LARGE GROUPS, PROPERTY (τ) AND THE HOMOLOGY GROWTH OF SUBGROUPS

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1. Introduction

In this paper, we investigate the homology of finite index subgroups G_i of a given finitely presented group G. We fix a prime p, denote the field of order p by $\mathbb{F}_p$, and define $d_p(G_i)$ to be the dimension of $H_1(G_i; \mathbb{F}_p)$. We will be interested in the situation where $d_p(G_i)$ grows fast as a function of the index $[G : G_i]$. Specifically, we say that a collection of finite index subgroups $\{G_i\}$ has *linear growth of mod p homology* if $\inf_i d_p(G_i)/[G : G_i]$ is positive. This is a natural and interesting condition that arises in several different contexts. For example, the main theorem of [9] states that when G is a lattice in $\mathrm{PSL}(2, \mathbb{C})$ with non-trivial torsion (equivalently, G is the fundamental group of a finite-volume hyperbolic 3-orbifold with non-empty singular locus), then G has such a sequence of subgroups. Another major class of groups G having such a collection of subgroups are those that are *large*. By definition, this means that G has a finite index subgroup that admits a surjective homomorphism onto a free non-abelian group. Large groups have many nice properties, for example super-exponential subgroup growth and infinite virtual first Betti number. One might wonder whether largeness is *equivalent* to the existence of some nested sequence of finite index subgroups $\{G_i\}$ with linear growth of mod p homology for some prime p. If so, this would establish that lattices in $\mathrm{PSL}(2, \mathbb{C})$ with non-trivial torsion are large, which would be a major breakthrough in low-dimensional topology.

We will relate this question to an important group-theoretic concept known as Property (τ). This was first defined by Lubotzky and Zimmer [16]. We recall its definition now. Let G be a finitely generated group, and let $\{G_i\}$ be a collection of finite index subgroups. Let S be a finite generating set for G, and let $X(G/G_i; S)$ be the Schreier coset graph for G/G_i with respect to S. Property (τ) is defined in terms of the geometry of these graphs. Specifically, we will look at subsets A of their vertex set and consider ∂A, which is defined to be the set of edges with one endpoint in A and one not in A (see Figure 1). The *Cheeger constant* $h(X)$ of a finite graph X is defined to be

$$h(X) = \min \left\{ \frac{|\partial A|}{|A|} : A \subset V(X) \text{ and } 0 < |A| \leq |V(X)|/2 \right\},$$

where $V(X)$ is the vertex set of X. Then G is said to have *Property (τ) with respect to $\{G_i\}$* if $\inf_i h(X(G/G_i; S))$ is strictly positive, for some finite generating set S for G.

It turns out that if this holds for some finite generating set then it holds for any finite generating set (see Lemma 2.3 in [7] for example).

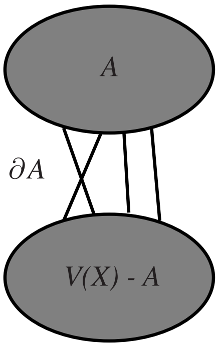

Figure 1

Property (τ) relates to many other areas of mathematics, including graph theory, representation theory and differential geometry. (See [10] for a detailed survey.) An equivalent condition is that $\lambda_1(X(G/G_i; S))$ is bounded away from zero, where λ_1 is the smallest positive eigenvalue of the Laplacian of a finite graph. To show that certain groups G have Property (τ) with respect to certain collections of finite index subgroups $\{G_i\}$ has been a major aim of many mathematicians, for example, Clozel [4] and Bourgain and Gamburd [2]. Such a conclusion has many applications, in areas such as group theory [13] and number theory [3]. The failure of Property (τ) also has some interesting consequences. For example, the Lubotzky-Sarnak conjecture [12] proposes that any lattice in $\mathrm{PSL}(2, \mathbb{C})$ has a sequence of finite index subgroups without Property (τ), and this is part of a programme initiated in [7] to prove the virtually Haken conjecture, which is a major unsolved problem in 3-manifold theory.

The aim of this paper is relate largeness for groups to Property (τ) and linear growth of mod p homology. Our main theorem is as follows.

Theorem 1.1. *Let G be a finitely presented group, let p be a prime and suppose that $G \geq G_1 \triangleright G_2 \triangleright \dots$ is a nested sequence of finite index subgroups, such that each G_{i+1} is normal in G_i and has index a power of p. Suppose that $\{G_i\}$ has linear growth of mod p homology. Then, at least one of the following must hold:*

(i) some G_i admits a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$ and some normal subgroup of G_i, with index a power of p, admits a surjective homomorphism onto a non-abelian free group; in particular, G is large;

(ii) G has Property (τ) with respect to $\{G_i\}$.

The two possible conclusions in this theorem can be viewed as a 'win/win' scenario, because largeness and Property (τ) are both extremely useful.

As an almost immediate consequence of Theorem 1.1, we obtain the following characterisation of large finitely presented groups. We will give a proof of this, assuming Theorem 1.1, in Section 2.

Theorem 1.2. *Let G be a finitely presented group. Then the following are equivalent:*

(i) *G is large;*

(ii) *there exists a sequence of finite index subgroups, $G \geq G_1 \triangleright G_2 \triangleright \dots$, and a prime p such that*

  (1) *G_{i+1} is normal in G_i and has index a power of p, for each i;*

  (2) *G does not have Property (τ) with respect to $\{G_i\}$; and*

  (3) *$\{G_i\}$ has linear growth of mod p homology.*

Theorem 1.1 can also be used to provide a substantial class of groups that have Property (τ) with respect to some nested sequence of finite index subgroups.

Theorem 1.3. *Let G be a finitely presented group and let p be a prime. Suppose that G has an infinite nested sequence of subnormal subgroups, each with index a power of p, and with linear growth of mod p homology. Then G has such a sequence that also has Property (τ).*

Theorem 1.1 bears a strong resemblance to another result of the author. In [8], the following was proved:

Theorem 1.4. *Let G be a finitely presented group, and let $\{G_i\}$ be a nested sequence of finite index normal subgroups. Then at least one of the following holds:*

(i) *G_i is an amalgamated free product or HNN extension for all sufficiently large i;*

(ii) *G has Property (τ) with respect to $\{G_i\}$;*

(iii) *$\inf_i d(G_i)/[G : G_i]$ is zero.*

Here, $d(\ )$ is the rank of a group, which is the minimal size of a generating set. In this paper, $d_p(\ )$ plays this rôle; using $d_p(\ )$ rather than $d(\ )$, we strengthen (i) to deduce that G is large. Not only are the statements of Theorems 1.1 and 1.4 very similar, but also their proofs follow similar lines, although the proof of Theorem 1.1 is more complicated. The geometry and topology of Schreier coset graphs play a central rôle in both arguments. The main difference is that a key application of the Seifert - van Kampen theorem in the proof of the Theorem 1.4 is replaced by the Mayer - Vietoris theorem with mod p coefficients in the proof of Theorem 1.1.

There is an interesting application of Theorem 1.1 to low-dimensional topology and geometry. A major area of research in this field is the study of lattices in $\mathrm{PSL}(2, \mathbb{C})$ (or, equivalently, finite-volume hyperbolic 3-orbifolds). An important unsolved problem asks whether any such lattice is a large group. As mentioned above, it was shown in [9] that if such a lattice contains a non-trivial torsion element then it has a nested sequence $\{G_i\}$ of finite index subgroups with linear growth of mod p homology, for some prime p. Moreover, these subgroups are all normal in G_1 and have index a power of p. Thus, we deduce from Theorem 1.1 that *either* G has Property (τ) with respect to $\{G_i\}$ *or* that G is large. In [9], we show that the following conjecture of Lubotzky and Zelmanov, which we have termed the GS-τ Conjecture, implies that we can arrange that the former possibility does not arise.

Conjecture 1.5. (GS-τ Conjecture) *Let G be a group with finite presentation $\langle X|R\rangle$, and let p be a prime. Suppose that $d_p(G)^2/4 > |R| - |X| + d_p(G)$. Then G does not have Property (τ) with respect to some infinite nested sequence $\{G_i\}$ of normal subgroups with index a power of p.*

Thus, Theorem 1.1 and the argument in [9] give the following result.

Theorem 1.6. *The GS-τ Conjecture implies that any lattice in* $\mathrm{PSL}(2, \mathbb{C})$ *with non-trivial torsion is large.*

After an earlier version of this paper was first distributed, Ershov [5] discovered some examples of finitely generated groups satisfying an inequality similar to that in Conjecture 1.5, but which have Property (τ) with respect to all sequences of finite index subgroups (indeed, they have Property (T)). This casts some doubt on Conjecture 1.5, but it nevertheless remains open as stated. Moreover, one does not need the full conjecture to deduce that lattices in $\mathrm{PSL}(2, \mathbb{C})$ with non-trivial torsion are large. For more details, see [9].

It is natural to ask which finitely generated groups G have a sequence of subnormal subgroups, each with index a power of p and with linear growth of mod p homology. We prove a stronger version of the following result in Section 8, which gives an alternative characterisation of these groups.

Theorem 1.7. *Let G be a finitely generated group, and let p be a prime. Then the following are equivalent:*

(i) G has an infinite nested sequence of subnormal subgroups, each with index a power of p, and with linear growth of mod p homology;

(ii) the pro-p completion of G has exponential subgroup growth.

Combining Theorems 1.3 and 1.7, we have the following interesting corollary.

Corollary 1.8. *Let G be a finitely presented group, and let p be a prime. Suppose that the pro-p completion of G has exponential subgroup growth. Then G has a nested sequence of subnormal subgroups, each with index a power of p, which has Property (τ).*

Property (τ) plays a prominent rôle in the statement of Theorem 1.1. But one might wonder to what extent it is needed. Might it be true that conclusion (i) of Theorem 1.1 always holds? We will see how this question relates to error-correcting codes. We will show that if (i) does not hold, then an infinite collection of linear codes can be constructed that are 'asymptotically good'. These are very important in the theory of error-correcting codes, because they have large rate and large Hamming distance. More details of this relationship can be found in Section 6.

We now briefly describe the plan of the paper. In Section 2, we recall the definition of Property (τ), and then go on to prove Theorems 1.2 and 1.3 from Theorem 1.1. In Section 3, we give a necessary and sufficient topological condition on a finite connected 2-complex (satisfying some generic conditions) for its fundamental group to admit a surjective homomorphism onto a non-abelian free group. This is a key step in the proof of Theorem 1.1, which is presented in Sections 4 and 5. Section 5 in particular is the heart of the paper. In Section 6, we establish a link between large groups and error-correcting codes. In Section 7, we show that the assumption of finite presentability in Theorems 1.1 and 1.3 cannot be weakened to being finitely generated. This is because the (generalised) lamplighter group $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$, which is finitely generated, satisfies the remaining hypotheses of Theorem 1.1 and 1.3 but satisfies none of their conclusions. Finally, in Section 8, we relate linear growth of mod p homology to the subgroup growth of the group's pro-p completion.

