Surface subgroups in dimension 3

Marc Lackenby

University of Oxford

Surfaces in 3-manifolds

Throughout: all surfaces are closed orientable and have positive genus.

Let S be a surface, and let M be a closed orientable 3-manifold.

A map $i\colon S \to M$ is π_1-injective if $i_*\colon \pi_1(S) \to \pi_1(M)$ is an injection.

M is irreducible if every embedded 2-sphere in M bounds a 3-ball.

M is Haken if it is irreducible and contains an embedded π_1-injective surface.

Equivalently: $\pi_1(M)$ splits as an amalgmated free product or HNN extension, but is not a free product or $\mathbb{Z}$.

Sample Theorem: [Waldhausen] If two closed Haken 3-manifolds have isomorphic π_1, they are homeomorphic.

Unfortunately, many 3-manifolds are non-Haken.

Some Conjectures

Suppose M is closed, orientable, irreducible and has infinite π_1

Conjecture: (The surface subgroup conjecture) $\pi_1(M)$ contains the fundamental group of a surface.

Equivalently: M contains an immersed π_1-injective surface.

$$\Uparrow$$

Conjecture: (Virtually Haken conjecture) M has a finite cover that is Haken.

$$\Uparrow$$

Conjecture: (Virtually positive b_1 conjecture) M has a finite cover with $b_1 > 0$.

Geometrisation

[Kneser, Milnor] Any closed orientable 3-manifold M is a connected sum of prime manifolds in a unique way.

[Jaco, Shalen, Johannson] Any closed orientable prime 3-manifold has a canonical collection of disjoint embedded π_1-injective tori which decompose it into atoroidal pieces (ie any embedded π_1-injective torus in one of these pieces is parallel to a boundary component).

[Thurston, Perelman] Each of the pieces is 'Seifert fibred' or 'hyperbolic'.

All of the above conjectures are known to hold unless M is a closed hyperbolic 3-manifold.

We then have the following stronger conjectures:

Stronger conjectures

Let M be a closed hyperbolic 3-manifold.

Conjecture: (Virtually infinite b_1 conjecture) M has a finite cover with b_1 arbitrarily large.

$$\Uparrow$$

Conjecture: (Largeness conjecture) M has a finite cover $\tilde{M}$ where $\pi_1(\tilde{M})$ has a non-abelian free quotient.

A group is large if it has a finite index subgroup with a non-abelian free quotient.

The surface subgroup conjecture

We'll focus on:

Surface subgroup conjecture: If M is a closed orientable irreducible 3-manifold and $\pi_1(M)$ is infinite, then $\pi_1(M)$ contains the fundamental group of a surface.

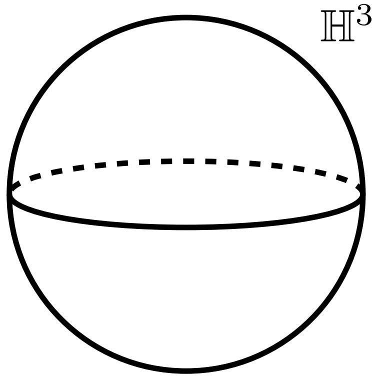
Equivalently: M contains an immersed π_1-injective surface.

Main Theorem 1.1: [L] Every arithmetic hyperbolic 3-manifold contains an immersed π_1-injective surface.

The main techniques

- 3-orbifolds
- Golod-Shafarevich inequality
- Perelman's solution to the geometrisation conjecture
- Cheeger constants
- The first eigenvalue of the Laplacian
- The critical exponent of Kleinian groups
- Some classical 3-manifold theory
- A little arithmetic machinery

Let $\mathbb{H}^3$ be hyperbolic 3-space.



$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$.

A discrete subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}^3)$ is a Kleinian group.

If Γ acts freely, then $\Gamma \backslash \mathbb{H}^3$ is an orientable hyperbolic 3-manifold.

