Surface subgroups in dimension 3

Lecture 4

An underlying manifold with infinite π_1

Recall:

Theorem 2.1: $O = \Gamma\backslash\mathbb{H}^3$ has a finite cover $\tilde{O}$ s.t.

1. $\tilde{O}$ has at least one singular vertex;
2. every arc and simple closed curve of $\mathrm{sing}(\tilde{O})$ has order 2;
3. $\pi_1(|\tilde{O}|)$ is infinite.

Proof outline:

Γ is a finitely generated linear group.

Theorem: (Selberg's lemma) Any f.g. linear group Γ has a finite index normal subgroup Γ_1 that is torsion free.

So, $\Gamma_1\backslash\mathbb{H}^3$ is a manifold (ie we've gone too far).

It is a regular cover of O with covering group Γ/Γ_1.

Consider the group $\Gamma_2 = \Gamma_1(\mathbb{Z}/2 \times \mathbb{Z}/2)$.

Let $O_2 = \Gamma_2 \backslash \mathbb{H}^3$.

Note: O_2 is the quotient of a manifold by a $\mathbb{Z}/2 \times \mathbb{Z}/2$ action.

So, every arc and simple closed curve of $\mathrm{sing}(O_2)$ has order 2.

Note: O_2 is covered by $(\mathbb{Z}/2 \times \mathbb{Z}/2)\backslash \mathbb{H}^3$ which has a singular vertex.

So, O_2 has at least one singular vertex.

But we have no guarantee that $\pi_1(|O_2|)$ is infinite.

Indeed, $|O_2|$ may be the 3-sphere.

The Golod-Shafarevich inequality

How do we show that a group is infinite?

Theorem: [Golod-Shafarevich] Let G be a finitely presented group $\langle X|R\rangle$. If

$$\frac{d_p(G)^2}{4} \geq d_p(G) - |X| + |R|,$$

where $d_p(G) = \dim(H_1(G;\mathbb{F}_p))$, then G is infinite.

Every 3-manifold group has a presentation where $|X| = |R|$.

Now use:

Theorem: [Lubotzky ?] If a finitely generated linear group Γ is not virtually soluble, then for any prime p, Γ has finite index subgroups Γ_1 where $d_p(\Gamma_1)$ is arbitrarily big.

So, we can certainly arrange that $\pi_1(|O_3|) = \infty$ for some finite cover $O_3 \to O_2$.

But to ensure that O_3 also has a singular vertex is a bit tricky.

Hyperbolic underlying space

We still have to prove:

Theorem 2.2: If a closed orientable 3-manifold M has infinite π_1, then either

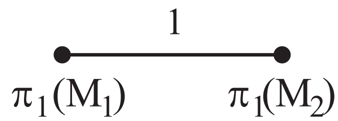
1. M is hyperbolic; or
2. M has a finite cover $\tilde{M}$ with $b_1 > 0$.

Proof outline:

This requires Perelman's solution to the geometrisation conjecture.

Case 1: M is a connected sum $M_1 \sharp M_2$.

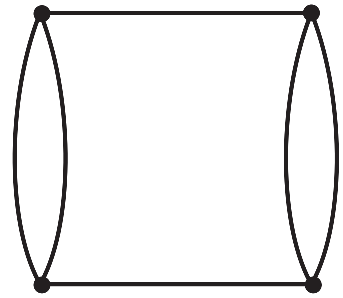
Then $\pi_1(M)$ is a graph of groups:



Fact: Any closed orientable 3-manifold has residually finite π_1.

So, we may find proper finite index subgroups $\Gamma_1 \leq \pi_1(M_1)$ and $\Gamma_2 \leq \pi_1(M_2)$.

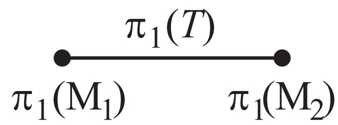
We have an associated cover $\tilde{M}$ of M with $\pi_1(\tilde{M})$ a graph of groups:



Since the graph has a cycle, $b_1(\tilde{M}) > 0$.

Case 2: M is prime but has an embedded π_1-injective torus T.

$M = M_1 \cup_T M_2$. So, $\pi_1(M) =$



Fact: Any compact orientable 3-manifold M_i with boundary a collection of π_1-injective tori has a finite cover $\tilde{M}_i$, which restricts on each component of ∂M_i to the characteristic p^2 cover, for each sufficiently big prime p.

