THE HEEGAARD GENUS OF AMALGAMATED 3-MANIFOLDS

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Abstract

Let M and M' be simple 3-manifolds, each with connected boundary of genus at least two. Suppose that M and M' are glued via a homeomorphism between their boundaries. Then we show that, provided the gluing homeomorphism is 'sufficiently complicated', the Heegaard genus of the amalgamated manifold is completely determined by the Heegaard genus of M and M' and the genus of their common boundary. Here, a homeomorphism is 'sufficiently complicated' if it is the composition of a homeomorphism from the boundary of M to some surface S, followed by a sufficiently high power of a pseudo-Anosov on S, followed by a homeomorphism to the boundary of M'. The proof uses the hyperbolic geometry of the amalgamated manifold, generalised Heegaard splittings and minimal surfaces.

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1. Introduction

When studying Haken 3-manifolds, one is led naturally to the following construction: the amalgamation of two 3-manifolds M and M' via a homeomorphism between their boundaries. In this paper, we study the behaviour of Heegaard genus under this operation. We show that, provided the gluing homeomorphism is 'sufficiently complicated' and M and M' satisfy some standard conditions, then the Heegaard genus of the amalgamated manifold is completely determined by the Heegaard genus of M and M' and the genus of their common boundary. Recall that a 3-manifold is *simple* if it is compact, orientable, irreducible, atoroidal, acylindrical and has incompressible boundary. We denote the Heegaard genus of a 3-manifold M by $g(M)$.

Main Theorem. *Let M and M' be simple 3-manifolds, and let $h\colon \partial M \to S$ and $h'\colon S \to \partial M'$ be homeomorphisms with some connected surface S of genus at least two. Let $\psi\colon S \to S$ be a pseudo-Anosov homeomorphism. Then, provided $|n|$ is sufficiently large,*

$$g(M \cup_{h'\psi^n h} M') = g(M) + g(M') - g(S).$$

Furthermore, any minimal genus Heegaard splitting for $M \cup_{h'\psi^n h} M'$ is obtained from splittings of M and M' by amalgamation, and hence is weakly reducible.

The amalgamation of two Heegaard splittings, referred to in the above theorem, was defined by Schultens [11]. We recall it here. Since ∂M and $\partial M'$ are assumed to be connected, the Heegaard splittings of M and M' divide each manifold into a compression body and a handlebody. Each compression body is a copy of $S \times I$ with 1-handles attached. Extend these 1-handles vertically through $S \times I$ so that they are attached to ∂M and $\partial M'$ respectively. We may assume that their attaching discs are disjoint when the manifolds are glued. Attach the boundaries of these 1-handles to the copy of S in $M \cup M'$, and remove the interiors of the attaching discs. The resulting surface is a Heegaard surface for $M \cup M'$, which is said to be obtained from the splittings of M and M' by *amalgamation*. (See Figure 1.) By calculating the genus of this surface, we obtain the inequality

$$g(M \cup M') \leq g(M) + g(M') - g(S).$$

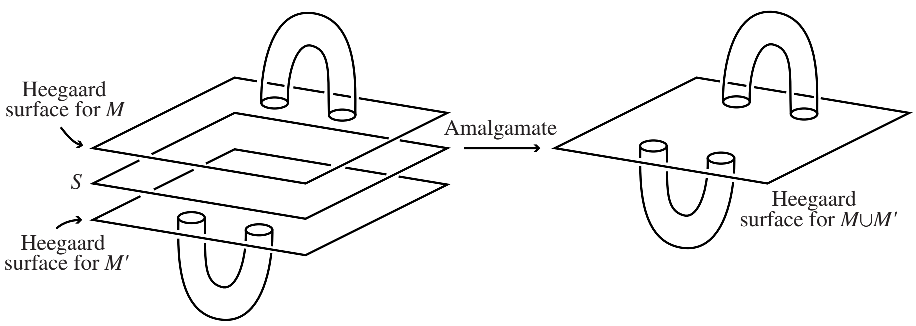

Figure 1.

Inequalities going in the other direction have been discovered by Johannson [3] and Schultens [12] who proved, respectively, that

$$g(M \cup M') \geq \frac{1}{5}g(M) + \frac{1}{5}g(M') - \frac{2}{5}g(S),$$
$$g(M \cup M') \geq \frac{1}{3}g(M) + \frac{1}{3}g(M') - \frac{4}{3}g(S) + \frac{5}{3}.$$

Clearly, in general, one will not be able to determine $g(M \cup M')$ precisely in terms of $g(M)$, $g(M')$ and $g(S)$. Inequalities such as the above will have to suffice. It is therefore slightly surprising that, for complicated gluings, an exact formula as in the main theorem should hold.

