies the unknot if and only if it 'obviously' does so. Another class of diagrams which satisfy this requirement is positive diagrams.

Positivity is defined for diagrams of oriented links. Positive and negative crossings are shown in Figure 13. A diagram is said to be *positive* if all its crossings are positive.

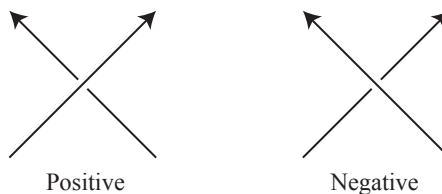


Figure 13: Positive and negative crossings

Cromwell [11] showed that a reduced positive diagram never represents the unknot. In fact, one can read off the genus of a positive knot from its positive diagram. This result was significantly generalised by Kronheimer and Mrowka [44], who showed that the smooth 4-ball genus of a positive knot can be read off from its positive diagram. Indeed, it is equal to the knot's genus.

However, positive knots are not totally well-behaved. For example, Stoimenow [93] found a positive knot which has no positive diagram with minimal crossing number. So, it seems unlikely that one can read off the crossing number of a positive knot in the way that one can for alternating knots. Nevertheless, they form an interesting class that is worthy of further investigation. In particular, it would be instructive to determine other topological and/or elementary invariants of positive knots. For example, is there a simple relationship between two positive diagrams of the same knot?

One can also consider, more specifically, closed positive braids. These appear to have even nicer properties than general positive knots. For example, they are fibred. They may be more amenable to analysis.

³ An algorithm to determine whether an alternating knot has unknotting number one has recently been given by McCoy [67].

7.3. OTHER INTERESTING CLASSES OF DIAGRAM

There are several other related classes of diagrams. *Homogeneous* diagrams form a common generalisation of positive diagrams and alternating diagrams. Cromwell [11] in fact established his formula for knot genus not just for positive knots but homogeneous ones.

Adequate diagrams form another interesting class. These are defined by considering ‘resolutions’ for each crossing, which remove the crossing. Each crossing may be removed in one of two ways, a ‘plus’ or a ‘minus’ resolution. If one resolves every crossing, the result is a collection of planar curves. At each crossing, the two new arcs may belong to the same simple curve or two distinct ones. A diagram is *plus-adequate* if, when all the crossings are resolved in a ‘plus’ way, each resulting simple closed curve runs through the remnants of each crossing at most once. *Minus-adequacy* is defined similarly using the ‘minus’ resolution at each crossing. A diagram is *adequate* if it is both plus-adequate and minus-adequate. Adequate knots have well-behaved Jones polynomial. Lickorish and Thistlethwaite [61] using this to prove that *if a knot has a reduced adequate diagram with at least one crossing, then it is not the unknot*. But adequacy has more topological applications, as shown in the recent monograph of Futer, Kalfagianni and Purcell [20]. Adequate links surely merit further investigation.

8. EPILOGUE

The theory of 3-manifolds has seen remarkable progress over the past few years, with the solution of many key conjectures, for example, the Poincaré conjecture [81, 82, 83] and the Virtually Haken Conjecture [2], to name just two. One might be led to the conclusion that the related field of knot theory might be devoid of interesting unsolved problems. The purpose of the present article is to emphasise that this is not the case. There remain many fundamental and interesting challenges ahead.

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Mathematical Institute, University of Oxford,
 Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, United Kingdom.