Moreover, $|\partial M_i| \geq 2$.

So, we get a cover $\tilde{M}$ s.t. $\pi_1(\tilde{M})$ has a graph of groups decomposition, in which each vertex has valence ≥ 2. Again, there is a cycle.

So, $b_1(\tilde{M}) > 0$.

Case 3: M is Seifert fibred.

ie M is a 'circle bundle' over a 2-orbifold F.

$\pi_1(M)$ infinite $\Rightarrow$ F has a finite surface cover $\tilde{F}$.

We get an induced S^1-bundle $\tilde{M}$ over $\tilde{F}$:

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \tilde{F} & \longrightarrow & F \end{array}$$

$\pi_1(M)$ infinite and $M \neq S^2 \times S^1$ or $\mathbb{R}P^3 \sharp \mathbb{R}P^3$
$\Rightarrow \tilde{F} \neq S^2, \mathbb{R}P^2 \Rightarrow b_1(\tilde{F}) > 0 \Rightarrow b_1(\tilde{M}) > 0.$

Case 4: M is hyperbolic.

We are done. □

We've now proved:

Main Theorem 1.2: [L] Any finitely generated Kleinian group Γ containing a finite non-cyclic subgroup is either finite, virtually free or contains a surface subgroup.

But we haven't shown:

Main Theorem 1.1: [L] Every arithmetic hyperbolic 3-manifold contains an immersed π_1-injective surface.

which relied on:

Theorem 1.3: [L-Long-Reid] Any arithmetic Kleinian group is commensurable with one that contains $\mathbb{Z}/2 \times \mathbb{Z}/2$.

I'll give a outline of this now.

Arithmetic 3-manifolds

Non-standard definition: A hyperbolic 3-manifold M is non-arithmetic if there is a hyperbolic orbifold O s.t. every 3-orbifold commensurable with M finitely covers O.

The usual definition is in terms of

number fields, quaternion algebras and orders

or

integral points in algebraic subgroups of semi-simple Lie groups.

Let g, h be non-commuting elements in the fundamental group of an arithmetic hyperbolic 3-manifold M

Fact 1. The commensurability class of M can be recovered from g, h and gh.

Fact 2. Any Kleinian group Γ generated by two elements g and h has an involution

$$g \mapsto g^{-1}, \qquad h \mapsto h^{-1}.$$

This is realised by an isometry of $\Gamma\backslash\mathbb{H}^3$ with non-empty fixed-point set.

Theorem 1.3 is proved by upgrading this involution to all of M, after first passing to some commensurable orbifold O.

This $\Rightarrow \mathbb{Z}/2 \leq \pi_1(O)$.

With more work, we get $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq \pi_1(O)$.

Where next?

Conjecture: [Lubotzky-Sarnark] Any closed hyperbolic 3-manifold has a sequence of finite covers M_i s.t. $h(M_i) \to 0$.

Theorem 4.1: [L] The Lubotzky-Sarnak conjecture implies that every co-compact Kleinian group containing a finite non-cyclic subgroup is large. Hence, the Lubotzky-Sarnak conjecture implies that every arithmetic Kleinian group is large.

Recall: a group Γ is large if some finite index subgroup has a non-abelian free quotient.

Proof

Let Γ = cocompact Kleinian subgroup, containing a finite non-cyclic subgroup. Let $O = \Gamma \backslash \mathbb{H}^3$.

Simplifying assumption: $\mathbb{Z}/2 \times \mathbb{Z}/2$

2.1 $\Rightarrow$ we may pass to a finite cover $\tilde{O}$ s.t.

