Surface subgroups
in dimension 3

Lecture 4
Recall:

**Theorem 2.1:** $O = \Gamma \backslash \mathbb{H}^3$ has a finite cover $\tilde{O}$ s.t.

1. $\tilde{O}$ has at least one singular vertex;
2. every arc and simple closed curve of $\text{sing}(\tilde{O})$ has order 2;
3. $\pi_1(\tilde{O})$ is infinite.

**Proof outline:**

$\Gamma$ is a finitely generated linear group.

**Theorem:** (Selberg’s lemma) Any f.g. linear group $\Gamma$ has a finite index normal subgroup $\Gamma_1$ that is torsion free.

So, $\Gamma_1 \backslash \mathbb{H}^3$ is a manifold (ie we’ve gone too far).

It is a regular cover of $O$ with covering group $\Gamma/\Gamma_1$. 
Consider the group $\Gamma_2 = \Gamma_1(\Z/2 \times \Z/2)$.

Let $O_2 = \Gamma_2 \backslash \mathbb{H}^3$.

Note: $O_2$ is the quotient of a manifold by a $\Z/2 \times \Z/2$ action.

So, every arc and simple closed curve of $\text{sing}(O_2)$ has order 2.

Note: $O_2$ is covered by $(\Z/2 \times \Z/2) \backslash \mathbb{H}^3$ which has a singular vertex.

So, $O_2$ has at least one singular vertex.

But we have no guarantee that $\pi_1(|O_2|)$ is infinite.

Indeed, $|O_2|$ may be the 3-sphere.
**The Golod-Shafarevich inequality**

How do we show that a group is infinite?

**Theorem: [Golod-Shafarevich]** Let $G$ be a finitely presented group $\langle X | R \rangle$. If

$$\frac{d_p(G')^2}{4} \geq d_p(G) - |X| + |R|,$$

where $d_p(G) = \dim(H_1(G; \mathbb{F}_p))$, then $G$ is infinite.

Every 3-manifold group has a presentation where $|X| = |R|$.

Now use:

**Theorem: [Lubotzky ?]** If a finitely generated linear group $\Gamma$ is not virtually soluble, then for any prime $p$, $\Gamma$ has finite index subgroups $\Gamma_1$ where $d_p(\Gamma_1)$ is arbitrarily big.
So, we can certainly arrange that $\pi_1(|O_3|) = \infty$ for some finite cover $O_3 \to O_2$.

But to ensure that $O_3$ also has a singular vertex is a bit tricky.
We still have to prove:

**Theorem 2.2:** If a closed orientable 3-manifold $M$ has infinite $\pi_1$, then either

1. $M$ is hyperbolic; or
2. $M$ has a finite cover $\tilde{M}$ with $b_1 > 0$.

**Proof outline:**

This requires Perelman’s solution to the geometrisation conjecture.
Case 1: $M$ is a connected sum $M_1 \# M_2$.

Then $\pi_1(M)$ is a graph of groups:

\[
\begin{array}{c}
\pi_1(M_1) \\
\bullet
\end{array} \quad \begin{array}{c}
\pi_1(M_2) \\
\bullet
\end{array}
\]

\[
\text{with } 1
\]

Fact: Any closed orientable 3-manifold has residually finite $\pi_1$.

So, we may find proper finite index subgroups $\Gamma_1 \leq \pi_1(M_1)$ and $\Gamma_2 \leq \pi_1(M_2)$.

We have an associated cover $\tilde{M}$ of $M$ with $\pi_1(\tilde{M})$ a graph of groups:
Since the graph has a cycle, $b_1(\tilde{M}) > 0$. 
Case 2: \( M \) is prime but has an embedded \( \pi_1 \)-injective torus \( T \).

\[ M = M_1 \cup_T M_2. \]  
So, \( \pi_1(M) = \)

![Diagram](image)

**Fact:** Any compact orientable 3-manifold \( M_i \) with boundary a collection of \( \pi_1 \)-injective tori has a finite cover \( \tilde{M}_i \), which restricts on each component of \( \partial M_i \) to the characteristic \( p^2 \) cover, for each sufficiently big prime \( p \).

Moreover, \( |\partial M_i| \geq 2. \)

So, we get a cover \( \tilde{M} \) s.t. \( \pi_1(\tilde{M}) \) has a graph of groups decomposition, in which each vertex has valence \( \geq 2. \) Again, there is a cycle.

So, \( b_1(\tilde{M}) > 0. \)
Case 3: $M$ is Seifert fibred.

ie $M$ is a ‘circle bundle’ over a 2-orbifold $F$.

$\pi_1(M)$ infinite $\Rightarrow$ $F$ has a finite surface cover $\tilde{F}$.

We get an induced $S^1$-bundle $\tilde{M}$ over $\tilde{F}$:

$$
\begin{array}{ccc}
\tilde{M} & \longrightarrow & M \\
\downarrow & & \downarrow \\
\tilde{F} & \longrightarrow & F
\end{array}
$$

$\pi_1(M)$ infinite and $M \neq S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$

$\Rightarrow \tilde{F} \neq S^2, \mathbb{RP}^2 \Rightarrow b_1(\tilde{F}) > 0 \Rightarrow b_1(\tilde{M}) > 0$.

Case 4: $M$ is hyperbolic.

