Surface subgroups in dimension 3

Lecture 3

Today

Theorem 2.3: [L-Long-Reid] Let M be a closed hyperbolic 3-manifold. Then M has infinite-sheeted covers M_i such that $h(M_i) \to 0$.

This is a consequence of:

Theorem 2.5: [Bowen] $\Gamma = \pi_1(M)$ has a sequence of finitely generated free subgroups Γ_i such that $\delta(\Gamma_i) \to 2$.

Here $\delta(\Gamma_i)$ = the 'critical exponent' of Γ_i

Theorem 2.6: [Sullivan]

$$\lambda_1(\Gamma_i \backslash \mathbb{H}^3) = \begin{cases} \delta(\Gamma_i)(2 - \delta(\Gamma_i)) & \text{if } \delta(\Gamma_i) \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Here $\lambda_1(\Gamma_i \backslash \mathbb{H}^3)$ = the first eigenvalue of the Laplacian of $\Gamma_i \backslash \mathbb{H}^3$.

Theorem 2.7: [Cheeger] For any complete Riemannian manifold M_i, $\lambda_1(M_i) \geq h(M_i)^2/4$.

Theorem 2.7: [Cheeger] For any complete Riemannian manifold M,

$$\lambda_1(M) \geq h(M)^2/4.$$

Proof: (when M is closed)

Let f be an eigenfunction with eigenvalue λ_1. Let

$$M_+ = \{x \in M : f(x) \geq 0\}$$
$$M_- = \{x \in M : f(x) \leq 0\}$$

Wlog, $\mathrm{vol}(M_+) \leq \mathrm{vol}(M_-)$.

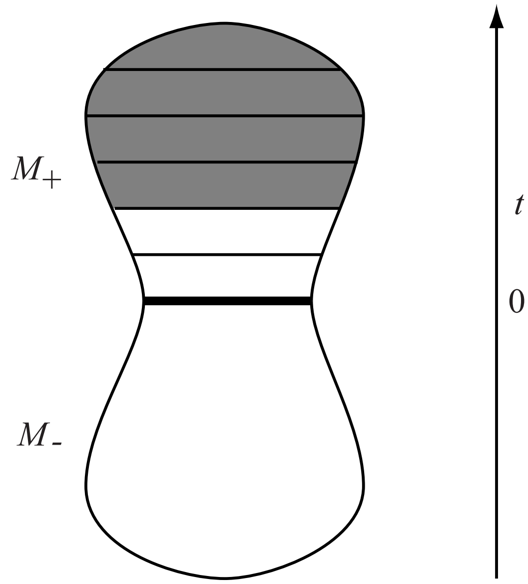
Focus on M_+ and take all integrals over M_+:

$$\begin{aligned}
\lambda_1 &= \frac{\int |df|^2}{\int |f|^2} \\
&= \frac{\int |df|^2 \int |f|^2}{\left(\int f^2\right)^2} \\
&\geq \frac{\left(\int |df|.|f|\right)^2}{\left(\int f^2\right)^2} \text{ by Cauchy-Schwarz} \\
&= \frac{1}{4}\frac{\left(\int |df^2|\right)^2}{\left(\int f^2\right)^2}
\end{aligned}$$

Claim: RHS $\geq h(M)^2/4$.

For any $t \geq 0$, let

$$\begin{aligned}
A(t) &= \mathrm{Area}(\{x \in M_+ : f(x)^2 = t\}) \\
V(t) &= \mathrm{Vol}(\{x \in M_+ : f(x)^2 \geq t\})
\end{aligned}$$



Co-area formula: $\int |df^2| = \int A(t)\ dt$

For each t, $A(t) \geq h(M)V(t)$, because at least half the volume lies in M_-.

So:

$$\begin{aligned}\int A(t)\ dt &\geq h(M) \int V(t)\ dt \\ &= h(M) \int_t \left(\int_{\{x: f(x)^2 \geq t\}} d\text{vol} \right) dt \\ &= h(M) \int f^2\ d\text{vol}\end{aligned}$$

So,

$$\lambda_1(M) \geq \frac{1}{4} \frac{\left(\int |df^2|\right)^2}{\left(\int f^2\right)^2} \geq \frac{h(M)^2}{4}. \qquad \square$$

Manifolds with infinite volume

If M is complete and has infinite volume,

$$h(M) = \inf_S \left\{ \frac{\text{Area}(S)}{\text{Vol}(M_1)} \right\},$$

as S varies over all codim 1 submanifolds bounding a finite volume submanifold M_1.

