Property (τ) and Geometry, II

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Main Topics:

- The consequences of the failure of Property (τ)
- Groups with fast subgroup growth and fast 'homology growth'
- Applications to 3-dimensional topology and geometry

Basic Principle

Let G be a finitely presented group.

Algebraic properties of its subgroups $\longleftrightarrow$ Geometry of its quotients

The quotients G/G_i (where G_i is a normal subgroup, usually finite index) can viewed as geometric objects:

Let S = a finite generating set for G

Let $X(G/G_i; S)$ = the Cayley graph of G/G_i with respect to S.
Its vertices: cosets $G_i g$
Its edges: $G_i g \leftrightarrow G_i gs$ $(s \in S \cup S^{-1})$
Make each edge have length 1.

Examples of geometric properties of the Cayley graph $X(G/G_i; S)$

Its diameter

Its Cheeger constant

Examples of algebraic properties of G_i

Amalgamated free product?

Positive b_1?

Free non-abelian quotient?

Its rank (ie minimal number of generators)

The rank of its first homology

The reason for the principle

Let K be a finite 2-complex with $\pi_1(K) = G$.

Let $K_i \to K$ be the covering space corresponding to the subgroup G_i.

$$\begin{array}{lcl}
\text{Geometric properties} & \longleftrightarrow & \text{Algebraic properties} \\
\text{of } K_i & & \text{of } \pi_1(K_i) = G_i \\
\quad\updownarrow & & \\
\text{Geometric properties} & & \\
\text{of } X(G/G_i; S) & &
\end{array}$$

Theorem 1.1: [L] Let G be a finitely presented group. Let $\{G_i\}$ be a nested sequence of finite index normal subgroups. Then one of the following must hold:

1. G_i is an amalgamated free product or HNN extension for all $i >> 0$;
2. G has Property (τ) with respect to $\{G_i\}$;
3. $\lim_{i \to \infty} (d(G_i) - 1)/[G : G_i] = 0$.

Here, $d(\,\cdot\,)$ is the rank of a group, ie the minimal number of generators.

(1) and (3) are algebraic properties of G_i

(2) is a geometric property of the Cayley graphs $X(G/G_i; S)$.

Possible applications to 3-manifold theory

Let G be the fundamental group of a closed orientable 3-manifold M.

Conjecture: (The virtually Haken conjecture)
If $|G| = \infty$, then some finite index subgroup G_i is an amalgamated free product or HNN extension.

(The covering space M_i of M corr. to G_i is known as Haken.)

$$\Uparrow$$

Conjecture: (The positive virtual b_1 conjecture)
If $|G| = \infty$, then $b_1(G_i) > 0$ for some finite index subgroup G_i.

$$\Uparrow$$

$$\Uparrow$$

(assuming the Geometrisation Conjecture)

Conjecture: (The infinite virtual b_1 conjecture)
If G is a lattice in $\mathrm{PSL}(2, \mathbb{C})$, then G has finite index subgroups with arbitrarily large b_1.

$$\Uparrow$$

Conjecture: (The largeness conjecture)
If G is a lattice in $\mathrm{PSL}(2, \mathbb{C})$, then G is large.

Recall: a group G is large if some finite index subgroup admits a surjective homomorphism onto a non-abelian free group.

Possible approach to the Virtually Haken conjecture:

Show that G has nested sequence of finite index normal subgroups $\{G_i\}$ s.t.

1. G does not have (τ) w.r.t. $\{G_i\}$
2. $\inf_i (d(G_i) - 1)/[G : G_i] > 0$.

Theorem 1.1 $\Rightarrow$ G_i is an amalgamated free product or HNN extension for all $i >> 0$.

(In fact, we'll follow a slight variant of this strategy.)

Rank gradient

The rank gradient of $(G, \{G_i\})$ is

$$\inf_i \frac{d(G_i) - 1}{[G : G_i]}.$$

Recall: (Reidermeister-Schreier)

$$d(G_i) - 1 \leq [G : G_i](d(G) - 1).$$

So, the rank gradient measures how far this is from being an equality.

Examples:

1. $G = \mathbb{Z}$, $G_i = i\mathbb{Z}$
Zero rank gradient

2. $G = \mathbb{Z} * \mathbb{Z}$, $G_i =$ any sequence of f.i. subgroups
Positive rank gradient.

Proof of Theorem 1.1

Suppose that

1. G does not have Property (τ) w.r.t. $\{G_i\}$, and
2. $(G, \{G_i\})$ has positive rank gradient.

We will show that G_i is an amalgamated free product or HNN extension, for all $i >> 0$.

Let K = a finite 2-complex with $\pi_1(K) = G$, with a single 0-cell.

Let K_i = covering space corr. to G_i.

Let X_i = 1-skeleton of K_i. This is $X(G/G_i; S)$, where S is the generating set for G arising from the 1-cells of K.