I am grateful to Jim Howie and Alex Lubotzky who suggested to me the examples in Section 7. I would also like to thank Andrei Jaikin who suggested an improvement to an earlier version of Proposition 4.2.

2. Property (τ)

In this section, we establish some elementary facts about Property (τ) and then go on to deduce Theorems 1.2 and 1.3 from Theorem 1.1.

The following two lemmas are elementary and well known.

Lemma 2.1. *Let G and K be finitely generated groups, and let $\phi\colon G \to K$ be a surjective homomorphism. Let $\{K_i\}$ be a collection of finite index subgroups of K. Then K has Property (τ) with respect to $\{K_i\}$ if and only if G has Property (τ) with respect to $\{\phi^{-1}(K_i)\}$.*

Proof. Let S be a finite generating set for G. Then $\phi(S)$ forms a finite generating set for K. Now, ϕ induces a bijection between the right cosets $G/\phi^{-1}(K_i)$ and K/K_i. This respects right multiplication by elements of G. Hence, the coset graphs $X(G/\phi^{-1}(K_i); S)$ and $X(K/K_i; \phi(S))$ are isomorphic. The lemma follows immediately. □

Lemma 2.2. *Let G be a finitely generated group, and let K be a finite index subgroup. Let $\{K_i\}$ be a collection of finite index subgroups of K. Then G has Property (τ) with respect to $\{K_i\}$ if and only if K has Property (τ) with respect to $\{K_i\}$.*

Proof. This is essentially contained in the proof of Lemma 2.5 in [7], but we include the proof here for the sake of completeness, and because we are explicitly dealing here with subgroups that need not be normal.

Let S be a finite generating set for G. Let T be a maximal tree in $X(G/K; S)$. Then the edges not in T form a finite generating set $\tilde{S}$ for K, by the Reidemeister-Schreier process. For any subgroup K_i of K, $X(G/K_i; S)$ is a covering space of $X(G/K; S)$. The inverse image of T in $X(G/K_i; S)$ is a forest F. If one were to collapse each component of this forest to a point, one would obtain $X(K/K_i; \tilde{S})$.

Let A be any non-empty subset of the vertex set of $X(K/K_i; \tilde{S})$. Its inverse image $\tilde{A}$ in $X(G/K_i; S)$ is a union of components of F. It is clear that $|\tilde{A}| = [G : K]|A|$ and $|\partial\tilde{A}| = |\partial A|$. Hence, $h(X(G/K_i; S)) \leq h(X(K/K_i; \tilde{S}))/[G : K]$. So if $h(X(K/K_i; \tilde{S}))$ has zero infimum, then so does $h(X(G/K_i; S))$.

Now consider a non-empty subset B of the vertex set of $X(G/K_i; S)$ such that $|\partial B|/|B| = h(X(G/K_i; S))$ and $|B| \leq |V(X(G/K_i; S))|/2$. Let $\overline{B}$ be the vertices of $X(G/K_i; S)$ lying in the union of those components of F that intersect B. Thus, $\overline{B}$ clearly contains B. If a component of F lies in $\overline{B}$ but does not lie entirely in B, then it contains an edge of ∂B. Hence,

$$|B| \leq |\overline{B}| \leq |B| + [G:K]|\partial B|.$$

If an edge lies in $\partial\overline{B}$ but not in ∂B, then it joins two different components of F, at least one of which contains an edge of ∂B. There are at most $2|S|[G:K]$ edges with an endpoint in this component of F. Hence,

$$|\partial\overline{B}| \leq (2|S|[G:K]+1)|\partial B|.$$

Now, $\overline{B}$ projects to a set of vertices in $X(K/K_i;\tilde{S})$ with size that has been scaled by a factor of $[G:K]^{-1}$ and with the same size boundary. Hence,

$$\begin{aligned} &h(X(K/K_i;\tilde{S})) \\ &\quad \leq \frac{|\partial\overline{B}|}{[G:K]^{-1}\min\{|\overline{B}|,|\overline{B}^c|\}} \\ &\quad \leq [G:K](2|S|[G:K]+1)\frac{|\partial B|}{\min\{|B|,|B^c|-[G:K]|\partial B|\}} \\ &\quad \leq [G:K](2|S|[G:K]+1)\max\left\{h,\frac{h}{1-[G:K]h}\right\}, \end{aligned}$$

where $h = h(X(G/K_i;S))$, and provided that $|B^c| - [G:K]|\partial B| > 0$. This assumption certainly holds if $h < [G:K]^{-1}$. So, if $h(X(G/K_i;S))$ has zero infimum, then so does $h(X(K/K_i;\tilde{S}))$. □

We are now in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. (ii) ⇒ (i) is an immediate consequence of Theorem 1.1. In the other direction, suppose that some finite index subgroup G_1 of G admits a surjective homomorphism ϕ_1 onto a non-abelian free group F. Let $\phi_2\colon F\to\mathbb{Z}$ be projection onto the first free summand. Now, $\mathbb{Z}$ does not have Property (τ) with respect to $\{p^i\mathbb{Z}\}$, by the earlier example. Let G_i be $\phi_1^{-1}\phi_2^{-1}(p^{i-1}\mathbb{Z})$. Then, for each i, G_{i+1} is normal in G_i and has index p. By Lemma 2.1, G_1 does not have Property (τ) with respect to $\{G_i\}$. By Lemma 2.2, G also does not have Property (τ) with respect to $\{G_i\}$. Now, $\phi_2^{-1}(p^{i-1}\mathbb{Z})$ forms a nested sequence of finite index subgroups in F, and any such sequence has linear growth of mod p homology. As each G_i surjects onto $\phi_2^{-1}(p^{i-1}\mathbb{Z})$, $d_p(G_i) \geq d_p(\phi_2^{-1}(p^{i-1}\mathbb{Z}))$. Hence, $\{G_i\}$ has linear growth of mod p homology. □

Proof of Theorem 1.3. If the given sequence of subgroups has Property (τ), we are done. If not, then Theorem 1.1 implies that some finite index subnormal subgroup of G, with index a power of p, admits a surjective homomorphism onto a non-abelian free group

F. By passing to a smaller subgroup of G if necessary, we may assume that F has arbitrarily large rank. We claim that F then has a sequence of normal subgroups, each with index a power of p, with linear growth of mod p homology and with Property (τ). Their inverse images in G form the required subgroups by Lemmas 2.1 and 2.2. There are many ways to prove this claim. One is to use the fact that $\mathrm{SL}(3, \mathbb{Z})$ has Property (τ) with respect to its principal congruence subgroups [10]. Let K_n denote the level p^n principal congruence subgroup. Then K_{n+1} is normal in K_n and has index a power of p, for all $n \geq 1$. If the rank of F is large enough, it admits a surjective homomorphism onto K_1. The inverse images of K_n in F then form the required subgroups. □

3. Cocycles and large groups

In this section, we will study connected finite 2-complexes K and give a necessary and sufficient topological condition for $\pi_1(K)$ to admit a free non-abelian quotient. *We make convention throughout this paper* that the attaching map of each 2-cell of K is cellular; that is, the boundary path of the 2-cell can be expressed as a concatenation of a finite sequence of paths, each of which is a homeomorphism onto a 1-cell of K.

The necessary and sufficient condition will be phrased in terms of *regular cocycles*. These are particularly nice representatives of elements of $H^1(K)$. We will show that any such cohomology class is represented by a regular cocycle.

A regular cocycle is just a non-empty finite graph Γ embedded within K in a certain way, together with orientation information. The graph must satisfy the following conditions:

(i) Γ is disjoint from the 0-skeleton of K;

(ii) its vertices $V(\Gamma)$ are the intersection of Γ with the 1-skeleton of K;

(iii) for any 2-cell with quotient map $i\colon D \to K$, where D is a 2-disc, $D \cap i^{-1}(\Gamma)$ is a finite collection of properly embedded disjoint arcs with endpoints precisely $\partial D \cap i^{-1}(V(\Gamma))$.

We then say that the graph is *regularly embedded*. A *regular cocycle* is a regularly embedded graph with a transverse orientation assigned to each arc in each 2-cell, with the requirement that near each vertex of Γ, these transverse orientations all coincide.

A regular cocycle determines an element of $H^1(K)$, as follows. It assigns to each oriented 1-cell of K a weight, which is just its signed intersection number with Γ. The total weight of the boundary of any 2-cell is clearly zero. This therefore gives a well-defined cellular cocycle and hence an element of $H^1(K)$.

Conversely, one may construct a representative regular cocycle for any element of $H^1(K)$, as follows. Pick a cellular cocycle representing the cohomology class. This is just an assignment of an integer weight to each oriented 1-cell, with the property that the weights of the boundary of any 2-cell sum to zero. For any 1-cell e, with weight $w(e)$, say, place $|w(e)|$ vertices of Γ on the interior of e. Give e an orientation, so that its weight is non-negative. Assign the same transverse orientation to the vertices on e. Since the total evaluation around each 2-cell is zero, there is a way to insert the transversely oriented edges of Γ into the 2-cells, forming a regular cocycle.

Note that a connected regular cocycle represents a non-trivial element of $H^1(K)$ if and only if it is non-separating. For, if it is separating, then its evaluation of any closed loop in K is zero, and hence it represents the trivial cohomology class. Conversely, if it is non-separating, then its evaluation on some closed loop is non-zero, and so the associated cohomology class is non-trivial.

We say that a point x in K is *locally separating* if it has a connected neighbourhood U such that $U - x$ is disconnected. The *valence* of a 1-cell of K is the total number of times the 2-cells of K run over it. In the second half of the following result, we consider only finite 2-complexes with no locally separating points and no 1-cells with valence 1. Note that any finite 2-complex can be transformed into a finite 2-complex with these properties, without changing its fundamental group. For, we may replace each 0-cell with a 2-sphere and each 1-cell with a tube. Thus, any finitely presented group arises as the fundamental group of a finite 2-complex with these properties.

For a group G and positive integer n, let $*^n G$ denote the free product of n copies of G. For a space X with a basepoint, let $\bigvee^n X$ denote the wedge of n copies of X glued along their basepoints.