We now consider the Laplacian

$$\begin{aligned} \Delta\colon L^2(M) &\to L^2(M) \\ f &\mapsto d^*df \end{aligned}$$

This has spectrum in $[0, \infty)$.

0 is no longer an eigenvalue of Δ, because no non-zero constant function is in $L^2(M)$.

$\lambda_1(M)$ is the infimum of the spectrum.

$$\lambda_1(M) = \inf \left\{ \frac{||df||^2}{||f||^2} : f \in L^2(M) \cap C^\infty(M) \right\}.$$

Theorem 2.7: [Cheeger] For any complete Riemannian manifold M,

$$\lambda_1(M) \geq h(M)^2/4.$$

Proof: (infinite volume case)

For any $t \geq 0$, let

$$A(t) = \text{Area}(\{x \in M : f(x)^2 = t\})$$
$$V(t) = \text{Vol}(\{x \in M : f(x)^2 \geq t\})$$

Then $V(t) < \infty$ because $f \in L^2$.

Same proof as before. $\square$

Theorem 2.6: [Sullivan] For $M = \Gamma\backslash\mathbb{H}^3$ a hyperbolic 3-manifold,

$$\lambda_1(M) = \begin{cases} \delta(\Gamma)(2-\delta(\Gamma)) & \text{if } \delta(\Gamma) \geq 1 \\ 1 & \text{otherwise} \end{cases}$$

We'll show $\leq$, and focus on $\delta(\Gamma) \geq 1$, as that is all we'll need.

$\delta(\Gamma)$ is the critical exponent of Γ, defined as follows.

Pick $x \in \mathbb{H}^3$.

$$\delta(\Gamma) = \inf \left\{ s : \sum_{\gamma \in \Gamma} \exp(-sd(x, \gamma x)) < \infty \right\}$$

ie if $N(r)$ is the number of γx within distance r of x, then

$$N(r) \simeq \exp(\delta(\Gamma) r).$$

Green's functions

Let

$$M = \text{Riemannian manifold}$$
$$p = \text{point in } M$$
$$\lambda > 0.$$

The associated λ-Green's function is a function $g_{p,\lambda}\colon M \to \mathbb{R}$ s.t.

$$(\Delta - \lambda) g_{p,\lambda} = \delta\text{-function at } p.$$

Fact: This is finite away from p, provided $\lambda < \lambda_1(M)$.

Green's functions on $\mathbb{H}^3$

$\lambda_1(\mathbb{H}^3) = 1.$

Provided $\lambda < 1$ and for $p \in \mathbb{H}^3$,

$$g_{p,\lambda}(x) \sim \text{const} \ . \ \exp(\alpha d(x,p)) \text{ as } d(x,p) \to \infty$$

where

$$\alpha = -1 - \sqrt{1-\lambda}.$$

(Solve a 2nd order ODE).

Proof of Sullivan's theorem

Let

$$M = \Gamma\backslash\mathbb{H}^3 \text{ a hyperbolic 3-manifold}$$

$$p \in M$$

$$\tilde{p} \in \text{inverse image of } p$$

$$\lambda < \lambda_1(M)$$

$$g_{p,\lambda} = \lambda - \text{Green's function on } M$$

$$\tilde{g}_{p,\lambda} = \text{pull-back to } \mathbb{H}^3$$

Then

$$\tilde{g}_{p,\lambda}(\tilde{p}) = \sum_{\gamma\in\Gamma} g_{\gamma\tilde{p},\lambda}(\tilde{p}) \simeq \sum_{\gamma\in\Gamma} \exp(\alpha d(\gamma\tilde{p},\tilde{p})) < \infty$$

$\Rightarrow \delta \leq -\alpha = 1 + \sqrt{1-\lambda}$.