2. Proof of the main theorem

The amalgamated manifold $M \cup M'$ is Haken, atoroidal and not Seifert fibred. So, by Thurston's geometrisation theorem [5], it admits a hyperbolic structure. In [13], Soma gave a careful analysis of its geometry. Soma proved that one can find a point x_n in each $M \cup_{h'\psi^n h} M'$ such that the based manifolds $(M \cup_{h'\psi^n h} M', x_n)$ converge in the Gromov-Hausdorff topology to the infinite cyclic cover of the hyperbolic fibred 3-manifold with monodromy ψ. Furthermore, any fibre in the limit space pulls back to a surface isotopic to the copy of S in $M \cup_{h'\psi^n h} M'$, provided $|n|$ is sufficiently large. Hence, we deduce that, provided $|n|$ is sufficiently large, one may find in $M \cup_{h'\psi^n h} M'$ an arbitrarily large number of parallel copies of S, such that any two adjacent copies have distance at least one from each other. We denote the product region in $M \cup_{h'\psi^n h} M'$ between the extreme copies of S by $S \times I$.

The injectivity radius of the fibred manifold with monodromy ψ is positive, since the manifold is compact. Injectivity radius does not decrease when passing to a covering space, and so the injectivity radius of the infinite cyclic cover is also positive. Since the geometry of $M \cup_{h'\psi^n h} M'$ increasingly approximates this space, on larger and larger balls about x_n, we may ensure, by picking the copies of S suitably, that there is an $\epsilon > 0$, independent of n, such that $S \times I$ lies in the ϵ-thick part of $M \cup_{h'\psi^n h} M'$. (Recall [14] that the ϵ-thick part of a Riemannian manifold is the set of points with the property that any homotopically non-trivial closed loop based at such a point has length more than ϵ.)

Now consider a minimal genus Heegaard surface F for $M \cup M'$. Note that $g(F) \leq g(M) + g(M') - g(S)$. From F, we construct (as in [8], [9] or [4]) a generalised Heegaard splitting $\{F_1, \ldots, F_m\}$ with the following properties:

- F_j is strongly irreducible, for each odd j;
- F_j is incompressible and has no 2-sphere components, for each even j;
- F_j and F_{j+1} are not parallel for any j;
- $\sum_{j=1}^{m} (-1)^j \chi(F_j) = -\chi(F)$;
- $|\chi(F_j)| \leq |\chi(F)|$ for each j.

Let $F_+ = F_1 \cup \ldots \cup F_m$. The third and fourth conditions imply that $m \leq |\chi(F)|$, and hence the fifth gives that $|\chi(F_+)| \leq |\chi(F)|^2$, a bound which is independent of n. One can obtain F back from F_+ by amalgamating F_1 and F_3, then amalgamating this with F_5, and so on.

By theorems of Schoen and Yau [10], Freedman, Hass and Scott [1] and Pitts and Rubinstein [6], each component of F_+ may be isotoped to a minimal surface or to the double cover of a minimal non-orientable surface (possibly with a small tube attached in the case of an odd surface). Furthermore, after these isotopies, any two components are either equal or disjoint. Each complementary region of F_+ after the isotopies corresponds to one before, but some product complementary regions may have been collapsed. In particular, each complementary region afterwards is a compression body.

We would like to apply Proposition 6.1 of [4], which gives a constant k, such that each component F' of F_+ has diameter at most $k|\chi(F')|$. (Here, we are using the path metric on F_+ arising from its induced Riemannian metric.) However, k depends on a positive lower bound for the injectivity radius of the ambient manifold. It is not immediately clear from Soma's paper whether there is such a bound that is independent of n. We will therefore present a variant of Proposition 6.1 of [4]. Let δ be $2\epsilon + 1 + \epsilon/\pi$. We claim that we can cover $F_+ \cap (S \times I)$ with regions, each of which has diameter at most δ in F_+, and so that the total number of regions is at most $k|\chi(F_+)|$, for some constant k independent of n. These regions will be of two types.

Let $(F_+)_{[\epsilon,\infty)}$ and $(F_+)_{(0,\epsilon]}$ be the ϵ-thick and ϵ-thin parts of F_+. Let Γ be a maximal collection of disjoint (not necessarily simple) closed geodesics in F_+, each with length less than ϵ. The first type of region will consist of those points within $\epsilon/2+\epsilon/(2\pi)+1/2$ of some component of Γ. Clearly, each such region has diameter at most δ. Claim 3 in the proof of Proposition 6.1 of [4] gives that there are at most $4|\chi(F_+)|$ geodesics in Γ, and hence at most $4|\chi(F_+)|$ such regions. The argument of Claims 2 and 1 there also gives that these regions cover $(F_+)_{(0,\epsilon]} \cap (S \times I)$. This uses the assumption that $S \times I$ lies in the ϵ-thick part of $M \cup M'$.

Now pick a maximal collection of points in $(F_+)_{[\epsilon,\infty)} \cap (S \times I)$, no two of which are less than ϵ apart in F_+. Then the ϵ-balls around these points cover $(F_+)_{[\epsilon,\infty)} \cap (S \times I)$, and there are at most $(\cosh(\epsilon/2) - 1)^{-1}|\chi(F_+)|$ such balls. Letting these be the other type of region, we have established the claim.