Theorem 3.1. *Let K be a finite connected 2-complex. Then $\pi_1(K)$ admits a surjective homomorphism onto $*^n\mathbb{Z}$ if K contains n disjoint regular cocycles whose union is non-separating. Furthermore, the converse also holds, provided K has no locally separating points and no 1-cells with valence 1.*

Proof. Suppose first that K contains n disjoint regular cocycles $C_1, \dots, C_n$ whose union is non-separating. These have disjoint product neighbourhoods $C_i \times [-1,1]$. Define a map $f\colon K \to \bigvee^n S^1$, as follows. Away from $\bigcup(C_i \times [-1,1])$, send everything to the central vertex of $\bigvee^n S^1$. On $C_i \times [-1,1]$, first project onto the second factor $[-1,1]$, and then compose this with the quotient map $[-1,1] \to S^1$ that identifies the endpoints of the interval, and then map this to the i^{th} circle of $\bigvee^n S^1$. Pick a basepoint b for K away from the neighbourhoods of the cocycles. We claim that the induced map $f_*\colon \pi_1(K, b) \to *^n\mathbb{Z}$ is a surjection. This is because the i^{th} free generator of $*^n\mathbb{Z}$ may be realised by a loop that starts at b, runs to C_i, crosses it transversely, and returns to b. We may ensure that this is the only point of intersection between the loop and $\bigcup C_i$, by the hypothesis that $\bigcup C_i$ is non-separating.

Conversely, suppose that $\pi_1(K)$ admits a surjective homomorphism onto $*^n\mathbb{Z}$. We will show that this is induced by a map $f\colon K \to \bigvee^n S^1$. Pick a basepoint b for K in the 0-skeleton. Pick a maximal tree T in the 1-skeleton of K. Let f send this tree to the central vertex of $\bigvee^n S^1$. Each remaining of edge e of K, when oriented, determines an element of $\pi_1(K, b)$, given by the path that starts at b, runs along T to the initial vertex of e, then along e, then back to b by a path in T. The image of this element of $\pi_1(K, b)$ under the given homomorphism is an element of $*^n\mathbb{Z}$, which we may take to be a reduced word. This then gives a path in $\bigvee^n S^1$. Define the restriction of f to e to be this path. Since we started with a homomorphism $\pi_1(K) \to *^n\mathbb{Z}$, the boundary of each 2-cell is sent a homotopically trivial loop in $\bigvee^n S^1$, and hence, there is a way to extend f over the 2-cells. Pick points $p_1, \ldots, p_n$, one in each circle of $\bigvee^n S^1$, disjoint from the central vertex. Then it is clear that we may ensure that, for each i, $f^{-1}(p_i)$ is a regularly embedded graph. Moreover, if we impose orientations on the circles, then these graphs inherit transverse orientations, making them regular cocycles $C_1, \ldots, C_n$, say. These cocycles are clearly disjoint, but their union may not yet be non-separating. The aim now is to modify f by a homotopy, thereby changing the cocycles C_i, to ensure that this is the case.

Define a graph Y, whose vertices correspond to the components of the complement of $\bigcup C_i$. Let its edges be in one-one correspondence with the components of $\bigcup C_i$, and where incidence between edges and vertices in Y is defined by topological incidence in K. The edges inherit an orientation from $\bigcup C_i$, and also inherit a label i. We will modify f, thereby giving new regular cocycles C_i, and hence a new graph Y. At each stage, the number of components of $\bigcup C_i$ will decrease, and so this process is guaranteed to terminate. The aim is to ensure that Y satisfies the following condition:

($*$) no vertex of Y has two edges pointing into it with the same label, or two edges pointing out of it with the same label.

Suppose now that ($*$) is violated. Let E_1 and E_2 be distinct components of C_i, say, both pointing into the same component X of $K - \bigcup C_i$. Since K contains no locally separating points, each 1-cell of K has non-zero valence. Hence, neither E_1 nor E_2 is a point. Pick an embedded arc α, with one endpoint on E_1 and the other endpoint on E_2, and with interior in X. Since every 1-cell of K has valence at least two, every vertex of the graphs E_1 and E_2 has valence at least two. So neither graph is a tree. Hence, each contains a point in the interior of an edge such that removing that point from E_i does not disconnect E_i. We may assume that the endpoints of α are these two points. Because K has no locally separating points, we may arrange for α to miss the 0-cells of K. We may ensure that α intersects each 1-cell in a finite collection of points, and each 2-cell in a finite collection of arcs, each of which is properly embedded, except the arc(s) containing the endpoints of α. Let $\alpha \times [-1, 1]$ be a thickening of α, so that $(\alpha \times [-1, 1]) \cap \bigcup C_i = \partial\alpha \times [-1, 1]$. We now modify f, leaving it unchanged away from a small regular neighbourhood of $\alpha \times [-1, 1]$. In $\alpha \times [-1, 1]$, modify f so that the intersection of the new C_i with $\alpha \times [-1, 1]$ is $\alpha \times \{-1, 1\}$, and the other C_j remain disjoint from $\alpha \times [-1, 1]$. There is an obvious way to extend this definition of f to a small neighbourhood of $\alpha \times [-1, 1]$, so that it remains unchanged outside of this neighbourhood. Note that this changes f on 2-cells D which intersect α only in points, with the introduction of a new arc of $C_i \cap D$ around each of these points. (See Figure 2.) Now, we have arranged that $E_1 - \partial\alpha$ and $E_2 - \partial\alpha$ are both connected. So, this operation has the effect of combining E_1 and E_2 into a single connected cocycle, thereby reducing the number of components of $\bigcup C_i$.

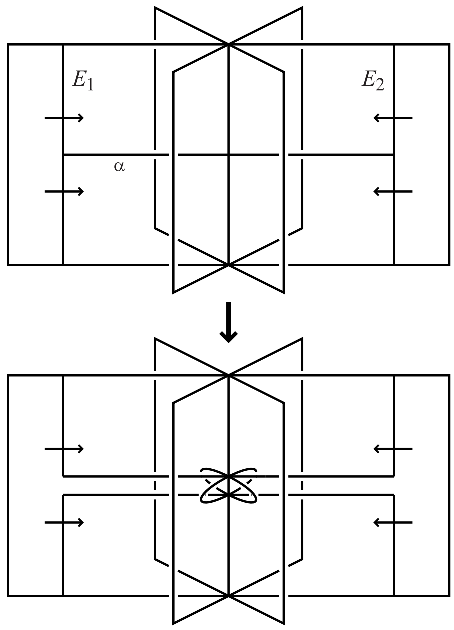

Figure 2

Hence, we may assume that $(*)$ holds, after possibly homotoping f. This homotopy has the effect of changing the induced homomorphism $f_*\colon \pi_1(K) \rightarrow *^n\mathbb{Z}$ by a conjugacy, but it remains a surjective homomorphism.

We claim that Y then has a single vertex, with n edges, labelled $1, \dots, n$. This will show that $\bigcup C_i$ is non-separating as required. To prove this claim, we use the hypothesis that f_* is surjective. This implies that there are loops $\ell_1, \dots, \ell_n$, based at the basepoint of K, that are sent to the free generators of $*^n\mathbb{Z}$. Pick these loops so that they have the fewest number of intersections with $\bigcup C_i$. The loops determine loops in the graph Y. No loop can travel over C_i in one direction, and then back across C_i in the other direction. For, by property $(*)$, it would have to return to the same component of C_i. We could then remove this sub-arc of the loop, and replace it by an arc in C_i, and then perform a small homotopy, reducing the number of intersections with $\bigcup C_i$ by two. The resulting loop still is sent to the same element of $*^n\mathbb{Z}$, which contradicts our minimality assumption. Hence, the word that ℓ_i spells, as it runs over $\bigcup C_i$, is a reduced word. It therefore runs over C_i exactly once, and is disjoint from the other cocycles. Hence, emanating from the vertex of Y that corresponds to the component of $K - \bigcup C_i$ containing the basepoint, there is an edge labelled i, for each i, and each such edge returns to this vertex. Therefore, Y is a bouquet of circles, as required. □

In this theorem, we worked with 2-complexes for convenience. We could just as easily have worked with smooth manifolds. In this case, transversely oriented, codimension one submanifolds play the rôle of regular cocycles. Essentially the same argument as for Theorem 3.1 gives the following.

Theorem 3.2. *Let M be a connected smooth manifold. Then $\pi_1(M)$ admits a surjective homomorphism onto $*^n\mathbb{Z}$ if and only if M contains n disjoint, transversely oriented, codimension one submanifolds whose union is non-separating.*

All of the above is fairly well known. What is possibly less widely known is that one can replicate much of this work using cohomology with coefficients in $\mathbb{F}_p$, the field of order a prime p. Therefore, fix a prime p.

A *regular mod p cocycle* has a similar definition to a regular cocycle. Again, it is a non-empty finite graph Γ embedded in K, with a little extra structure. It must be disjoint from the 0-skeleton of K. However, unlike the case of regular cocycles, it has two type of vertices, which we term *edge vertices* and *interior vertices*. The edge vertices are the intersection of Γ with the 1-skeleton of K. The vertices of Γ on the boundary of any 2-cell are therefore edge vertices, and we require them to have valence one in that 2-cell. Each interior vertex lies in the interior of a 2-cell of K. The edges of Γ are given a transverse orientation and a *weight*, which is a non-zero integer mod p. These must satisfy the following local conditions near the vertices. Near the edge vertices, the transverse orientations and the weights must all be locally equivalent. Around any interior vertex, the total weight (signed according to the transverse orientations) must be congruent to zero mod p. We also insist that each interior vertex has at least one edge incident to it. (See Figure 3.)

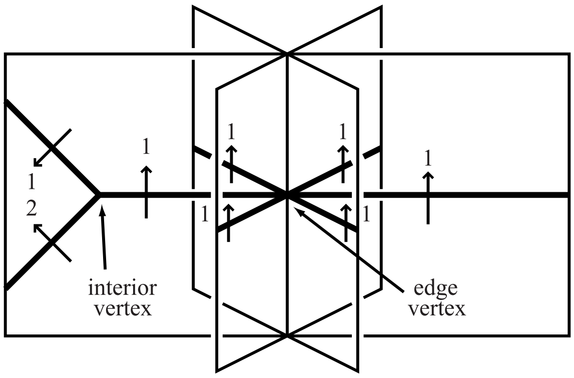

Figure 3

We will see that, as before, any element of $H^1(K;\mathbb{F}_p)$ is represented by a regular mod p cocycle, and conversely, a regular mod p cocycle determines a class in $H^1(K;\mathbb{F}_p)$. The following states that, for non-trivial cohomology classes, we may ensure that the regular mod p cocycle is also non-separating.