Hyperbolic 3-orbifolds

If Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$, not necessarily acting freely, then $\Gamma\backslash\mathbb{H}^3$ is an orientable hyperbolic 3-orbifold O.

One keeps track not just of the underlying space $|O|$ but also the isotropy data.

ie, for $x \in O$, consider $\tilde{x} \in$ inverse image of x, and define the local group of x to be $\mathrm{Stab}_\Gamma(\tilde{x})$.

The singular locus is the set of points in O with non-trivial local group.

The local group of any $x \in O$ is a finite subgroup of $SO(3)$ ie:

- cyclic,
- dihedral (including $\mathbb{Z}_2 \times \mathbb{Z}_2$)
- A_4, S_4, A_5.

More generally, an n-dimensional orbifold is a space O, where for each point $x \in O$, there is an open neighbourhood U of x and a finite subgroup $G \leq \mathrm{O}(n)$ (called the local group of x) s.t.

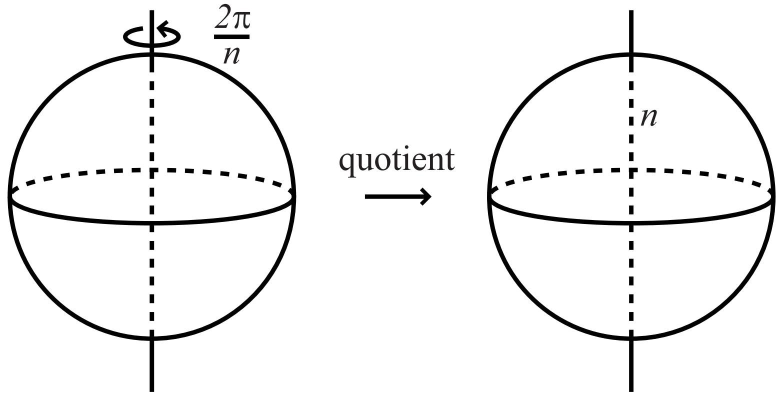
$$\begin{aligned} U &\xrightarrow{\cong} G\backslash\mathbb{R}^n \\ x &\mapsto 0 \end{aligned}$$

These neighbourhoods form 'charts' which must patch together correctly.

O is orientable if each copy of $\mathbb{R}^n$ has an orientation that G preserves, and these orientations patch together coherently under the chart transformations.

Example

Let Γ be the group generated by rotation of order n about a geodesic:

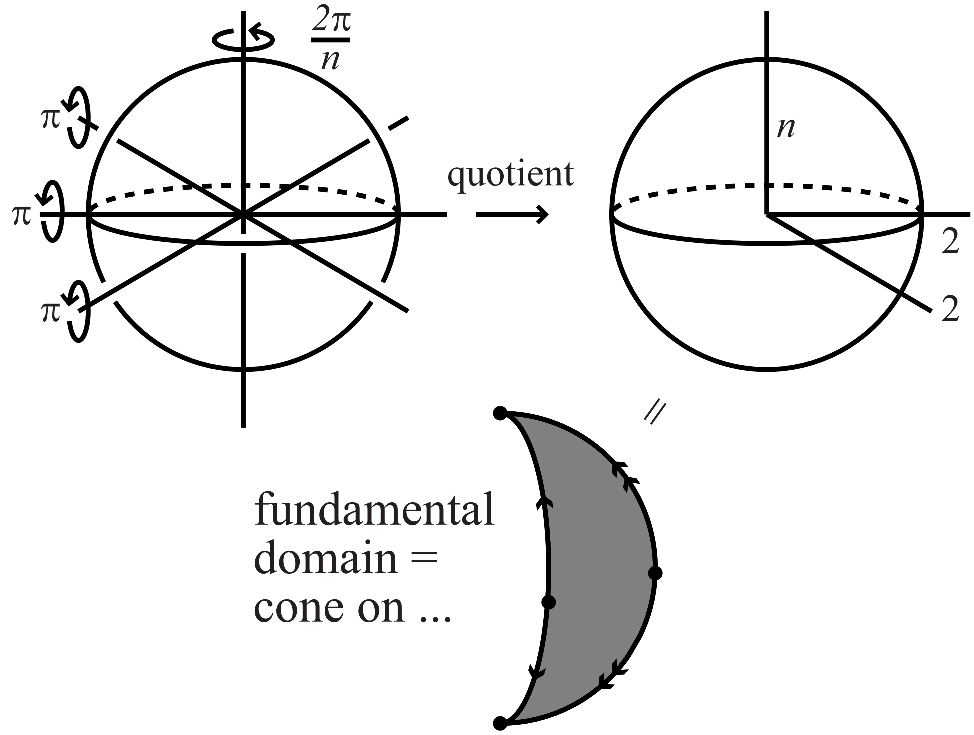


Then $\Gamma\backslash\mathbb{H}^3$ is a 3-ball.

Its singular locus is an arc.

Each singular point has cyclic local group of order n.

Dihedral local group:



In fact, when O is an orientable 3-orbifold, $|O|$ is a 3-manifold and $\text{sing}(O)$ is always a collection of simple closed curves and trivalent graphs.

Surface subgroups again

Main Theorem 1.2: [L] Any finitely generated Kleinian group Γ containing a finite non-cyclic subgroup is either finite, virtually free or contains a surface subgroup.

The main case is when Γ is co-compact.

Equivalently in this case: Any closed hyperbolic 3-orbifold that contains a singular vertex admits an immersed π_1-injective surface.

Commensurable groups

Two groups Γ_1 and Γ_2 are commensurable if there are finite index subgroups $\Gamma_1' \leq \Gamma_1$ and $\Gamma_2' \leq \Gamma_2$ such that $\Gamma_1' \cong \Gamma_2'$.

Γ_1 contains a surface subgroup iff Γ_2 does.

Theorem 1.3: [L-Long-Reid] Any arithmetic Kleinian group is commensurable with one that contains $\mathbb{Z}/2 \times \mathbb{Z}/2$.

We'll outline a proof of this in a later talk.

Note: Arithmetic Kleinian groups are neither finite nor virtually free.

Hence: 1.2 & 1.3 $\Rightarrow$ 1.1

Covering spaces of orbifolds

A covering map between orbifolds is a cts map $p\colon \tilde{O} \to O$ such that for each $x \in O$, there is an open neighbourhood U which is a copy of $G\backslash\mathbb{R}^n$ (where G is the local group of x) and each component of $p^{-1}(U)$ is a copy of $\tilde{G}\backslash\mathbb{R}^n$, for some subgroup $\tilde{G} \leq G$, and the restriction of p to this component is the canonical quotient map $\tilde{G}\backslash\mathbb{R}^n \to G\backslash\mathbb{R}^n$.

Example: For any discrete subgroup Γ in $\mathrm{Isom}(\mathbb{H}^3)$, $\mathbb{H}^3 \to \Gamma\backslash\mathbb{H}^3$ is a covering map.

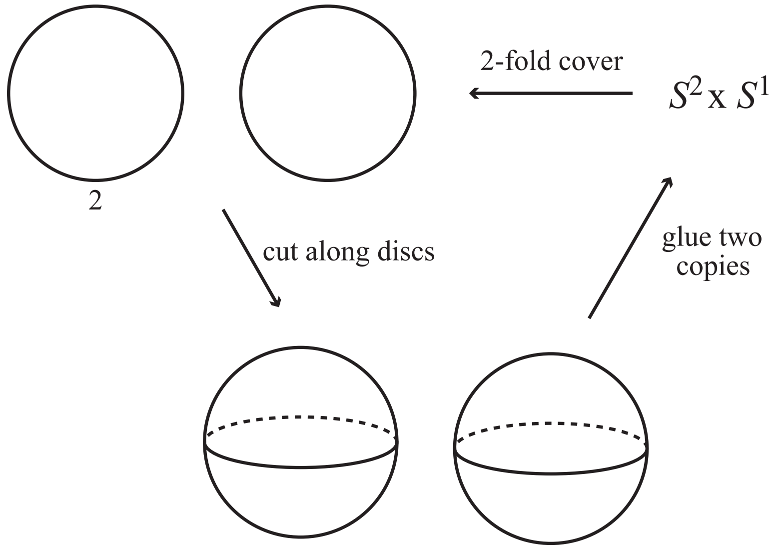
More generally, If Γ' is a subgroup of Γ, then