So, $\lambda \leq \delta(2-\delta)$. True for all $\lambda < \lambda_1(M)$. $\square$

$M = \Gamma \backslash \mathbb{H}^3$ a closed hyperbolic 3-manifold.

Theorem 2.5: [Bowen] $\Gamma = \pi_1(M)$ has a sequence of finitely generated free subgroups Γ_i such that $\delta(\Gamma_i) \to 2$.

This relies on the following well-known old result.

Theorem 3.1: There exist finitely generated free Kleinian groups F without parabolics and with $\delta(F)$ arbitrarily close to 2.

Outline of proof of 3.1.

Let F be a f.g. free group.

Let $\mathcal{R}$ be the space of all representations $\phi\colon F \to \mathrm{PSL}(2,\mathbb{C})$ (topologised as $(\mathrm{PSL}(2,\mathbb{C}))^n$ where $n = \mathrm{rank}(F)$).

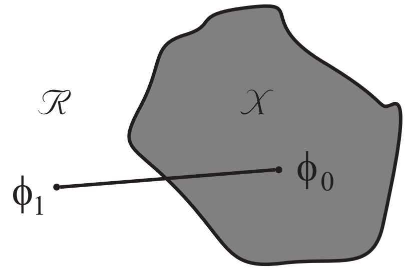
Let $\mathcal{X}$ be the representations ϕ in $\mathcal{R}$ that are discrete and faithful, where $\phi(F)$ contains no parabolics, and where $\delta(\phi(F)) < 2$. Such representations are called convex cocompact.

This is an open subset of $\mathcal{R}$.

There exists some $\phi_0 \in \mathcal{X}$.

There exists some indiscrete $\phi_1 \in \mathcal{R}$.

$\mathcal{R}$ is path-connected so there is a path from ϕ_0 to ϕ_1.



All $\phi \in \partial\mathcal{X}$ are discrete and faithful.

So, for all $\phi \in \partial\mathcal{X}$, either $\delta(\phi(F)) = 2$ or $\phi(F)$ contains parabolics.

Fact 1: If we remove the latter from $\mathcal{R}$, the space is still path-connected.

So, we can choose the path ϕ_t so that, at the first time T where $\phi_T \in \partial\mathcal{X}$, $\delta(\phi_T(F)) = 2$.

Fact 2: $\delta(\phi(F))$ is a continuous fn of $\phi \in \overline{\mathcal{X}}$.

As $t \to T$ from below, $\phi_t(F) \in \mathcal{X}$ and $\delta(\phi_t(F)) \to 2$. □

Theorem 2.5: [Bowen] $\Gamma = \pi_1(M)$ has a sequence of finitely generated subgroups Γ_i such that $\delta(\Gamma_i) \to 2$.

Theorem 3.1: There exist convex cocompact finitely generated free Kleinian groups F with $\delta(F)$ arbitrarily close to 2.

We now modify F a little so that it 'almost' sits inside Γ.

Let S be a finite generating set for F.

Definition. For $\epsilon > 0$, an ϵ-perturbation of F is a function $\phi\colon F \to \mathrm{Isom}^+(\mathbb{H}^3)$ such that for all $f \in F$ and all $s \in S$,

$$d(\phi(fs^{\pm 1}), \phi(f)s^{\pm 1}) \leq \epsilon.$$

Here $d(\ ,\ )$ is the left-invariant metric on $\mathrm{Isom}^+(\mathbb{H}^3)$.

Definition. $\phi: F \to \mathrm{Isom}^+(\mathbb{H}^3)$ is a virtual homomorphism if there is a finite index subgroup F' of F s.t. for all $f' \in F'$ and $f \in F$,

$$\phi(f'f) = \phi(f')\phi(f).$$

[In particular, $\phi|F'$ is a homomorphism.]
It is a virtual homomorphism into Γ if, in addition, $\phi(F') \leq \Gamma$.

Theorem 3.2. [Bowen] Let Γ be a cocompact Kleinian group. Let F be a convex cocompact free Kleinian group. Then, for any $\epsilon > 0$, there is an ϵ-perturbation of F that is a virtual homomorphism into Γ.

