Surface subgroups
in dimension 3

Lecture 2
Recall:

**Main Theorem 1.2:** [L] Any finitely generated Kleinian group $\Gamma$ containing a finite non-cyclic subgroup is either finite, virtually free or contains a surface subgroup.

**Lemma 1.4:** If $O$ is a compact orientable 3-orbifold, and each arc and circle of $\text{sing}(O)$ has order a prime $p$, then

$$\dim H_1(O; \mathbb{F}_p) \geq b_1(\text{sing}(O)).$$
**Endgame of proof of 1.2**

Let $\Gamma$ be a Kleinian group with a finite non-cyclic subgroup.

**Simplifying assumptions:**

1. $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq \Gamma$.

2. $\Gamma$ is cocompact

Then $O = \Gamma \backslash \mathbb{H}^3$ is a closed hyperbolic 3-orbifold.

**Goal:** find an infinite covering space $O_i$ of $O$, containing a compact 3-dimensional suborbifold $N_i$ such that:

1. every arc and circle of $\text{sing}(N_i)$ has order 2;

2. $b_1(\text{sing}(N_i)) > d_2(\partial N_i)$

By Lemma 1.4, $d_2(N_i) > d_2(\partial N_i)$. 
So, \( \ker H^1(N_i; \mathbb{F}_2) \to H_1(\partial N_i; \mathbb{F}_2) \) is non-trivial.

Let \( S_i \) be a surface properly embedded in \( N_i - \text{sing}(N_i) \) dual to a non-trivial element of this kernel, and that is disjoint from \( \partial N_i \).

Let \( \tilde{O}_i \) be the 2-fold cover of \( O \) dual to \( S_i \).
This has at least two ends.

We may find a finite manifold cover $M_i$ of $\tilde{O}_i$.

This also has at least two ends.

Now apply ...

**Lemma 1.5**: Let $M$ be an orientable hyperbolic 3-manifold with at least 2 ends. Then $\pi_1(M)$ contains a surface subgroup.

**Proof:**

Let $S$ be a closed orientable surface separating two ends of $M$.

Compress $S$ as much as possible to $\overline{S}$:
$\overline{S}$ is incompressible:

**Old theorem:** Any properly embedded orientable incompressible surface is $\pi_1$-injective.
Some component of $\overline{S}$ still separates two ends of $M$.

It’s $\pi_1$-injective.

It’s not a sphere, because $M$ is irreducible (as $M$ is hyperbolic).

Hence, $\pi_1(M)$ contains a surface subgroup. □ (1.5)

So, surface subgroup $\leq \pi_1(M_i) \leq \pi_1(O)$. □ (1.2)
THREE MAIN THEOREMS

Let $\Gamma$ be a cocompact Kleinian group with a finite non-cyclic subgroup.

Let $O = \Gamma \setminus \mathbb{H}^3$.

Simplifying assumption: $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq \Gamma$.

**Theorem 2.1:** $O$ has a finite cover $\tilde{O}$ s.t.

1. $\tilde{O}$ has at least one singular vertex;
2. every arc and simple closed curve of $\text{sing}(\tilde{O})$ has order 2;
3. $\pi_1(|\tilde{O}|)$ is infinite

Let $M = |\tilde{O}|$. 
**Theorem 2.2:** If a closed orientable 3-manifold $M$ has infinite $\pi_1$, then either

1. $M$ is hyperbolic; or
2. $M$ has a finite cover $\tilde{M}$ with $b_1 > 0$.

In case 2, there is an induced finite cover of $\tilde{O}$ with underlying space $\tilde{M}$. So, its $\pi_1 \to \pi_1(\tilde{M}) \to \mathbb{Z}$.

So, wlog $M$ is hyperbolic.

**Theorem 2.3:** [L-Long-Reid] Any closed hyperbolic 3-manifold $M$ has a sequence of infinite covers $M_i$ s.t. $h(M_i) \to 0$.

Here $h(M_i)$ is the ‘Cheeger constant’ of $M_i$. 