We are done. $\blacksquare$
We’ve now proved:

**Main Theorem 1.2**: [L] Any finitely generated Kleinian group $\Gamma$ containing a finite non-cyclic subgroup is either finite, virtually free or contains a surface subgroup.

But we haven’t shown:

**Main Theorem 1.1**: [L] Every arithmetic hyperbolic 3-manifold contains an immersed $\pi_1$-injective surface.

which relied on:

**Theorem 1.3**: [L-Long-Reid] Any arithmetic Kleinian group is commensurable with one that contains $\mathbb{Z}/2 \times \mathbb{Z}/2$.

I’ll give a outline of this now.
Non-standard definition: A hyperbolic 3-manifold $M$ is non-arithmetic if there is a hyperbolic orbifold $O$ s.t. every 3-orbifold commensurable with $M$ finitely covers $O$.

The usual definition is in terms of

number fields, quaternion algebras and orders

or

integral points in algebraic subgroups of semi-simple Lie groups.
Let $g, h$ be non-commuting elements in the fundamental group of an arithmetic hyperbolic 3-manifold $M$

**Fact 1.** The commensurability class of $M$ can be recovered from $g, h$ and $gh$.

**Fact 2.** Any Kleinian group $\Gamma$ generated by two elements $g$ and $h$ has an involution

$$ g \mapsto g^{-1}, \quad h \mapsto h^{-1}. $$

This is realised by an isometry of $\Gamma \backslash \mathbb{H}^3$ with non-empty fixed-point set.

Theorem 1.3 is proved by upgrading this involution to all of $M$, after first passing to some commensurable orbifold $O$.

This $\Rightarrow \mathbb{Z}/2 \leq \pi_1(O)$.

With more work, we get $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq \pi_1(O)$. 
**Where next?**

**Conjecture: [Lubotzky-Sarnark]** Any closed hyperbolic 3-manifold has a sequence of finite covers $M_i$ s.t. $h(M_i) \to 0$.

**Theorem 4.1: [L]** The Lubotzky-Sarnak conjecture implies that every co-compact Kleinian group containing a finite non-cyclic subgroup is large. Hence, the Lubotzky-Sarnak conjecture implies that every arithmetic Kleinian group is large.

Recall: a group $\Gamma$ is large if some finite index subgroup has a non-abelian free quotient.
Proof

Let $\Gamma = \text{cocompact Kleinian subgroup}$, containing a finite non-cyclic subgroup. Let $O = \Gamma \backslash \mathbb{H}^3$.

Simplifying assumption: $\mathbb{Z}/2 \times \mathbb{Z}/2$

2.1 $\Rightarrow$ we may pass to a finite cover $\tilde{O}$ s.t.

1. $\tilde{O}$ has at least one singular vertex;
2. every arc and simple closed curve of $\text{sing}(\tilde{O})$ has order 2;
3. $\pi_1(|\tilde{O}|)$ is infinite

Let $M = |\tilde{O}|$.

2.2 $\Rightarrow$ $M$ is hyperbolic or $M$ has a finite cover with $b_1 > 0$.

LS conjecture $\Rightarrow$ $M$ has a sequence of finite covers $M_i$ s.t. $h(M_i) \to 0$.

We get induced orbifold covers $O_i$ of $\tilde{O}$, where $|O_i| = M_i$. 

Each $O_i$ is divided into two sub-orbifolds $N_1$ and $N_2$.

We may ensure:

$$d_2(N_i) > d_2(\partial N_i) + 1, \quad i = 1, 2$$

Take 2 independent classes in $\ker(H^1(N_1; \mathbb{Z}/2) \to H^1(\partial N_1; \mathbb{Z}/2))$ and let $\tilde{O_i}$ be the associated 4-fold cover of $O_i$.

This contains 4 disjoint (possibly non-orientable) surfaces, whose union is non-separating.

So, we get a surjection $\pi_1(\tilde{O_i}) \to *^4(\mathbb{Z}/2)$.

Since $*^4(\mathbb{Z}/2)$ is virtually free non-abelian, $\pi_1(\tilde{O_i})$ is large. □
Note: If $M$ has a finite cover $\tilde{M}_1$ with $b_1(\tilde{M}_1) > 0$, then it has sequence of finite covers $\tilde{M}_i$ with $h(\tilde{M}_i) \to 0$.

Take cyclic covers of $\tilde{M}_1$:

Thus:

**Theorem 4.2:** [L-Long-Reid] The positive virtual $b_1$ conjecture for closed hyperbolic 3-manifolds implies that every arithmetic Kleinian group is large.
A group $\Gamma$ is **LERF** if, for every finitely generated subgroup $H \leq G$, and for every $g \in G - H$, there is a finite index subgroup $K \leq G$ s.t. $H \leq K$ and $g \not\in K$.

**Old conjecture:** Every finitely generated Kleinian group is LERF.

**Topological consequence:** Suppose $\pi_1(M)$ is LERF, where $M$ is compact. For every cover $M_1 \to M$ where $\pi_1(M_1)$ is finitely generated, and for every compact subset $C \subset M_1$, there is a finite cover $M_2$ of $M$ s.t.

$$M_1 \overset{p}{\longrightarrow} M_2 \longrightarrow M$$

and $p|C$ is an embedding.

So, 2.3 $\Rightarrow$

**Theorem 4.3:** [L-Long-Reid] If $M$ is a closed hyperbolic 3-manifold and $\pi_1(M)$ is LERF, it satisfies the Lubotzky-Sarnak conjecture.