Part III Hyperbolic Manifolds

Lent 1999

Examples Sheet 1

1. Show that the three interior angles of a triangle in the hyperbolic plane add up to less than π.

2. If α_1 and α_2 are disjoint geodesics in $\mathbb{H}^3$ which do not share a point at infinity, show that there is a geodesic α_3 which intersects both α_1 and α_2 orthogonally.

3. For an element $A \in PSL(2,\mathbb{C})$, let $\mathrm{tr}(A)$ denote its trace (which is defined up to sign). If $A \neq \pm\mathrm{id}$, show that the corresponding isometry of $\mathbb{H}^3$ is

(i) parabolic if $\mathrm{tr}(A) = \pm 2$

(ii) elliptic if $\mathrm{tr}(A) \in (-2,2) \subset \mathbb{R}$,

(iii) loxodromic if $\mathrm{tr}(A) \in \mathbb{C} - [-2,2]$.

In cases (ii) and (iii), show that $\mathrm{tr}(A)$ determines the conjugacy class of the isometry. In these cases, show how the angle of rotation and the hyperbolic translation length along the invariant geodesic can be calculated from $\mathrm{tr}(A)$.

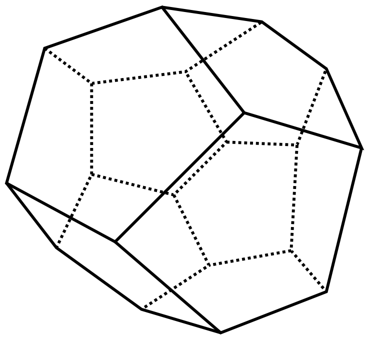
4. Show that any two non-degenerate ideal triangles in $\mathbb{H}^2$ are isometric. Is the same true for ideal quadrilaterals? What about ideal tetrahedra in $\mathbb{H}^3$?

5. Show that, for *any* knot K in S^3, $S^3 - K$ admits an incomplete hyperbolic structure.

6. Let P be a non-degenerate hyperbolic polyhedron. Show that ∂P is the union of the facets of P and that P is the convex hull of its vertices.

7. Show that if P is non-degenerate polyhedron in $\mathbb{H}^3$ and V is the vertices of P, then $\partial P - V$ inherits an (incomplete) hyperbolic structure. Does this extend to a hyperbolic structure on all of ∂P? Show that if P' is a non-degenerate ideal polyhedron in $\mathbb{H}^3$, then $\partial P'$ inherits a complete hyperbolic structure.

8. Let P be a dodecahedron, namely the polyhedron with twelve pentagonal faces, shown overleaf.



Let M be the space obtained by gluing each facet of P to the one opposite it, via a clockwise twist of $3\pi/5$. This is the Seifert-Weber dodecahedral space. Impose a hyperbolic structure on it. [This is necessarily complete and finite volume, and hence unique up to isometry, by Mostow Rigidity.]

9. Construct a complete hyperbolic structure on $S^1 \times \mathbb{R}^{n-1}$.

10. Show that the thrice-punctured 2-sphere S admits a complete hyperbolic structure, obtained by gluing two ideal triangles along their edges via isometries. [This is in fact the unique complete hyperbolic structure on S, up to isometry.] Show, however, that for 'most' ways of gluing the ideal triangles via isometries, the result is an incomplete hyperbolic structure on S.

11. [Hard] Generalise the technique for the construction of the hyperbolic structure on the figure-eight knot complement given in the lectures: construct a hyperbolic structure on the complements of the following links (which, if you do it correctly, will be complete and have finite volume):