**Cheeger constants**

Let $M$ be a complete Riemannian manifold.

If $M$ has finite volume, then its **Cheeger constant** $h(M)$ is

$$
\inf_S \left\{ \frac{\text{Area}(S)}{\min\{\text{Vol}(M_1), \text{Vol}(M_2)\}} \right\},
$$

as $S$ ranges over all embedded codimension 1 submanifolds that separate $M$ into $M_1$ and $M_2$. 
If $M$ has infinite volume, then its **Cheeger constant** $h(M)$ is

$$
\inf_S \left\{ \frac{\text{Area}(S)}{\text{Vol}(M_1)} \right\},
$$

as $S$ ranges over all embedded codimension 1 submanifolds that bound a finite volume submanifold $M_1$. 

![Diagram of submanifolds $M_1$ and $S$](image)
Let $X$ be a graph, with vertex set $V(X)$.

For $A \subseteq V(X)$, $\partial A$ is the set of edges with one endpoint in $A$ and one endpoint not in $A$.

If $V(X)$ is finite,

$$h(X) = \min \left\{ \frac{|\partial A|}{|A|} : A \subset V(X), 0 < |A| \leq |V(X)|/2 \right\}.$$
If $V(X)$ is infinite,

$$h(X) = \inf \left\{ \frac{\partial A}{|A|} : A \subset V(X), 0 < |A| < \infty \right\}.$$

Let

$M = \text{closed Riemannian manifold}$

$\Gamma = \pi_1(M)$

$S = \text{finite generating set for } \Gamma$

$M_i = \text{covering space of } M$

$\Gamma_i = \pi_1(M_i)$

$X_i = \text{coset diagram of } \Gamma/\Gamma_i \text{ w.r.t. } S$

**Theorem 2.4:** There are constants $c, C > 0$ s.t. for all covers $M_i \to M$,

$$c \ h(X_i) \leq h(M_i) \leq C \ h(X_i).$$
Let $\Gamma$ be a cocompact Kleinian group with a finite non-cyclic subgroup.
Let $O = \Gamma \setminus \mathbb{H}^3$.

**Simplifying assumption:** $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq \Gamma$.

2.1 $\Rightarrow$ we may pass to a finite cover $\tilde{O}$ s.t.
1. $\tilde{O}$ has at least one singular vertex;
2. every arc and simple closed curve of $\text{sing}(\tilde{O})$ has order 2;
3. $\pi_1(|\tilde{O}|)$ is infinite

Let $M = |\tilde{O}|$.

2.2 $\Rightarrow$ wlog $M$ is hyperbolic.
Let $T$ be a triangulation of $M$ with one vertex. Wlog, this vertex is a singular vertex of $\tilde{\mathcal{O}}$.

Its edges (when oriented) $\longrightarrow$ a generating set $S$ for $\pi_1(M)$.

In the interior of each edge, pick a ‘midpoint’.
In each face, pick three arcs running between the midpoints.
In each tetrahedron, pick triangles and squares with these arcs as edges:

Wlog each triangle and square intersects $\text{sing}(\tilde{\mathcal{O}})$ transversely.
2.3 ⇒ $M$ has covers $M_i$ with $h(M_i) \to 0$.

Let $X_i = \text{coset diagram of } \pi_1(M)/\pi_1(M_i) \text{ w.r.t. } S$.

$X_i = 1$-skeleton of $M_i$.

2.4 ⇒ $h(X_i) \to 0$.

Let $A_i$ be a finite subset of $V(X_i)$ s.t.

\[
\frac{|\partial A_i|}{|A_i|} \to 0 \text{ as } i \to \infty.
\]
Construction of $N_i$:

Let $|\partial N_i| \cap X_i = \partial A_i$.
Join up these points using lifts of arcs, triangles and squares.
This bounds a compact 3-dimensional suborbifold $N_i$.

**Must check:** $b_1(\text{sing}(N_i)) > d_2(\partial N_i)$.

$\text{sing}(N_i)$ is a graph with:
- $\sim |A_i|$ trivalent vertices;
- $\preceq |\partial A_i|$ univalent vertices.