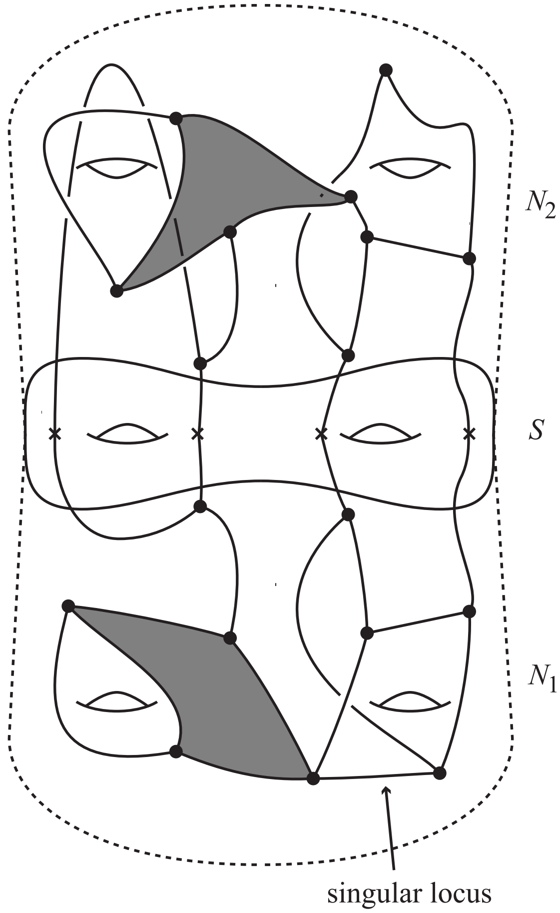
1. $\tilde{O}$ has at least one singular vertex;
2. every arc and simple closed curve of $\text{sing}(\tilde{O})$ has order 2;
3. $\pi_1(|\tilde{O}|)$ is infinite

Let $M = |\tilde{O}|$.

2.2 $\Rightarrow$ M is hyperbolic or M has a finite cover with $b_1 > 0$.

LS conjecture $\Rightarrow$ M has a sequence of finite covers M_i s.t. $h(M_i) \to 0$.

We get induced orbifold covers O_i of $\tilde{O}$, where $|O_i| = M_i$.



N_2
S
N_1
singular locus

Each O_i is divided into two sub-orbifolds N_1 and N_2.

We may ensure:

$$d_2(N_i) > d_2(\partial N_i) + 1, \qquad i = 1, 2$$

Take 2 independent classes in $\ker(H^1(N_1; \mathbb{Z}/2) \to H^1(\partial N_1; \mathbb{Z}/2))$

and let $\tilde{O}_i$ be the associated 4-fold cover of O_i.

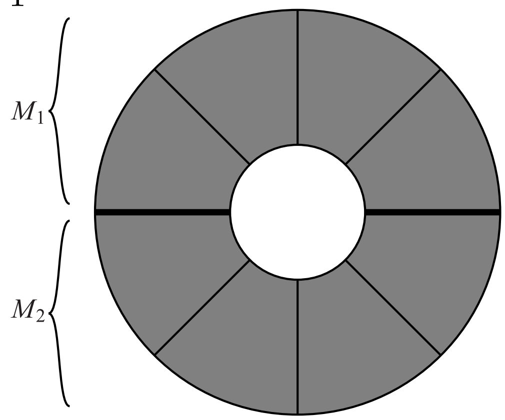
This contains 4 disjoint (possibly non-orientable) surfaces, whose union is non-separating.

So, we get a surjection $\pi_1(\tilde{O}_i) \to *^4(\mathbb{Z}/2)$.

Since $*^4(\mathbb{Z}/2)$ is virtually free non-abelian, $\pi_1(\tilde{O}_i)$ is large. □

Note: If M has a finite cover $\tilde{M}_1$ with $b_1(\tilde{M}_1) > 0$, then it has sequence of finite covers $\tilde{M}_i$ with $h(\tilde{M}_i) \to 0$.

Take cyclic covers of $\tilde{M}_1$:



Thus:

Theorem 4.2: [L-Long-Reid] The positive virtual b_1 conjecture for closed hyperbolic 3-manifolds implies that every arithmetic Kleinian group is large.

Approaches to the Lubotzky-Sarnak conjecture

A group Γ is LERF if, for every finitely generated subgroup $H \leq G$, and for every $g \in G - H$, there is a finite index subgroup $K \leq G$ s.t. $H \leq K$ and $g \notin K$.

Old conjecture: Every finitely generated Kleinian group is LERF.

Topological consequence: Suppose $\pi_1(M)$ is LERF, where M is compact. For every cover $M_1 \to M$ where $\pi_1(M_1)$ is finitely generated, and for every compact subset $C \subset M_1$, there is a finite cover M_2 of M s.t.

$$M_1 \xrightarrow{p} M_2 \longrightarrow M$$

and $p|C$ is an embedding.

So, 2.3 $\Rightarrow$

Theorem 4.3: [L-Long-Reid] If M is a closed hyperbolic 3-manifold and $\pi_1(M)$ is LERF, it satisfies the Lubotzky-Sarnak conjecture.