We claim that one of the parallel copies of S is disjoint from F_+, when $|n|$ is sufficiently large. Since each of the regions into which we have divided $F_+ \cap (S \times I)$ has uniformly bounded diameter, there is a uniform upper bound on the number of copies of S it can intersect. There is also a uniform upper bound on the number of such regions. Hence, there is a uniform upper bound on the number of copies of S that F_+ can intersect. When $|n|$ is sufficiently large, there are more copies of S than this bound. This proves the claim. (See Figure 2.)

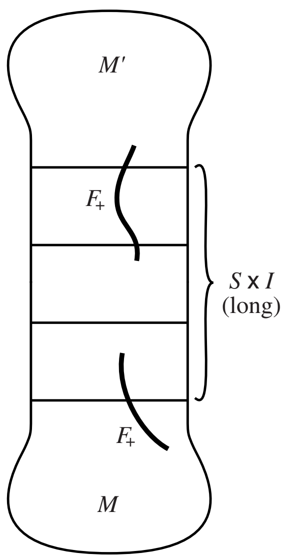

Figure 2.

So, some copy of S lies in the complement of F_+, which is a collection of compression bodies. Since S is incompressible, it must be parallel to a component of F_j for some even j. Thus, if we were to cut $M \cup M'$ along S, we would obtain generalised Heegaard splittings for M and M'. Amalgamate each of these, to form Heegaard surfaces $\tilde{F}$ and $\tilde{F}'$ for M and M'. Then, F is obtained by amalgamating $\tilde{F}$ and $\tilde{F}'$ along S.

This implies that $g(F) = g(\tilde{F}) + g(\tilde{F}') - g(S) \geq g(M) + g(M') - g(S)$. Since we already have the opposite inequality, the theorem is proved. □

3. Generalisations

The main theorem is not the most general possible statement one can make. In fact, the proof gives the following stronger result.

Theorem. *Let M, M', S, h, h' and ψ be as in the main theorem. Then for each $g > 0$, there is an $N > 0$ with the following property: if $|n| \geq N$, then any genus g splitting for $M \cup_{h'\psi^n h} M'$ is obtained from splittings of M and M' by amalgamation. In particular, it is weakly reducible.*

There is a related way of building a Haken 3-manifold via gluing: one can start with a single simple 3-manifold M, and glue two of its boundary components via an orientation-reversing homeomorphism. In this case, we obtain a similar result to the main theorem, but do not obtain a precise equality.

Theorem. *Let M be a simple 3-manifold, and let Y and Y' be distinct boundary components of M. Suppose that there is an orientation-preserving homeomorphism $h: Y \to S$ and an orientation-reversing homeomorphism $h': S \to Y'$, where S is some surface of genus at least two. Let $\psi: S \to S$ be a pseudo-Anosov homeomorphism, and let $M/\!\sim$ be the manifold obtained by gluing Y and Y' via $h'\psi^n h$. Then, provided $|n|$ is sufficiently large,*

$$g(M) - g(S) + 1 \leq g(M/\!\sim) \leq g(M) + g(S) + 1.$$

The proof is very similar, but not identical, to that of the main theorem. To achieve the upper bound on $g(M/\!\sim)$, one starts with a minimal genus splitting for M, and uses it to construct a splitting for $M/\!\sim$. One might have to modify the surface in M to ensure that it does not separate Y from Y'. This may increase its genus by $g(S)$. Then, to construct a Heegaard surface for $M/\sim$, one attaches a tube that runs through S. This increases the genus of the surface by one. Hence, we obtain the upper bound. An instructive example is where M is the product $S \times I$ of a closed orientable surface and an interval, and where $M/\sim$ fibres over the circle. (Of course, though, M is not simple in this case.) Then, $g(M) = g(S)$, but in general, $g(M/\sim)$ may be as much as $2g(S) + 1$. (See [7] for example).

To achieve the lower bound on $g(M/\sim)$, one starts with a minimal genus Heegaard surface F for $M/\sim$. One untelescopes it to a generalised Heegaard splitting satisfying the five conditions given earlier. Using the geometry of $M/\sim$, one can show that this is disjoint from a copy of S in $M/\sim$, provided $|n|$ is sufficiently large. Thus, it gives a generalised Heegaard splitting for M, which can be amalgamated to form a Heegaard surface. One calculates its genus to be $g(F) + g(S) - 1$.

The same issues arise when gluing simple manifolds M and M' but when ∂M and $\partial M'$ are disconnected. Again, one does not obtain an exact equality.

It should be possible to generalise the main theorem even further. One can consider the manifold $M \cup_{h'\psi h} M'$, where $\psi \colon S \to S$ is some homeomorphism. It should be true that, under the hypotheses of the main theorem, and provided the distance of ψ is sufficiently large, then the conclusion of the main theorem holds. Here, distance is as measured by the action of ψ on the curve complex of S. This would indeed represent a generalisation, since the distance of ψ^n, for a given pseudo-Anosov ψ, is arbitrarily large, provided $|n|$ is sufficiently large [2]. One might also try to drop the assumptions that M and M' are acylindrical, or even that they have incompressible boundary. But one would then need to make further hypotheses on ψ. To prove these more general results using the techniques of this paper, one would need to establish geometric control on $M \cup M'$, using the theory of Kleinian groups.

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