Proposition 3.3. *Let K be a finite connected 2-complex and let p be a prime. Then any non-trivial element of $H^1(K;\mathbb{F}_p)$ is represented by a non-separating regular mod p cocycle.*

Proof. Any element of $H^1(K;\mathbb{F}_p)$ is represented by a cellular 1-dimensional cocycle c. This is an assignment to each oriented 1-cell e of an integer mod p which we denote by $c(e)$, with the proviso that the sum of the integers around any 2-cell is zero mod p. From this, we build a regular mod p cocycle Γ as follows. Into each 1-cell e for which $c(e)$ is non-zero mod p, we place an edge vertex of Γ with weight $c(e)$. If a 2-cell contains a 1-cell with non-zero weight in its boundary, insert into it a single interior vertex. Join this vertex to each edge vertex in the boundary of the 2-cell. The fact that the total weight of c around the 2-cell is zero mod p implies that the local condition near the interior vertex is satisfied. Thus, it is trivial that any element of $H^1(K;\mathbb{F}_p)$ is represented by a regular mod p cocycle Γ.

The aim now is to ensure that Γ is non-separating when the cohomology class is non-zero. To establish this, we will perform a sequence of alterations to Γ. Each will reduce the number of edge vertices, and so this sequence is guaranteed to terminate. Suppose that Γ is separating, and let K_1 be some component of $K - \Gamma$. Then, there is some edge vertex in the boundary of K_1 that is incident to another component of $K - \Gamma$. Let Γ' be the component of Γ minus its interior vertices that contains this edge vertex. Then all the edges of Γ' are compatibly oriented and have the same weight w, say. Remove Γ' from Γ. For each edge in the boundary of K_1 but not in Γ', add or subtract w to its weight, according to whether the transverse orientation of the edge points into or out of K_1. If both sides of the edge lie in K_1, then leave its weight unchanged. If this procedure changes the weight of any edge to zero mod p, then remove it. If any interior vertices become isolated, remove them. The result is a new regular mod p cocycle, representing the same cohomology class, and with fewer edge vertices. Repeating this process a sufficient number of times, we therefore end with a non-separating regular mod p cocycle. □

The proof of the above result gives the following extra information which will be useful later.

Addendum 3.4. *Let K be a finite 2-complex and let p be a prime. If a regular mod p cocycle Γ represents a non-trivial element of $H^1(K; \mathbb{F}_p)$, then some subgraph of Γ is a regular mod p cocycle (with possibly different weights) which represents the same cohomology class and is non-separating in K.*

There is also a more technical version of Proposition 3.3 that deals with subcomplexes.

Proposition 3.5. *Let K be a finite 2-complex and let p be a prime. Let L be a subcomplex of K. Suppose that there is a non-trivial element α in the kernel of $H^1(K; \mathbb{F}_p) \to H^1(L; \mathbb{F}_p)$, the map induced by inclusion. Then α is represented by a regular mod p cocycle that is non-separating in K and disjoint from L.*

Proof. Pick a cellular cochain c that represents α. Since the restriction of c to L is cohomologically trivial, it is a coboundary in L. Subtracting this coboundary from c does not change the class it represents, but afterwards its evaluation on any 1-cell in L is trivial. Thus, when the construction in the proof of Proposition 3.3 is performed, a regular mod p cocycle Γ is created that is disjoint from L. Applying Addendum 3.4, we can ensure that Γ is non-separating in K and still disjoint from L. □

There is also a corresponding version of Theorem 3.1 for regular mod p cocycles, which works best when $p = 2$. This will be a crucial tool in proving that certain groups are large.

Theorem 3.6. *Let K be a finite connected 2-complex, and let p be a prime. Then $\pi_1(K)$ admits a surjective homomorphism onto $*^n(\mathbb{Z}/p\mathbb{Z})$ if K contains n disjoint regular mod p cocycles whose union is non-separating. Furthermore the converse holds when $p = 2$ and K contains no locally separating points and no 1-cells with valence 1.*

Proof. The proof is very similar to that of Theorem 3.1, and so we will only focus on those parts where the details differ.

Suppose first that K contains n disjoint regular mod p cocycles whose union is non-separating. Then we construct a map $f\colon K \to \bigvee^n L(p)$, where $L(p)$ is the 2-complex consisting of a single 0-cell, a single 1-cell, and a 2-cell that winds p times around the 1-skeleton. Outside of a small regular neighbourhood of the cocycles, everything is sent by f to the central vertex of $\bigvee^n L(p)$. On product neighbourhoods of the edges and edge vertices of the cocycles, f is defined to collapse these products onto an interval and then map this interval w times around the relevant 1-cell of $\bigvee^n L(p)$, where w is the weight of the edge. Finally, near the interior vertices of the cocycles, f maps onto the relevant 2-cell of $\bigvee^n L(p)$. The proof that $f_*\colon \pi_1(K) \to *^n(\mathbb{Z}/p\mathbb{Z})$ is a surjection is similar to the corresponding proof for Theorem 3.1.

Suppose now that $\pi_1(K)$ admits a surjective homomorphism onto $*^n(\mathbb{Z}/p\mathbb{Z})$. Suppose also that $p = 2$ and K contains no locally separating points and no 1-cells with valence 1. Then, exactly as in the proof of Theorem 3.1, this homomorphism is induced by a map $f\colon K \to \bigvee^n L(2)$. Let α_i be the regular mod 2 cocycle in $\bigvee^n L(2)$ that has exactly one edge vertex in the i^{th} 1-cell and exactly one interior vertex in the i^{th} 2-cell. Then we may arrange that $f^{-1}(\alpha_i)$ forms a regular mod 2 cocycle C_i for each i. We may also arrange that each interior vertex of $\bigcup C_i$ has valence 2. However, as in the proof of Theorem 3.1, the union of these cocycles may not yet be non-separating in K. We may need to modify f by a homotopy before this condition is satisfied.

Define a graph Y whose vertices correspond to complementary components of $\bigcup C_i$, and whose edges correspond to the components of $\bigcup C_i$. It may not be the case that a component of $\bigcup C_i$ has a regular neighbourhood that is a product. If it is not a product, then using the fact that $p = 2$, it is adjacent to a single complementary region of $\bigcup C_i$, and we therefore define the corresponding edge of Y to be a loop. The edges of Y come with an integer label between 1 and n, depending on which cocycle C_i they came from. However, they do not necessarily come with a well-defined orientation. Again, we will homotope f, to ensure that a certain condition holds:

($*'$) no vertex of Y has two distinct edges adjacent to it with the same label.

Each modification will reduce the number of components of $\bigcup C_i$, and so they are guaranteed to terminate. The modifications are exactly as before, except now the transverse orientations of E_1 and E_2 at the endpoints of α might not point towards each other or away from each other. However, this is easily rectified by the introduction of two interior vertices near one of the endpoints of α. The argument now proceeds exactly as in the proof of Theorem 3.1. □

The following consequence of Theorem 3.6 gives a method for proving that certain groups are large.

Theorem 3.7. *Let K be a finite cell complex, and let A and B be subcomplexes such that $K = A \cup B$. Let p be a prime and let $\mathbb{F}_p$ be the field of order p. Suppose that both of the maps*

$$H^1(A; \mathbb{F}_p) \to H^1(A \cap B; \mathbb{F}_p)$$
$$H^1(B; \mathbb{F}_p) \to H^1(A \cap B; \mathbb{F}_p)$$

induced by inclusion are not injections. In the case $p = 2$, suppose also that the kernel of at least one of these maps has dimension more than one. Then $\pi_1(K)$ admits a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$. Furthermore, some normal subgroup of $\pi_1(K)$ with index a power of p admits a surjective homomorphism onto a non-abelian free group. Hence, $\pi_1(K)$ is large.

Proof. We may restrict attention to the 2-skeleton of K, since this has the same fundamental group as K, and since the relevant homomorphisms between cohomology groups are unchanged. Thus, we may assume that K is a 2-complex.

Pick a non-trivial element of the kernel of $H^1(A; \mathbb{F}_p) \to H^1(A \cap B; \mathbb{F}_p)$. By Proposition 3.5, this is represented by a regular mod p cocycle that is disjoint from $A \cap B$ and that is non-separating in A. It is therefore a regular mod p cocycle in K. The same argument gives a non-separating regular mod p cocycle in B that is disjoint from $A \cap B$. Hence, we obtain two disjoint regular mod p cocycles in K whose union is non-separating. By Theorem 3.6, this implies that $\pi_1(K)$ admits a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$. When p is odd, $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$ contains a free non-abelian normal subgroup with index a power of p. The inverse image of this subgroup in $\pi_1(K)$ is also normal and has index a power of p. It surjects on this free non-abelian group. This therefore proves the theorem when p is odd.

Now, $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ does not have a free non-abelian group as a subgroup, and so the theorem is not yet fully proved when $p = 2$. In this case, however, we are assuming that the kernel of one of the maps, say $H^1(A; \mathbb{F}_p) \to H^1(A \cap B; \mathbb{F}_p)$, has dimension at least two. Construct the finite-sheeted covering space of A corresponding to this kernel. The inverse image of $A \cap B$ is a disjoint union of at least four copies of $A \cap B$. Attach to each of these a copy of B. The result is a finite-sheeted regular cover $\tilde{K}$ of K with degree a power of 2. In each copy of B, there is a non-separating regular mod 2 cocycle. The union of these is therefore non-separating in $\tilde{K}$. Thus, by Theorem 3.6, $\pi_1(\tilde{K})$ admits a surjective homomorphism onto $*^4(\mathbb{Z}/2\mathbb{Z})$. This contains a normal free non-abelian subgroup, with index a power of 2. Its inverse image in $\pi_1(\tilde{K})$ surjects onto this non-abelian free group. By passing a further subgroup if necessary, we may assume that this is normal in $\pi_1(K)$ and has index a power of 2 in $\pi_1(K)$. □

Thus, one route to proving that a cell complex K has large fundamental group is to find a decomposition into subcomplexes A and B where $|H_1(A; \mathbb{F}_p)|$ and $|H_1(B; \mathbb{F}_p)|$ are both bigger than $2|H_1(A \cap B; \mathbb{F}_p)|$. This suggests the following definition.

Definition. Let K be a finite cell complex. Consider all ways of decomposing K into two sets A and B, where A and B are subcomplexes in some subdivision of the cell structure on K. Let the *mod p Cheeger constant* of K, denoted $h_p(K)$, be

$$\inf \left\{ \frac{|H_1(A \cap B; \mathbb{F}_p)|}{\min\{|H_1(A; \mathbb{F}_p)|, |H_1(B; \mathbb{F}_p)|\}} \right\}.$$

Theorem 3.7 has the following immediate corollary. (See Lemma 2.2 in [11] for a related result.)

Corollary 3.8. *Let K be a finite connected cell complex, and let p be a prime. Suppose that*

$$h_p(K) < \begin{cases} 1 & \text{if } p \text{ is odd;} \\ 1/2 & \text{if } p = 2. \end{cases}$$

Then $\pi_1(K)$ admits a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$. Furthermore, some normal subgroup of $\pi_1(K)$ with index a power of p admits a surjective homomorphism onto a non-abelian free group. Hence, $\pi_1(K)$ is large.

