Part III Hyperbolic Manifolds

Lent 1999

Examples Sheet 2

1. Let $\tilde{M}$ be the universal cover of a Riemannian manifold M. Verify that a path $\alpha: I \to M$ is a geodesic if and only if some lift of α is a geodesic.

2. Show that if α is a closed geodesic in a complete hyperbolic manifold, then $\eta([\alpha])$ is loxodromic. Deduce that there are no simple closed geodesics in the hyperbolic structure on the thrice-punctured sphere given in Question 10 on Example Sheet 1. (A geodesic is *closed* if it factors through $\mathbb{R} \to S^1 \to M$. A non-closed geodesic $\alpha: \mathbb{R} \to M$ is *simple* if α is injective. A closed geodesic is *simple* if the associated map $S^1 \to M$ is injective.)

3. Show that each homotopically non-trivial closed curve α in a compact hyperbolic n-manifold is freely homotopic to a unique closed geodesic β. In the case $n = 2$, show that β is simple if α was simple. (A *free homotopy* between two closed curves $\alpha_0, \alpha_1: S^1 \to M$ is a homotopy $H: S^1 \times [0,1] \to M$ such that $H|_{S^1 \times \{t\}} = \alpha_t$, for $t = 0$ and 1. The word 'free' is used to emphasise that no basepoints are involved.)

4. Construct a simple non-closed geodesic on each compact orientable hyperbolic 2-manifold.

5. Define a *Euclidean n-manifold* to be a Riemannian manifold, each point of which has an open neighbourhood isometric to an open subset of $\mathbb{E}^n$, where $\mathbb{E}^n$ is $\mathbb{R}^n$ with the standard Euclidean metric. Adapt the techniques of the lectures to show that the universal cover of any complete Euclidean n-manifold is isometric to $\mathbb{E}^n$. A theorem of Bieberbach asserts that any group of isometric covering transformations for $\mathbb{E}^n$ contains a finite index subgroup consisting only of translations. Deduce that any compact Euclidean n-manifold is finitely covered by $S^1 \times \ldots \times S^1$. [One can also define a *spherical n-manifold* to be a Riemannian manifold locally modelled on S^n. Again, any complete spherical manifold has universal cover S^n. However, the techniques of the lectures do not immediately give this fact: where do they break down?]

6. If M is any open subset of $\mathbb{H}^n$ and $\tilde{M}$ is its universal cover, what are the

possible images for $D(\tilde{M})$, where D is a developing map for $\tilde{M}$? [Hint: prove and use the fact that a local isometry $h\colon N \to N'$ between connected Riemannian manifolds is determined by $h(x)$ and $(Th)_x$ for any $x \in N$.]

7. Recall the hyperbolic structure on the compact orientable surface F_k ($k > 1$) given in Theorem 3.2.2, obtained by gluing the facets of a hyperbolic $4k$-gon P. Show that, for a suitable choice of basepoint, a fundamental domain for F_k is P.

8. Show that neither $S^1 \times S^1 \times (0,1)$ nor $S^1 \times (0,1) \times (0,1)$ admits a complete finite volume hyperbolic structure. Show however that they both admit an uncountable number of non-isometric complete (infinite volume) hyperbolic structures.

9. Show that if M is any complete hyperbolic 3-manifold, then $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is not a subgroup of $\pi_1(M)$.

10. For sufficiently small $\epsilon > 0$, determine $\mathrm{inj}^{-1}((0, \epsilon])$ for the complete hyperbolic structure on the figure-eight knot complement given in the lectures. Your description should be both topological and geometric.

11. Let Γ be the set of elements

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R})$$

such that a and d are odd integers, and b and c are even integers. Verify that Γ forms a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Show that it is a group of isometric covering transformations and hence that $\mathbb{H}^2/\Gamma$ inherits a complete hyperbolic structure. Show that this is isometric to the hyperbolic structure on the thrice-punctured sphere S given in Question 10 on Example Sheet 1. [Hint: $\pi_1(S)$ is a free group on two generators. Show that (a suitable choice of) η sends these generators to

$$\pm \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \pm \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Hence, $\eta(\pi_1(S))$ is a subgroup of Γ. So, S covers $\mathbb{H}^2/\Gamma$. Since S has finite volume, this is a finite cover. Now show that this cover must be the identity.]

12. Construct, for each compact orientable surface F, a non-identity homeomorphism $h\colon F \to F$ such that $h \circ h$ is the identity. Let M be the result of $F \times [0, 1]$ after gluing $F \times \{0\}$ to $F \times \{1\}$ via h. Show that there is a cover $M \to M$ which

has finite index greater than one. Deduce that M has zero Gromov norm and hence does not admit a hyperbolic structure. Why is M not a counter-example to the conjecture of Thurston given before the start of Section 1?