Theorem 3.3. [Bowen] Let F be a convex cocompact free Kleinian group. For all $\epsilon' > 0$, there is an $\epsilon > 0$ such that if ϕ is an ϵ-perturbation of F that is a virtual homomorphism, then $|\delta(\phi(F)) - \delta(F)| < \epsilon'$.

Note: If F' is the finite index subgroup of F, then $\delta(\phi(F')) = \delta(\phi(F))$.

We'll focus on Theorem 3.2.

Let:

$$\begin{aligned} \Gamma &= \text{ the cocompact Kleinian group} \\ M &= \Gamma\backslash\mathbb{H}^3 \\ F &= \text{ the convex cocompact free group} \\ S &= \text{ a free generating set} \\ \mu &= \text{ Haar measure on } \mathrm{Isom}^+(\mathbb{H}^3) \end{aligned}$$

We may assume that M is a manifold.

Let $\delta > 0$ satisfy:

$$\forall g_1, g_2 \in \mathrm{Isom}^+(\mathbb{H}^3) \text{ s.t. } d(g_1, \mathrm{id}) \leq \delta, d(g_2, \mathrm{id}) \leq \delta, \text{ and } \forall s \in S,$$

$$d(g_1 s g_2, s) \leq \epsilon.$$

Let $B = \Gamma \backslash \text{Isom}(\mathbb{H}^3)$ [the frame bundle over M].

Partition B into a finite collection of subsets $B_1, \ldots, B_n$, each with diameter $\leq \delta$.

Arrange that $\mu(\partial B_i) = 0 \ \forall i$.

Pick $\beta_i \in B_i$.

Wlog $\beta_1 = \text{id}$ coset.

Construction of graph Y in B:

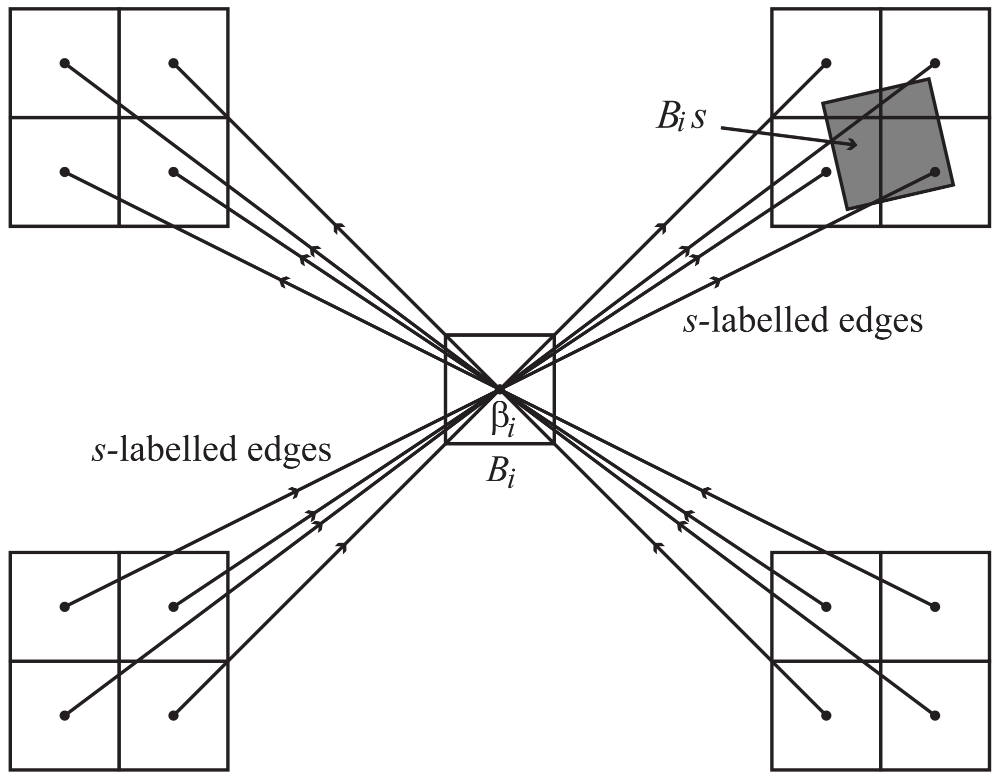
A vertex at each β_i.