The following result summarises much of what has been done in this section.

Theorem 3.9. *Let K be a finite connected 2-complex with fundamental group G. Suppose that K has no locally separating points and no 1-cells with valence 1. Then the following are equivalent:*

(i) G is large;

(ii) in some finite-sheeted covering space $\tilde{K}$ of K, there are two disjoint regular cocycles whose union is non-separating;

(iii) in some finite-sheeted covering space $\tilde{K}$ of K, there are two disjoint regular mod p cocycles whose union is non-separating, for some odd prime p;

(iv) in some finite-sheeted covering space $\tilde{K}$ of K, there are three disjoint regular mod 2 cocycles whose union is non-separating.

Proof. Note first that condition of having no locally separating points and no 1-cells with valence 1 is preserved under finite covers.

(i) $\Rightarrow$ (ii): Since G is large, some finite index subgroup of G admits a surjective homomorphism onto $\mathbb{Z} * \mathbb{Z}$. Let $\tilde{K}$ be the covering space of K corresponding to this subgroup. By Theorem 3.1, it has two disjoint regular cocycles whose union is non-separating.

(ii) $\Rightarrow$ (iii): This is obvious, because a regular cocycle becomes a regular mod p cocycle when every edge is given weight 1.

(iii) $\Rightarrow$ (i): By Theorem 3.6, the fundamental group of $\tilde{K}$ admits a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$. But this contains a non-abelian free group as a finite index normal subgroup. Hence, G is large.

(i) $\Rightarrow$ (iv): This is very similar to (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Since G is large, some finite index subgroup admits a surjective homomorphism onto $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. The corresponding covering space has three disjoint regular cocycles whose union is non-separating. Each is, by definition, a regular mod 2 cocycle when every edge is given weight 1.

(iv) $\Rightarrow$ (i): This proof is essentially the same as (iii) $\Rightarrow$ (i), using the fact that $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ has a free non-abelian group as a finite index normal subgroup. □

4. Cheeger decompositions of coset diagrams

The following result was a key technical lemma in [7] (Lemma 2.1 there).

Lemma 4.1. *Let X be a Cayley graph of a finite group, and let D be a non-empty subset of $V(X)$ such that $|\partial D|/|D| = h(X)$ and $|D| \leq |V(X)|/2$. Then $|D| > |V(X)|/4$. Furthermore, the subgraphs induced by D and its complement D^c are connected.*

This was useful when analysing finite index normal subgroups H of a group G, because then a finite generating set for G determines a Cayley graph of G/H. However, in this paper, we wish to consider subgroups that are not necessarily normal. Thus, the following generalisation will be necessary.

Proposition 4.2. *Let G be a group with a finite generating set S, and let $\{G_i\}$ be a sequence of finite index subgroups, where each G_i is normal in G_{i-1}. Let S be a finite set of generators for G, and let X_i be $X(G/G_i; S)$. Then $h(X_i)$ is a non-increasing sequence. Suppose that, for some i, $h(X_i) < h(X_{i-1})$. Then, there is some non-empty subset D of $V(X_i)$ such that $|\partial D|/|D| = h(X_i)$ and $|V(X_i)|/4 < |D| \leq |V(X_i)|/2$.*

Proof. The fact that $h(X_i)$ is non-increasing is trivial. Therefore, let us concentrate on the second part of the proposition. Consider a non-empty subset D of $V(X_i)$ such that $|\partial D|/|D| = h(X_i)$ and $|D| \leq |V(X_i)|/2$. Pick D so that $|D|$ is as large as possible subject to these two conditions. Let us suppose that $|D| \leq |V(X_i)|/4$, with the aim of reaching a contradiction. Now, G_i is normal in G_{i-1} and so G_{i-1}/G_i acts on X_i by covering transformations. Let g be any element of G_{i-1}/G_i. We consider $g(D) \cup D$. It is shown in [7] (see the proof of Lemma 2.1 there) that

$$|\partial(g(D) \cup D)| = |\partial D| + |\partial g(D)| - |\partial(g(D) \cap D)| - 2e(g(D) - D, D - g(D)),$$

where $e(g(D) - D, D - g(D))$ denotes the number of edges joining $g(D) - D$ to $D - g(D)$. By the definition of $h(X_i)$, we must have that $|\partial(g(D) \cap D)| \geq h(X_i)|g(D) \cap D|$. Thus,

$$|\partial(g(D) \cup D)| \leq h(X_i)(|D| + |g(D)| - |g(D) \cap D|) = h(X_i)|g(D) \cup D|.$$

Now, $g(D) \cup D$ is at most half the vertices of X_i, by our assumption that $|D| \leq |V(X_i)|/4$. As $|D|$ was assumed to be maximal, $|g(D) \cup D|$ must be equal to $|D|$ and hence $g(D) = D$. This is true for each $g \in G_{i-1}/G_i$. Thus, D is invariant under the action of G_{i-1}/G_i on X_i, and therefore descends to a subset D' of $V(X_{i-1})$. Now, $|\partial D'| = |\partial D|/[G_{i-1} : G_i]$ and $|D'| = |D|/[G_{i-1} : G_i]$. Hence,

$$h(X_{i-1}) \leq |\partial D'|/|D'| = |\partial D|/|D| = h(X_i) \leq h(X_{i-1}).$$

Thus, these must be equalities, which contradicts our hypothesis that $h(X_i) < h(X_{i-1})$. Hence, it must have been the case that $|D| > |V(X_i)|/4$. □

5. Proof of the main theorem

In this paper, we will be concentrating on groups G having a sequence of finite index subgroups $\{G_i\}$ with linear growth of mod p homology, for some prime p. It will be helpful to introduce a quantity that measures the growth rate of $d_p(G_i)$. This is the *mod p homology gradient* which is defined to be

$$\inf_i \frac{(d_p(G_i) - 1)}{[G : G_i]}.$$

This quantity is most relevant when each G_{i+1} is normal in G_i and has index a power of p. In this case, we have the following well-known proposition.

Proposition 5.1. *Let G be a finitely generated group, and let H be a subnormal subgroup with index a power of a prime p. Then*

$$d_p(H) - 1 \leq [G : H](d_p(G) - 1).$$

This appears as Proposition 3.7 in [9] for example. It implies that when each G_{i+1} is normal in G_i and has index a power of p, $(d_p(G_i) - 1)/[G : G_i]$ is a non-increasing function of i. In particular, the infimum in the definition of mod p homology gradient is a limit.

We will, in fact, need the following stronger result.

Proposition 5.2. *Let K be a connected 2-complex, and let Γ be a connected union of 1-cells such that the map $H_1(\Gamma; \mathbb{F}_p) \to H_1(K; \mathbb{F}_p)$ induced by inclusion is a surjection, for some prime p. Let $\tilde{K} \to K$ be a finite-sheeted covering such that $\pi_1(\tilde{K})$ is subnormal in $\pi_1(K)$ and has index a power of p. Let $\tilde{\Gamma}$ be the inverse image of Γ in $\tilde{K}$. Then $\tilde{\Gamma}$ is connected and the map $H_1(\tilde{\Gamma}; \mathbb{F}_p) \to H_1(\tilde{K}; \mathbb{F}_p)$ induced by inclusion is a surjection.*

To prove this, we will require the following.

Lemma 5.3. *Let Γ be a path-connected subset of a path-connected space L such that the map $\pi_1(\Gamma) \to \pi_1(L)$ induced by inclusion is surjection. Let $q\colon \tilde{L} \to L$ be a covering map, and $\tilde{\Gamma}$ be the inverse image of Γ in $\tilde{L}$. Then $\tilde{\Gamma}$ is path-connected and the map $\pi_1(\tilde{\Gamma}) \to \pi_1(\tilde{L})$ induced by inclusion is a surjection.*

Proof. Let b be a basepoint for L in Γ. The restriction of q to any path-component of $\tilde{\Gamma}$ is a covering map onto Γ, which is therefore surjective. Thus, to show that $\tilde{\Gamma}$ is path-connected, it suffices to show that any two points of $q^{-1}(b)$ lie in the same path-component of $\tilde{\Gamma}$. We may assume that one of these points is a basepoint $\tilde{b}$ of $\tilde{L}$. Pick a path from $\tilde{b}$ to the other point in $q^{-1}(b)$. This projects to a loop ℓ in L based at b. Since $\pi_1(\Gamma, b) \to \pi_1(L, b)$ is a surjection, ℓ is homotopic, relative to its endpoints, to a loop in Γ. This lifts to a path in $\tilde{\Gamma}$ joining the two points of $q^{-1}(b)$.

We now show that $\pi_1(\tilde{\Gamma}, \tilde{b}) \to \pi_1(\tilde{L}, \tilde{b})$ is a surjection. Given any loop $\tilde{\ell}$ in $\tilde{L}$ based at $\tilde{b}$, we project it to a loop ℓ in L. This is homotopic relative to its endpoints to a loop in Γ. This homotopy lifts to a homotopy, relative to endpoints, between $\tilde{\ell}$ and a loop in $\tilde{\Gamma}$. □

Proof of Proposition 5.2. Note first that an obvious induction allows us to reduce to the case where $\pi_1(\tilde{K})$ is a normal subgroup of $\pi_1(K)$ with index a power of p.

Pick a maximal tree in Γ and extend it to a maximal tree T in the 1-skeleton of K. Let $\overline{K}$ be obtained from K by collapsing T to a point, and let $\overline{\Gamma}$ be the image of Γ in $\overline{K}$. Then clearly the map $H_1(\overline{\Gamma}; \mathbb{F}_p) \to H_1(\overline{K}; \mathbb{F}_p)$ induced by inclusion is a surjection. Suppose that we could prove the theorem for $\overline{K}$ and $\overline{\Gamma}$. Then this would clearly imply the theorem for K and Γ. Thus, we may assume that K has a single 0-cell. It therefore specifies a presentation for $\pi_1(K)$, once we have picked an orientation on each of the 1-cells of K.

Let G and H denote the groups $\pi_1(K)$ and $\pi_1(\tilde{K})$ respectively. Let H' denote $[H, H]H^p$, the subgroup of H generated by the commutators and p^{th} powers of H. This is a characteristic subgroup of H, with index a power of p. We are assuming that H is a normal subgroup of G with index a power of p. Hence, H' is a normal subgroup of G with index a power of p. In other words, G/H' is a finite p-group.