As $|\partial A_i|/|A_i| \to 0$, $b_1(\text{sing}(N_i)) \sim |A_i|$. 

$d_2(|\partial N_i|) \preceq |\partial A_i|$

$|\text{sing}(\partial N_i)| \preceq |\partial A_i|$

$\Rightarrow d_2(\partial N_i) \preceq |\partial A_i|$

So, for $i >> 0$, $b_1(\text{sing}(N_i)) > d_2(\partial N_i)$, as required.
Cheeger constants

Theorem 2.3: [L-Long-Reid] Let $M$ be a closed hyperbolic 3-manifold. Then $M$ has infinite-sheeted covers $M_i$ such that $h(M_i) \to 0$.

This is a consequence of:

Theorem 2.5: [Bowen] $\Gamma = \pi_1(M)$ has a sequence of finitely generated free subgroups $\Gamma_i$ such that $\delta(\Gamma_i) \to 2$.

Here $\delta(\Gamma_i) =$ the ‘critical exponent’ of $\Gamma_i$

Theorem 2.6: [Sullivan] 

$$\lambda_1(\Gamma_i \setminus \mathbb{H}^3) = \begin{cases} 
\delta(\Gamma_i) (2 - \delta(\Gamma_i)) & \text{if } \delta(\Gamma_i) \geq 1 \\
1 & \text{otherwise.}
\end{cases}$$

Here $\lambda(\Gamma_i \setminus \mathbb{H}^3) =$ the first eigenvalue of the Laplacian of $\Gamma_i \setminus \mathbb{H}^3$.

Theorem 2.7: [Cheeger] For any complete Riemannian manifold $M_i$, $\lambda_1(M_i) \geq h(M_i)^2/4$. 


Let $M$ be a closed $n$-dimensional Riemannian manifold.

Let $C^\infty(M)$ be the smooth functions $M \to \mathbb{R}$.

There is an inner product on $C^\infty(M)$:

$$\langle f, g \rangle = \int_M f g \, d\text{vol}.$$ 

Let $*$ be the Hodge star operator on differential forms on $M$:

$$*: \Omega^k(M) \to \Omega^{n-k}(M)$$

If $dx_1, \ldots, dx_n$ forms an orthonormal basis at a point of $T^*(M)$. Then at this point

$$*(dx_1 \wedge \ldots \wedge dx_k) = dx_{k+1} \wedge \ldots \wedge dx_n.$$
Then there is an inner product on differential $k$-forms:

$$\langle \omega_1, \omega_2 \rangle = \int_M \omega_1 \wedge * \omega_2 \, d\text{vol}. $$

Stokes theorem $\Rightarrow$ for $\omega_1 \in \Omega^{k-1}(M), \omega_2 \in \Omega^k(M),$

$$\langle d\omega_1, \omega_2 \rangle = \langle \omega_1, (-1)^{k(n-k)} d * \omega_2 \rangle$$

And so $(-1)^{k(n-k)} d*$ is the formal adjoint of $d$.

We denote it by $d^*$.

$$
\begin{align*}
\Omega^0(M) & \leftrightarrow \Omega^1(M) & \leftrightarrow \ldots & \leftrightarrow \Omega^n(M) \\
\downarrow d^* & \quad & \quad & \quad \\
\Omega^0(M) & \leftrightarrow \Omega^1(M) & \leftrightarrow \ldots & \leftrightarrow \Omega^n(M)
\end{align*}
$$
The Laplacian is

\[ \Delta : C^\infty(M) \to C^\infty(M) \]

\[ f \mapsto d^* df \]

This is self-adjoint:

\[ \langle f, d^* dg \rangle = \langle df, dg \rangle = \langle d^* df, g \rangle \]

There is an orthonormal set of smooth eigenfunctions \( u_n \) such that any \( f \in C^\infty(M) \) is

\[ f = \sum_n \mu_n u_n. \]

Say that

\[ \Delta u_n = \lambda_n u_n, \]

where

\[ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots \]
Definition:

\[ \lambda_1(M) = \lambda_1 \]

Note: \( u_0 \) is the constant function \( 1/\sqrt{\text{vol}(M)} \).