Each edge has a label $s \in S$.

β_i and β_j are joined by an s-labelled edge e iff

$$\exists \beta_i' \in \text{int}(B_i), \beta_j' \in \text{int}(B_j) \text{ s.t. } \beta_i' s = \beta_j'.$$

Link of a vertex β_i:



For each edge e define $\psi(e)$ which will be an approximation to s:

Since $\operatorname{diam}(B_i), \operatorname{diam}(B_j) \leq \delta$, there are elements

$$g_i \in \operatorname{Isom}^+(\mathbb{H}^3), \text{ s.t. } d(g_i, \mathrm{id}) \leq \delta \text{ and } \beta_i g_i = \beta_i'$$
$$g_j \in \operatorname{Isom}^+(\mathbb{H}^3), \text{ s.t. } d(g_j, \mathrm{id}) \leq \delta \text{ and } \beta_j g_j = \beta_j'$$

Let

$$\psi(e) = g_i s g_j^{-1}.$$

Then

$$\beta_i \psi(e) = \beta_j$$

$$d(\psi(e), s) \leq \epsilon$$

We'll define functions

$$w\colon V(Y) \to (0,\infty)$$

$$w\colon E(Y) \to (0,\infty)$$

s.t. for all $v \in V(Y)$ and all $s \in S$,

$$\sum_{\substack{s-\text{labelled edges}\\ e \text{ entering v}}} w(e) = w(v)$$

$$\sum_{\substack{s-\text{labelled edges}\\ e \text{ exiting v}}} w(e) = w(v).$$

This is called a weighting on Y.

For a vertex β_i, define

$$w(\beta_i) = \mu(B_i).$$

For an edge e with label s running from β_i to β_j, define

$$w(e) = \mu(\{\beta_i' \in B_i : \beta_i' s \in B_j\}).$$

Clearly

$$\sum_{\substack{s-\text{labelled edges} \\ e \text{ exiting } \beta_\text{i}}} w(e) = w(v).$$

Each edge, starting at β_j entering β_i has weight

$$\begin{aligned} \mu(\{\beta'_j \in B_j : \beta'_j s \in B_i\}) &= \mu(\{\beta'_j s : \beta'_j \in B_j \text{ and } \beta'_j s \in B_i\}) \\ &= \mu(\{\beta'_i \in B_i : \beta'_i s^{-1} \in B_j\}). \end{aligned}$$

[The last equality uses the invariance of Haar measure under right multiplication.]

So,

$$\sum_{\substack{s-\text{labelled edges} \\ e \text{ entering } \beta_\text{i}}} w(e) = w(v).$$

For functions

$$w\colon V(Y) \to (0,\infty)$$

$$w\colon E(Y) \to (0,\infty)$$

to give a weighting they must satisfy linear equations with integer coefficients. So

$$\exists \text{ weighting } w \Rightarrow \exists \text{ integral weighting } w'.$$

Replace

$$\text{each vertex } v \in V(Y) \text{ by } w'(v) \text{ vertices}$$

$$\text{each edge } e \in E(Y) \text{ by } w'(e) \text{ edge}$$

Join them up so that, for each vertex v and each $s \in S$,

$$\text{exactly one } s\text{-labelled edge enters } v, \text{ and}$$

$$\text{exactly one } s\text{-labelled edge exits } v.$$

Let X = this graph.

Make one of the vertices at β_1 be its basepoint.

Definition of $\phi\colon F \to \mathrm{Isom}^+(\mathbb{H}^3)$:

Each $f \in F$ is a word $s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$ (where $s_i \in S$ and $\epsilon_i \in \{-1, 1\}$).

This gives a path $e_1^{\epsilon_1} \dots e_n^{\epsilon_n}$ in X starting at the basepoint.

Define

$$\phi(f) = \psi(e_1)^{\epsilon_1} \dots \psi(e_n)^{\epsilon_n}.$$

This is an ϵ-perturbation of F.

Define F' to be those $f \in F$ that give loops in X.

Then, for all $f' \in F$, $f \in F$,

$$\phi(f'f) = \phi(f')\phi(f),$$

$$\phi(f') \in \Gamma \qquad \square$$