Now, $H_1(G/H'; \mathbb{F}_p)$ is isomorphic to $H_1(G; \mathbb{F}_p)$. Hence, the 1-cells of Γ form a generating set for $H_1(G/H'; \mathbb{F}_p)$. It is a well known fact that in any finite p-group C, a set of elements forms a generating set for C if and only if they form a generating set for $H_1(C; \mathbb{F}_p)$. Thus, the 1-cells of Γ form a generating set for G/H'. Let L be the 2-complex obtained from K by attaching a 2-cell along each word in H'. Then L has fundamental group G/H'. The map $\pi_1(\Gamma) \to \pi_1(L)$ induced by inclusion is a surjection. Let $\tilde{L}$ be the covering space of L corresponding to the subgroup H/H'. This is obtained from $\tilde{K}$ by attaching various 2-cells. But one may view their 1-skeletons as the same. By Lemma 5.3, the inverse image of Γ in $\tilde{L}$ is a connected graph. This is a copy of $\tilde{\Gamma}$, and so $\tilde{\Gamma}$ is connected. The map $\pi_1(\tilde{\Gamma}) \to \pi_1(\tilde{L})$ induced by inclusion is a surjection, by Lemma 5.3. The natural map $\pi_1(\tilde{L}) \to H_1(\tilde{L}; \mathbb{F}_p)$ is a surjection. This implies that the map $H_1(\tilde{\Gamma}; \mathbb{F}_p) \to H_1(\tilde{L}; \mathbb{F}_p)$ is a surjection. The map $H_1(\tilde{K}; \mathbb{F}_p) \to H_1(\tilde{L}; \mathbb{F}_p)$ induced by inclusion is an isomorphism. Hence, $H_1(\tilde{\Gamma}; \mathbb{F}_p) \to H_1(\tilde{K}; \mathbb{F}_p)$ is a surjection, as required. □

Before we prove Theorem 1.1, we introduce some terminology. If K is a topological space and p is a prime, then $d_p(K)$ denotes the dimension of $H_1(K; \mathbb{F}_p)$.

Proof of Theorem 1.1. Suppose that $\{G_i\}$ has linear growth of mod p homology, and that G does not have Property (τ) with respect to $\{G_i\}$. Our aim is to show that some G_i admits a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$ and that some normal subgroup of G_i, with index a power of p, admits a surjective homomorphism onto a non-abelian free group.

We fix ϵ to be some real number strictly between 0 and $\sqrt{10}/3 - 1$, but where we view it as very small. Since the mod p homology gradient of $\{G_i\}$ is non-zero, there is some j such that $(d_p(G_j) - 1)/[G : G_j]$ is at most $(1 + \epsilon)$ times the mod p homology gradient of $\{G_i\}$. The mod p homology gradient of $\{G_i : i \geq j\}$ (viewed as subgroups of G_j) is $[G : G_j]$ times the mod p homology gradient of $\{G_i\}$ (viewed as subgroups of G). So, $d_p(G_j) - 1$ is at most $(1 + \epsilon)$ times the mod p homology gradient of $\{G_i : i \geq j\}$. Hence, by replacing G by G_j, and replacing $\{G_i\}$ by $\{G_i : i \geq j\}$, we may assume that $d_p(G) - 1$ is at most $(1 + \epsilon)$ times the mod p homology gradient of $\{G_i\}$. We may also assume (by replacing G by G_1) that the index of each G_i in G is a power of p.

Let S be a set of elements of G that forms a basis for $H_1(G; \mathbb{F}_p)$. Extend this to a finite generating set S_+ for G. Let K be a finite 2-complex having fundamental group G, arising from a finite presentation of G with generating set S_+. Thus, K has a single vertex and $|S_+|$ edges. Let L be the sum of the lengths of the relations in this presentation. Let $K_i \to K$ be the covering corresponding to G_i. Our aim is to show that its mod p Cheeger constant satisfies the inequality $h_p(K_i) < 1/2$ for all sufficiently large i. Corollary 3.8 will then prove the theorem.

Let X_i be the 1-skeleton of K_i. Then $X_i = X(G/G_i; S_+)$. Let Γ_i be the subgraph of X_i consisting of those edges labelled by S. By Proposition 5.2, Γ_i is connected and the inclusion $\Gamma_i \to K_i$ induces a surjection $H_1(\Gamma_i; \mathbb{F}_p) \to H_1(K_i; \mathbb{F}_p)$.

Since we are assuming that G does not have Property (τ) with respect to $\{G_i\}$, $\inf_i h(X_i) = 0$. Since the subgroups G_i are nested, $h(X_i)$ is a non-increasing sequence. Hence $h(X_i) \to 0$. Let us focus on those values of i for which $h(X_i) < h(X_{i-1})$. This occurs infinitely often. Proposition 4.2 asserts that there is a non-empty subset D_i of $V(X_i)$ such that $|\partial D_i|/|D_i| = h(X_i)$ and $|V(X_i)|/4 < |D_i| \leq |V(X_i)|/2$. We will use D_i to construct a decomposition of K_i into two overlapping subsets. Let A_i (respectively, B_i) be the closure of the union of those cells in K_i that intersect D_i (respectively, D_i^c). Let C_i be $A_i \cap B_i$. The edges of $A_i \cap \Gamma_i$ are of three types (that are not mutually exclusive):

(i) those edges with both endpoints in D_i,

(ii) those edges in ∂D_i,

(iii) those edges in the boundary of a 2-cell that intersects both D_i and D_i^c.

If we consider the $d_p(G)$ oriented edges of Γ_i emanating from each vertex in D_i, we will cover every edge in (i), and possibly others. Hence, there are at most $|D_i|d_p(G)$ edges of type (i) in $A_i \cap \Gamma_i$.

Any type (iii) edge lies in a 2-cell that intersects both D_i and D_i^c. This 2-cell therefore intersects an edge in ∂D_i. Consider one of the endpoints of the latter edge. At most L 2-cells run over this vertex. Each 2-cell runs over at most L edges. So, there are no more than $|\partial D_i|L^2$ type (iii) edges. There are $|\partial D_i|$ type (ii) edges, and so, there are at most $|\partial D_i|(L^2 + 1)$ type (ii) and (iii) edges in total. A similar argument gives that there are at most $|\partial D_i|(L^2 + 2)$ vertices in C_i.

We claim that each component of $A_i \cap \Gamma_i$ and $B_i \cap \Gamma_i$ contains a vertex in C_i. Consider any component of $A_i \cap \Gamma_i$. Since Γ_i is connected, there is a path in Γ_i from this component to $B_i \cap \Gamma_i$. The first point in this path that lies in B_i is the required vertex in C_i. The argument for components of $B_i \cap \Gamma_i$ is similar. So, $|A_i \cap \Gamma_i|$ and $|B_i \cap \Gamma_i|$ are both at most $|\partial D_i|(L^2 + 2)$.

Now, the following is an excerpt from the Mayer-Vietoris sequence applied to $\Gamma_i \cap A_i$ and $\Gamma_i \cap B_i$:

$$H_1(\Gamma_i \cap A_i; \mathbb{F}_p) \oplus H_1(\Gamma_i \cap B_i; \mathbb{F}_p) \to H_1(\Gamma_i; \mathbb{F}_p) \to H_0(\Gamma_i \cap C_i; \mathbb{F}_p).$$

The exactness of this sequence implies that the subspace of $H_1(\Gamma_i; \mathbb{F}_p)$ generated by the images of $H_1(\Gamma_i \cap A_i; \mathbb{F}_p)$ and $H_1(\Gamma_i \cap B_i; \mathbb{F}_p)$ has codimension at most the number of components of $\Gamma_i \cap C_i$. This is at most the number of vertices in C_i, which is at most $|\partial D_i|(L^2 + 2)$. Let $\mathrm{Im}(H_1(\Gamma_i \cap A_i; \mathbb{F}_p))$ denote the image of $H_1(\Gamma_i \cap A_i; \mathbb{F}_p)$ in $H_1(K_i; \mathbb{F}_p)$, and define $\mathrm{Im}(H_1(\Gamma_i \cap B_i; \mathbb{F}_p))$ and $\mathrm{Im}(H_1(\Gamma_i; \mathbb{F}_p))$ similarly. Note that this latter group is all of $H_1(K_i; \mathbb{F}_p)$ by Proposition 5.2. We deduce that the sum of the subspaces $\mathrm{Im}(H_1(\Gamma_i \cap A_i; \mathbb{F}_p))$ and $\mathrm{Im}(H_1(\Gamma_i \cap B_i; \mathbb{F}_p))$ has codimension at most $|\partial D_i|(L^2 + 2)$ in $H_1(K_i; \mathbb{F}_p)$.

Now, $\Gamma_i \cap A_i$ has at most $|D_i| d_p(G) + |\partial D_i|(L^2+1)$ edges. It has at least $|D_i|$ vertices. Hence,

$$\begin{aligned}
d_p(\Gamma_i \cap A_i) &= -\chi(\Gamma_i \cap A_i) + |\Gamma_i \cap A_i| \\
&\leq |D_i| d_p(G) + |\partial D_i|(L^2+1) - |D_i| + |\partial D_i|(L^2+2) \\
&= |D_i|(d_p(G) - 1 + h(X_i)(2L^2+3)) \\
&\leq \tfrac{1}{2}[G : G_i](d_p(G) - 1 + h(X_i)(2L^2+3)) \\
&\leq \tfrac{1}{2}(1+\epsilon)[G : G_i](d_p(G) - 1) \text{ when } h(X_i) \text{ is sufficiently small} \\
&\leq \tfrac{1}{2}(1+\epsilon)^2(d_p(G_i) - 1).
\end{aligned}$$

A similar sequence of inequalities holds for $d_p(\Gamma_i \cap B_i)$ but with $|D_i|$ replaced throughout by $|D_i^c|$ and with $\frac{1}{2}$ replaced throughout by $\frac{3}{4}$. Here, we are using the fact that $|D_i^c| \leq \frac{3}{4}[G : G_i]$. So, when $h(X_i)$ is sufficiently small, $\mathrm{Im}(H_1(\Gamma_i \cap A_i; \mathbb{F}_p))$ and $\mathrm{Im}(H_1(\Gamma_i \cap B_i; \mathbb{F}_p))$ each have dimension at most $\frac{3}{4}(1+\epsilon)^2 d_p(G_i)$. Note that $\frac{3}{4}(1+\epsilon)^2 < \frac{5}{6}$, by our assumption that $\epsilon < \sqrt{10}/3 - 1$. We saw above that the sum of $\mathrm{Im}(H_1(\Gamma_i \cap A_i; \mathbb{F}_p))$ and $\mathrm{Im}(H_1(\Gamma_i \cap B_i; \mathbb{F}_p))$ has codimension at most $|\partial D_i|(L^2+2)$, which equals $h(X_i)|D_i|(L^2+2)$, and this is small compared with $d_p(G_i)$. Therefore, when $h(X_i)$ is sufficiently small, $\mathrm{Im}(H_1(\Gamma_i \cap A_i; \mathbb{F}_p))$ and $\mathrm{Im}(H_1(\Gamma_i \cap B_i; \mathbb{F}_p))$ each have dimension at least $d_p(G_i)/6$. Since $H_1(\Gamma_i \cap A_i; \mathbb{F}_p) \to H_1(K_i; \mathbb{F}_p)$ factors through $H_1(A_i; \mathbb{F}_p)$, this must also have dimension at least $d_p(G_i)/6$. When $h(X_i)$ is sufficiently small, this is significantly more than $d_p(C_i)$. Thus, we deduce that, when i is sufficiently large, $d_p(C_i)$ is less than both $d_p(A_i) - 1$ and $d_p(B_i) - 1$. The mod p Cheeger constant of K_i is therefore less than $1/2$. Corollary 3.8 then implies that G_i admits a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$. Furthermore, some normal subgroup of G_i with index a power of p admits a surjective homomorphism onto a non-abelian free group. Hence, G is large. □