Note:

\[ \langle f, f \rangle = \sum_n \mu_n^2. \]

\[ \langle df, df \rangle = \langle f, \Delta f \rangle = \sum_n \mu_n^2 \lambda_n. \]

So:

\[ \lambda_1(M) = \inf \left\{ \frac{\|df\|^2}{\|f\|^2} : f \in C^\infty(M) \text{ and } \int_M f = 0 \right\}. \]
Theorem 2.7: [Cheeger] For any complete Riemannian manifold $M$, \[ \lambda_1(M) \geq h(M)^2/4. \]

Proof: (when $M$ is closed)

Let $f$ be an eigenfunction with eigenvalue $\lambda_1$. Let
\[ M_+ = \{ x \in M : f(x) \geq 0 \} \]
\[ M_- = \{ x \in M : f(x) \leq 0 \} \]

Wlog, $\text{vol}(M_+) \leq \text{vol}(M_-)$. 

Focus on $M_+$ and take all integrals over $M_+$:

$$\lambda_1 = \frac{\int |df|^2}{\int |f|^2}$$

$$= \frac{\int |df|^2 \int |f|^2}{(\int f^2)^2}$$

$$\geq \frac{(\int |df| \cdot |f|)^2}{(\int f^2)^2} \text{ by Cauchy-Schwarz}$$

$$= \frac{1}{4} \left( \frac{\int |df^2|}{\int f^2} \right)^2$$

**Claim:** RHS $\geq h(M)^2/4$.

For any $t \geq 0$, let

$$A(t) = \text{Area}\{x \in M_+: f(x)^2 = t\}$$

$$V(t) = \text{Vol}\{x \in M_+: f(x)^2 \geq t\}$$
Co-area formula: \( \int |df^2| = \int A(t) \, dt \)

For each \( t \), \( A(t) \geq h(M)V(t) \), because at least half the volume lies in \( M_- \).
So:

\[
\int A(t) \, dt \geq h(M) \int V(t) \, dt
\]

\[
= h(M) \int_t \left( \int_{\{x: f(x)^2 \geq t\}} \, d\text{vol} \right) \, dt
\]

\[
= h(M) \int f^2 \, d\text{vol}
\]

So,

\[
\lambda_1(M) \geq \frac{1}{4} \left( \int |df^2| \right)^2 \geq \frac{h(M)^2}{4}.
\]
If $M$ is complete and has infinite volume,

$$h(M) = \inf_S \left\{ \frac{\text{Area}(S)}{\text{Vol}(M_1)} \right\},$$

as $S$ varies over all codim 1 submanifolds bounding a finite volume submanifold $M_1$.

We now consider the Laplacian

$$\Delta: L^2(M) \to L^2(M)$$

$$f \mapsto d^* df$$

This has spectrum in $[0, \infty)$.

0 is no longer an eigenvalue of $\Delta$, because no non-zero constant function is in $L^2(M)$. 
\( \lambda_1(M) \) is the infimum of the spectrum.

\[
\lambda_1(M) = \inf \left\{ \frac{\|df\|^2}{\|f\|^2} : f \in L^2(M) \cap C^\infty(M) \right\}.
\]

**Theorem 2.7**: [Cheeger] For any complete Riemannian manifold \( M \),

\[
\lambda_1(M) \geq h(M)^2/4.
\]

**Proof: (infinite volume case)**

For any \( t \geq 0 \), let

\[
A(t) = \text{Area}(\{x \in M : f(x)^2 = t\})
\]

\[
V(t) = \text{Vol}(\{x \in M : f(x)^2 \geq t\})
\]

Then \( V(t) < \infty \) because \( f \in L^2 \).

Same proof as before. \( \blacksquare \)