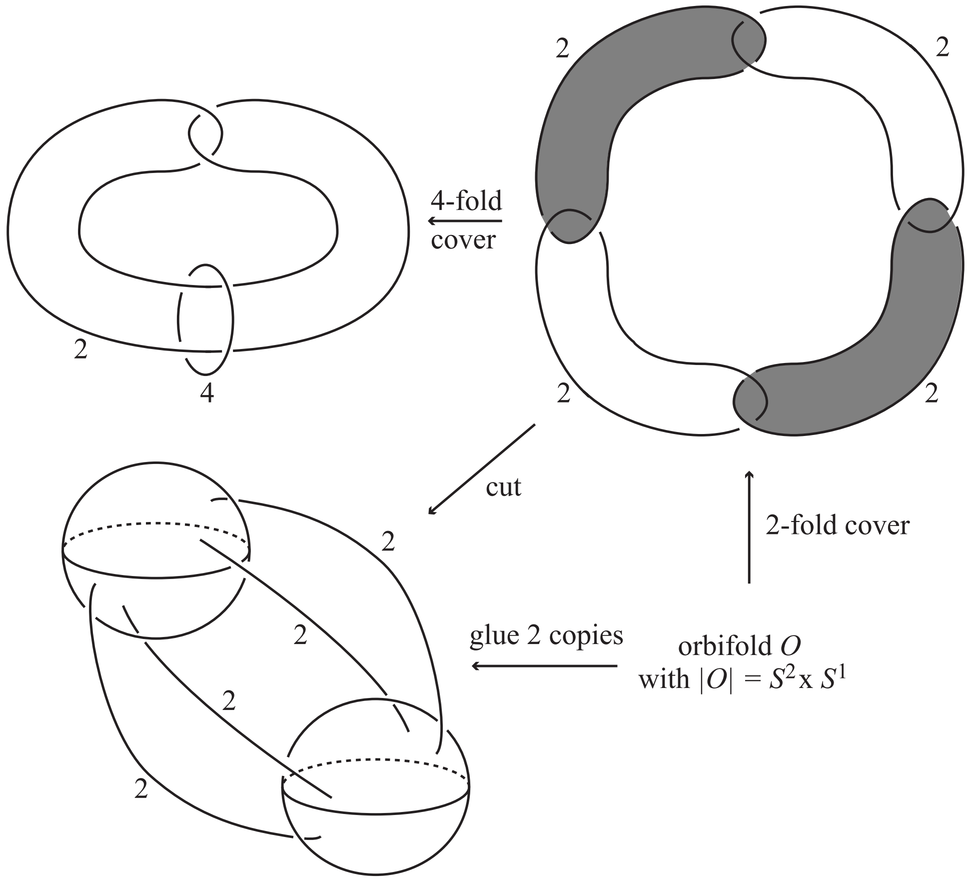
$$\Gamma'\backslash\mathbb{H}^3 \to \Gamma\backslash\mathbb{H}^3$$

is a covering map.

Fact: Any cover of a hyperbolic orbifold arises in this way.