6. Error-correcting codes and large groups

Let G be a finitely presented group, and let $\{G_i\}$ be a nested sequence of finite index subgroups. Suppose that $\{G_i\}$ has linear growth of mod p homology. Does this imply that G is large? Let K be a finite 2-complex with fundamental group G, and let K_i be the covering space corresponding to the subgroup G_i. Then one might suspect that the sheer number of elements of $H^1(K_i; \mathbb{F}_p)$ might force the existence of two regular mod p cocycles that are disjoint and whose union is non-separating. Hence, by Theorem 3.6, G_i would admit a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$, establishing (i), at least when p is odd. However, it appears not to be possible to turn this reasoning into a proof, due to the intervention of error-correcting codes. In this section, we explain how these codes play a rôle.

We first introduce a new concept: the *relative size* of a cohomology class. Let K be a finite cell complex. For a cellular 1-dimensional cocycle c on K, let its support $\mathrm{supp}(c)$ be those 1-cells with non-zero evaluation under c. For an element $\alpha \in H^1(K;\mathbb{F}_p)$, consider the following quantity. The *relative size* of α is

$$\frac{\min\{|\mathrm{supp}(c)| : c \text{ is a cellular cocycle representing } \alpha\}}{\text{Number of 1-cells of } K}.$$

The relevance of this quantity is apparent in the following result.

Theorem 6.1. *Let K be a finite connected 2-complex, and let $\{K_i \to K\}$ be a collection of finite-sheeted covering spaces. Suppose that $\{\pi_1(K_i)\}$ has linear growth of mod p homology for some prime p. Then one of the following must hold:*

(i) $\pi_1(K_i)$ admits a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$ for infinitely many i, and $\pi_1(K)$ is large, or

(ii) there is some $\epsilon > 0$ such that the relative size of any non-trivial class in $H^1(K_i;\mathbb{F}_p)$ is at least ϵ, for all i.

The following will be useful in the proof of this.

Lemma 6.2. *Let K be a finite 2-complex. Let M be the maximal valence of any its 1-cells. Let c be a cellular cocycle representing a class α in $H^1(K;\mathbb{F}_p)$, for some prime p. Then α is represented by a regular mod p cocycle Γ containing at most $M|\mathrm{supp}(c)|$ edges, at most $|\mathrm{supp}(c)|$ edge vertices.*

Proof. Recall from Proposition 3.3 the construction of a regular mod p cocycle Γ from the cellular cocycle c. Each 1-cell in $\mathrm{supp}(c)$ is assigned an edge vertex of Γ. Each such edge vertex is adjacent to at most M edges. Also, every edge is adjacent to some edge vertex. Hence, Γ contains at most $M|\mathrm{supp}(c)|$ edges. □

Proof of Theorem 6.1. Suppose that (ii) does not hold. Then there exist non-trivial elements of $H^1(K_i;\mathbb{F}_p)$ with arbitrarily small relative size. Let Γ be a regular mod p cocycle representing one of these cohomology classes, and let $e(\Gamma)$ and $ev(\Gamma)$ denote its number of edges and edge vertices. By Lemma 6.2, we may ensure that the ratios of $e(\Gamma)$ and $ev(\Gamma)$ to the number of 1-cells of K_i is arbitrarily close to zero. (Note that the maximal valence of the 1-cells of K_i is the same for all i.) By Addendum 3.4, we may arrange that Γ is non-separating, without increasing its number of edges and edge vertices. Note that $e(\Gamma)$ forms an upper bound on $d_p(\Gamma)$. Let $N(\Gamma)$ be a thin regular neighbourhood of Γ. Then $\partial N(\Gamma)$ is a graph with as many edges as Γ, and at most $2ev(\Gamma)$ components. Thus, $d_p(\partial N(\Gamma))$ is bounded above by $e(\Gamma)$. We are assuming that $\{\pi_1(K_i)\}$ has linear growth of mod p homology. Hence, the ratios of $e(\Gamma)$ and $ev(\Gamma)$ to

$d_p(K_i)$ are both arbitrarily close to zero. Consider the Mayer-Vietoris sequence applied to $N(\Gamma)$ and $K_i - \mathrm{int}(N(\Gamma))$:

$$H_1(\partial N(\Gamma); \mathbb{F}_p) \to H_1(N(\Gamma); \mathbb{F}_p) \oplus H_1(K_i - \mathrm{int}(N(\Gamma)); \mathbb{F}_p)$$
$$\to H_1(K_i; \mathbb{F}_p) \to H_0(\partial N(\Gamma); \mathbb{F}_p).$$

Now, the dimensions of $H_1(\partial N(\Gamma); \mathbb{F}_p)$, $H_1(N(\Gamma); \mathbb{F}_p)$ and $H_0(\partial N(\Gamma); \mathbb{F}_p)$ are all small compared with $d_p(K_i)$. Hence, the ratio of $d_p(K_i)$ and $d_p(K_i - \mathrm{int}(N(\Gamma)))$ tends to 1. So, the ratio of $d_p(\partial N(\Gamma))$ and $d_p(K_i - \mathrm{int}(N(\Gamma)))$ tends to zero. Therefore, the map $H^1(K_i - \mathrm{int}(N(\Gamma)); \mathbb{F}_p) \to H^1(\partial N(\Gamma); \mathbb{F}_p)$ induced by inclusion has non-trivial kernel. Subdivide K_i so that $N(\Gamma)$ is a subcomplex. By Proposition 3.5, there is a regular mod p cocycle Γ' in $K_i - \mathrm{int}(N(\Gamma))$ such that Γ' is non-separating in $K_i - \mathrm{int}(N(\Gamma))$ and disjoint from $\partial N(\Gamma)$. So, $\Gamma \cup \Gamma'$ is non-separating in K_i. By Theorem 3.6, $\pi_1(K_i)$ admits a surjective homomorphism onto $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$. When $p > 2$, this gives (i). So, let us suppose now that $p = 2$. We may assume that the kernel of $H^1(K_i - \mathrm{int}(N(\Gamma)); \mathbb{F}_p) \to H^1(\partial N(\Gamma); \mathbb{F}_p)$ has dimension at least two. Pick two linearly independent elements in this kernel, and consider the cover of $K_i - \mathrm{int}(N(\Gamma))$, with order 4, dual to these two elements. This extends to a cover $\tilde{K}_i$ of K_i. The inverse image of Γ in $\tilde{K}_i$ has at least 4 components. The complement of their union is, by construction, connected. So, by Theorem 3.6, $\pi_1(\tilde{K}_i)$ admits a surjective homomorphism onto $*^4(\mathbb{Z}/2\mathbb{Z})$, and hence G is large. □

Theorem 6.1 leads naturally to the following question: how can the relative sizes of the non-trivial cohomology classes of K_i not have zero infimum? The answer is: when they form error correcting codes with large Hamming distance.

Recall that a *linear code* is a subspace $\mathcal{C}$ of a finite vector space $(\mathbb{F}_p)^n$. The *rate* r of the code is $\dim(\mathcal{C})/n$. The *Hamming distance* d of $\mathcal{C}$ is the smallest number of non-zero co-ordinates in a non-trivial element of $\mathcal{C}$. One of the main goals of coding theory is to construct codes with large rate and large Hamming distance. Specifically, an infinite collection of codes is known as *asymptotically good* if r/n and d/n are both bounded away from zero. The construction of asymptotically good sequences of codes is an interesting and difficult problem. They were first proved to exist using probabilistic methods, but explicit constructions are now available ([6],[17]).

In our situation, the ambient vector space V of the code is the space of cellular 1-dimensional mod p cochains on K_i. It has a natural basis, where each basis element is supported on a single 1-cell. Hence, its dimension is equal to the number of 1-cells of K_i. Pick a basis for $H^1(K_i; \mathbb{F}_p)$, and represent each element by a cellular cocycle. The subspace of V spanned by these cocycles we view as the code $\mathcal{C}_i$. Let n_i be the dimension of V, and let r_i and d_i be the rate and Hamming distance of $\mathcal{C}_i$. The assumption that $\{\pi_1(K_i)\}$ has linear growth of mod p homology is equivalent to the statement that r_i/n_i is bounded away from zero. The quantity d_i/n_i simply measures the smallest ratio between the support size of a non-trivial cocycle in $\mathcal{C}_i$ and the number of 1-cells of K_i. Hence, it is an upper bound for the smallest relative size of a non-trivial class in $H^1(K_i; \mathbb{F}_p)$. Thus, we have the following.

Theorem 6.3. *Let K be a finite connected 2-complex, and let $\{K_i \to K\}$ be a collection of finite-sheeted covering spaces. Suppose that $\{\pi_1(K_i)\}$ has linear growth of mod p homology for some prime p. Suppose also that there is some $\epsilon > 0$ such that the relative size of any non-trivial class in $H^1(K_i; \mathbb{F}_p)$ is at least ϵ. Then the codes $\mathcal{C}_i$ described above are asymptotically good.*

Combining Theorems 6.1 and 6.3, we have the following result.

Theorem 6.4. *Let K be a finite connected 2-complex, and let $\{K_i \to K\}$ be a collection of finite-sheeted covering spaces. Suppose that $\{\pi_1(K_i)\}$ has linear growth of mod p homology for some prime p. Then either $\pi_1(K)$ is large or the codes $\mathcal{C}_i$ described above are asymptotically good.*

7. Finitely generated versus finitely presented

In Theorem 1.1, we assumed that G was finitely presented. The remaining hypotheses make sense when G is only finitely generated. So, it is natural to enquire whether Theorem 1.1 remains true when the hypothesis of being finite presented is weakened to being finitely generated. In this section, we show that the answer is 'no', by analysing a collection of examples. These were suggested to the author by Jim Howie. Using the same examples, we also show that the hypothesis of finite presentability cannot be weakened in Theorem 1.3 and Corollary 1.8. The argument here was supplied by Alex Lubotzky.

The groups we will study are the generalised lamplighter groups $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$. (When $p = 2$, this is the usual lamplighter group.) Each is a semi-direct product $(\oplus_{-\infty}^{\infty}(\mathbb{Z}/p\mathbb{Z})) \rtimes \mathbb{Z}$. Here, an arbitrary element of $\oplus_{-\infty}^{\infty}(\mathbb{Z}/p\mathbb{Z})$ is required to have only finitely many non-zero co-ordinates. To define the semi-direct product, we must specify the action of $\mathbb{Z}$ on $\oplus_{-\infty}^{\infty}(\mathbb{Z}/p\mathbb{Z})$. The action of an integer n in $\mathbb{Z}$ on $\oplus_{-\infty}^{\infty}(\mathbb{Z}/p\mathbb{Z})$ simply shifts the indexing set n to the right. These groups are finitely generated but not finitely presented [1]. Indeed, each is generated by two elements a and b, where a shifts the indexing set one to the right, and b lies in $\oplus_{-\infty}^{\infty}(\mathbb{Z}/p\mathbb{Z})$, with a single non-zero entry which takes the value

1 in the zero copy of $\mathbb{Z}/p\mathbb{Z}$.

Proposition 7.1. *The generalised lamplighter group $G = (\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$ has a nested sequence of finite index normal subgroups $\{G_i\}$, each with index a power of p, with the following properties:*

(i) G does not have Property (τ) with respect to $\{G_i\}$, and

(ii) $\{G_i\}$ has linear growth of mod p homology.

But G is not large.

Proof. By the definition of the semi-direct product, G admits a surjective homomorphism ϕ onto $\mathbb{Z}$. Let G_i be $\phi^{-1}(p^i\mathbb{Z})$. Then G_i is normal and has index p^i. Clearly, these subgroups are nested.

(i): Lemma 2.1 states that G has property (τ) with respect to G_i if and only if $\mathbb{Z}$ has property (τ) with respect to $\{p^i\mathbb{Z}\}$. But, we have already seen in the example in Section 2 that this is not the case.

(ii): We claim that $d_p(G_i) \geq [G : G_i]$. To do this, we will find p^i linearly independent homomorphisms $G_i \to \mathbb{F}_p$. Now, G_i is the subgroup of G generated by $\oplus_{-\infty}^{\infty}(\mathbb{Z}/p\mathbb{Z})$ and a^{p^i}. Each homomorphism will send a^{p^i} to the identity. To define such a homomorphism, it suffices to define a homomorphism $\oplus_{-\infty}^{\infty}(\mathbb{Z}/p\mathbb{Z}) \to \mathbb{F}_p$ which is invariant under the action of a^{p^i}. Let j be an integer between 0 and $p^i - 1$. Define

$$\begin{aligned} \oplus_{-\infty}^{\infty}(\mathbb{Z}/p\mathbb{Z}) &\xrightarrow{\phi_j} \mathbb{F}_p \\ (n_k)_{k=-\infty}^{\infty} &\mapsto \sum_{k=-\infty}^{\infty} n_{p^i k+j}. \end{aligned}$$

These are clearly linearly independent, as required.

Finally, G is not large, because it is soluble. □

We now show that Theorem 1.3 does not remain true for finitely generated, infinitely presented groups.

Proposition 7.2. *The generalised lamplighter group $G = (\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$ does not have Property (τ) with respect to any infinite collection of finite index subgroups.*

However, as we have seen in Proposition 7.1, G does have a nested sequence of normal subgroups, each with index a power of p, that have linear growth of mod p homology. Hence, by Corollary 1.8, the pro-p completion of G has exponential subgroup growth.

Proof of Proposition 7.2. Now, G is amenable, and Theorem 3.1 of [15] asserts that a finitely generated amenable group does not have Property (τ) with respect to any infinite family of finite index normal subgroups. However, the assumption of normality is not required in the proof of that theorem. The proposition now follows. □

I am grateful to Alex Lubotzky who informed me of his work with Weiss [15], which formed the basis for this proof.

8. Subgroup growth and linear growth of homology

Throughout this paper, the main focus has been on groups having a sequence of subnormal subgroups, each with index a power of a prime p, and with linear growth of mod p homology. In this section, we show how the existence of such a sequence of subgroups has equivalent characterisations in terms of subgroup growth.

For a group G, let $s_n(G)$ be the number of subgroups with index at most n, and let $a_n(G)$ be the number of subgroups with index precisely n. Let $s_n^{\lhd\lhd}(G)$ and $a_n^{\lhd\lhd}(G)$ be the number of subnormal subgroups with index at most n and precisely n, respectively. A group is said to have *(at least) exponential subgroup growth* if

$$\limsup_n \frac{\log s_n(G)}{n} > 0.$$

If p is a prime, let $\hat{G}_{(p)}$ be the pro-p completion of G. It turns out that the subgroup growth of a finitely generated pro-p group is at most exponential. In other words, $\limsup_n \log s_n(\hat{G}_{(p)})/n$ is finite (Theorem 3.6 of [14]).

The following is a stronger version of Theorem 1.7, which was stated in the Introduction.

Theorem 8.1. *Let G be a finitely generated group, and let p be a prime. Then the following are equivalent:*

(i) G has an infinite nested sequence of subnormal subgroups, each with index a power of p, and with linear growth of mod p homology;

(ii) $\hat{G}_{(p)}$ has exponential subgroup growth;

(iii) $\limsup_n (\log a_{p^n}^{\lhd\lhd}(G))/p^n > 0$;

(iv) $\inf_{n\geq 1}(\log a_{p^n}^{\lhd\lhd}(G))/p^n > 0$.

Proof. (ii)⇔(iii): There is a one-one correspondence between subnormal subgroups of G with index p^n and subgroups of $\hat{G}_{(p)}$ with index p^n. In addition, any finite index subgroup of $\hat{G}_{(p)}$ has index a power of p. Thus, the equivalence of (ii) and (iii) is consequence of the following general fact. Any sequence of non-negative integers c_j has at least exponential growth (that is, $\limsup_j (\log c_j)/j > 0$) if and only if the partial sums $\sum_{i=0}^{j} c_i$ have at least exponential growth. In this case, the sequence c_j is $a_j^{\lhd\lhd}(G)$ if j is a power of p, and zero otherwise.

(i)⇒(iv): Suppose that G has a sequence of subgroups $G = G_1 \rhd G_2 \rhd \dots$ such that G_n/G_{n+1} is a non-trivial finite p-group for each n, and with linear growth of mod p homology. Let λ be $\inf_n (d_p(G_n) - 1)/[G : G_n]$, the mod p homology gradient, which is therefore positive. Now, any finite p-group has a subnormal series, where successive quotients are cyclic of order p. Thus, by refining the sequence $\{G_n\}$ if necessary, we may assume that each G_n/G_{n+1} is cyclic of order p. By Proposition 5.1, $(d_p(G_n)-1)/[G : G_n]$ is a non-increasing function of n. Thus, $\inf_n (d_p(G_n)-1)/[G : G_n]$ is still λ. Any normal subgroup of G_n with index p arises as the kernel of a non-trivial homomorphism $G_n \to \mathbb{Z}/p\mathbb{Z}$. There are $p^{d_p(G_n)} - 1$ such homomorphisms. The number of homomorphisms with a given kernel is $p - 1$. Thus, there are $(p^{d_p(G_n)} - 1)/(p - 1)$ normal subgroups of G_n with index p. Each gives a subnormal subgroup of G with index $[G : G_n]p = p^n$. Hence, when $n \geq 1$, $a_{p^n}^{\lhd\lhd}(G)$ is at least

$$\frac{p^{\lambda p^{n-1}+1} - 1}{p - 1},$$

and so we deduce that $\liminf_n (\log a_{p^n}^{\lhd\lhd}(G))/p^n$ is positive. Finally, note that $a_{p^n}^{\lhd\lhd}(G)$ is always more than 1, when $n \geq 1$, and so $(\log a_{p^n}^{\lhd\lhd}(G))/p^n$ is strictly positive. Thus, we deduce (iv).

(iv)⇒(iii): This is trivial.

(iii)⇒(i): Define

$$r_n = \max\{d_p(H) : H \lhd\lhd G \text{ and } [G : H] = p^n\}.$$

Let us suppose that (i) does not hold. We claim that $\limsup_n r_n/p^n = 0$. For otherwise, $\limsup_n r_n/p^n$ is positive, and therefore so is $\limsup_n (r_n - 1)/p^n$. Let λ be this latter value. Note that, by Proposition 5.1, $(r_n - 1)/p^n$ is a non-increasing function of n. Thus, λ is actually the infimum and limit of this sequence. Hence, for each n, there is a subnormal subgroup G_n, with index p^n such that $d_p(G_n) - 1 \geq \lambda p^n$. For each n, we may find a subnormal sequence

$$G = G_{n,1} \rhd G_{n,2} \rhd \dots \rhd G_{n,n} = G_n$$

such that $G_{n,i}/G_{n,i+1}$ is cyclic of order p for each i. Now, $(d_p(G_{n,i}) - 1)/p^i \geq \lambda$ by Proposition 5.1. Since G has only finitely many subgroups of index p, we may find a subsequence of the G_n where $G_{n,2}$ is a fixed group G_2. By passing to a further subsequence, we may assume that $G_{n,3}$ is a fixed group G_3, and so on. Thus, we obtain a sequence of subnormal subgroups $G = G_1 \triangleright G_2 \triangleright \ldots$, each with index p in its predecessor, and with linear growth of mod p homology. This is condition (i), which we are assuming does not hold. This contradiction proves the claim: $\limsup_n r_n/p^n = 0$. Hence, $\lim_{n\to\infty} \left(\sum_{i=0}^{n} r_i\right)/p^n = 0$. Now, any subnormal subgroup of G with index p^n is a normal subgroup of some subnormal subgroup of G with index p^{n-1}. Hence,

$$a^{\triangleleft\triangleleft}_{p^n}(G) \leq a^{\triangleleft\triangleleft}_{p^{n-1}}(G)p^{r_{n-1}}.$$

Thus, by induction,

$$a^{\triangleleft\triangleleft}_{p^n}(G) \leq p^{\sum_{i=0}^{n-1} r_i}.$$

Taking logs:

$$\log a^{\triangleleft\triangleleft}_{p^n}(G) \leq (\log p)\sum_{i=0}^{n-1} r_i.$$

Therefore,

$$\left(\log a^{\triangleleft\triangleleft}_{p^n}(G)\right)/p^n \to 0,$$

which means that (iii) does not hold, as required. □

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