Examples




2
4
4-fold
cover
2
2
2
2
cut
2-fold cover
2
2
2
2
glue 2 copies
orbifold O
with $|O| = S^2 \times S^1$

The fundamental group of an orbifold

Fact: Every orbifold O has a 'universal cover' $\tilde{O}$.

In the case of hyperbolic 3-orbifolds, it is $\mathbb{H}^3$.

The covering map $\tilde{O} \to O$ is obtained by quotienting $\tilde{O}$ by a group of covering transformations Γ.

Definition: The fundamental group $\pi_1(O) = \Gamma$.

So, when O is a hyperbolic orbifold $\Gamma \backslash \mathbb{H}^n$, $\pi_1(O) = \Gamma$.

General fact: There is a one-one correspondence

$$\{\text{subgroups of } \pi_1(O)\} \longleftrightarrow \{\text{covering spaces of } O\}$$

A method of computing $\pi_1(O)$

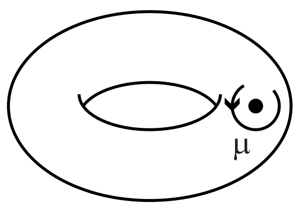
Suppose that O is orientable.

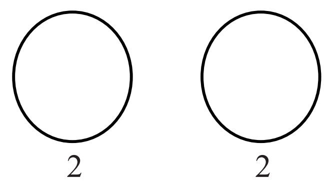
Let $S_1, \ldots, S_k$ be the codim 2 components of $\mathrm{sing}(O)$ with local groups of order $n_1, \ldots, n_k$.

Let $\mu_1, \ldots, \mu_k$ be meridian curves for $S_1, \ldots, S_k$. Then:

$$\pi_1(O) = \frac{\pi_1(|O| - \mathrm{sing}(O))}{\langle\langle \mu_1^{n_1}, \ldots, \mu_k^{n_k} \rangle\rangle}.$$

Examples:





$\pi_1(O) = \pi_1(\text{punc torus})/\langle\langle \mu^2 \rangle\rangle.$

$\pi_1(O) = \mathbb{Z}/2 * \mathbb{Z}/2$

Homology of 3-orbifolds

If O is an orbifold, define

$$\begin{aligned} H_1(O;\mathbb{Z}) &= H_1(\pi_1(O);\mathbb{Z}) \\ &= \pi_1(O)/[\pi_1(O),\pi_1(O)] \\ H_1(O;\mathbb{R}) &= H_1(\pi_1(O);\mathbb{R}) \\ &\cong H_1(|O|;\mathbb{R}) \text{ when } O \text{ is orientable} \\ b_1(O) &= \dim(H_1(O;\mathbb{R})) \\ H_1(O;\mathbb{F}_p) &= H_1(\pi_1(O);\mathbb{F}_p) \\ &= \pi_1(O)/([\pi_1(O),\pi_1(O)]\pi_1(O)^p) \\ d_p(O) &= \dim(H_1(O;\mathbb{F}_p)) \end{aligned}$$

More generally, for a group Γ,

$$d_p(\Gamma) = \dim(H_1(\Gamma;\mathbb{F}_p)).$$

Homology of 3-orbifolds

Important classical fact: If M is a compact orientable 3-manifold, $b_1(M) \geq b_1(\partial M)/2$.

An orbifold version:

Lemma 1.4: If O is a compact orientable 3-orbifold, and each arc and circle of $\mathrm{sing}(O)$ has order a prime p, then

$$\dim\, H_1(O; \mathbb{F}_p) \geq b_1(\mathrm{sing}(O)).$$

Proof:

Let $X = O - \mathrm{int}(N(\mathrm{sing}(O)))$.

X is a compact orientable 3-manifold $\Rightarrow$

$$b_1(X) \geq b_1(\partial X)/2 \geq b_1(\mathrm{sing}(O)).$$

Killing $\mu_1^p, \dots, \mu_k^p$ doesn't change $H_1(\ \ ; \mathbb{F}_p)$. $\square$