Positive rank gradient $\Rightarrow$

for all $i >> 0$,

$$d(G_i) \sim \text{rank gradient}(G_i, \{G_j : j \geq i\})$$

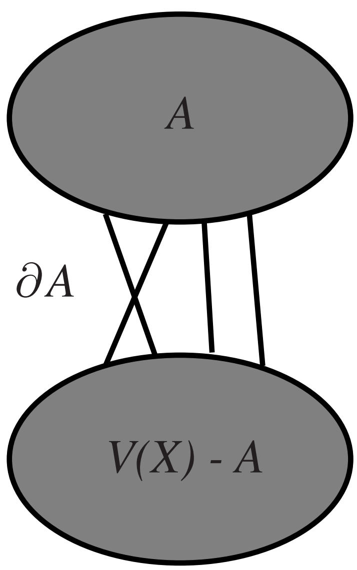
So, by passing to G_i, we can assume

$$d(G) \sim \text{rank gradient}(G, \{G_i\})$$

Not $(\tau) \Rightarrow h(X_i) \to 0$.

Recall:

$$h(X_i) = \min \left\{ \frac{|\partial A_i|}{|A_i|} : \begin{array}{c} A_i \subset V(X_i), \\ 0 < |A_i| \leq |V(X_i)|/2 \end{array} \right\}.$$



Elementary facts:

Let A_i realise the above minimum. Then

1. A_i and A_i^c are connected;

2. $|A_i| > |V(X_i)|/4$.

Use A_i and A_i^c to decompose K_i into two pieces B_i and C_i, each of which is connected.

Case 1: $B_i \cap C_i$ is disconnected.

Then $\pi_1(K_i) = G_i$ is an HNN extension. $\square$

Case 2: $B_i \cap C_i$ is connected.

Seifert - van Kampen theorem:

$\pi_1(K_i) = G_i$ is the push-out of

$$\begin{array}{ccc} \pi_1(B_i \cap C_i) & \rightarrow & \pi_1(B_i) \\ \downarrow & & \\ \pi_1(C_i) & & \end{array}$$

Taking images in $\pi_1(K_i)$, we see that $\pi_1(K_i) = G_i$ is the push-out of

$$\begin{array}{ccc} \mathrm{Im}(\pi_1(B_i \cap C_i)) & \rightarrow & \mathrm{Im}(\pi_1(B_i)) \\ \downarrow & & \\ \mathrm{Im}(\pi_1(C_i)) & & \end{array}$$

So, either G_i is the amalgmated free product

$$\mathrm{Im}(\pi_1(B_i)) *_{\mathrm{Im}(\pi_1(B_i \cap C_i))} \mathrm{Im}(\pi_1(C_i))$$

or $G_i = \mathrm{Im}(\pi_1(B_i))$ or $\mathrm{Im}(\pi_1(C_i))$

$\Rightarrow d(G_i) \leq d(\pi_1(B_i))$ or $d(\pi_1(C_i))$.

But the 1-skeleton of B_i gives an explicit generating set for $\pi_1(B_i)$: it has size at most (roughly)

$$\begin{array}{cc} (d(G)-1)[G : G_i]/2 & +h(X_i)[G : G_i] \\ \wr & \uparrow \\ d(G_i)/2 & \text{small} \end{array}$$

So, $d(G_i) \lesssim d(G_i)/2$.

Contradiction $\square$

Natural questions

How can we ensure that a group G has a nested sequence of finite index normal subgroups $\{G_i\}$

- without Property (τ)?
- with positive rank gradient?

These questions are related!

We'll examine Q2 next time. Let's consider Q1.

Groups without Property (τ)

Proposition 1.2: Suppose that G has a finite index subgroup G_1 such that $b_1(G_1) > 0$. Then G does not have Property (τ).

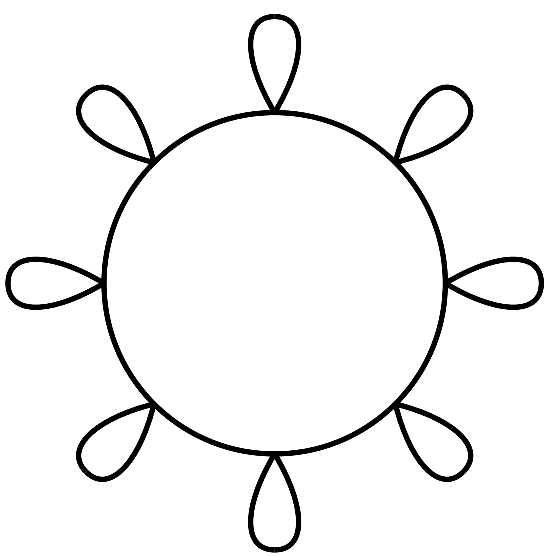
Proof: We have $\phi\colon G_1 \twoheadrightarrow \mathbb{Z}$. Let $G_i = \phi^{-1}(i\mathbb{Z})$.

G has (τ) w.r.t.$\{G_i\} \Leftrightarrow G_1$ has (τ) w.r.t.$\{G_i\}$.

Pick a finite set of generators $\{s_1, \dots, s_n\}$ for G_1 such that

$$\phi(s_1) = 1, \qquad \phi(s_i) = 0 \ \forall i > 1.$$

Then $X(G_1/G_i; S)$ is:



$h \to 0$

Question: Does the converse of Proposition 1.2 hold for finitely presented groups?

$$\begin{array}{ccc} (\tau) & \overset{?!}{\Longleftrightarrow} & \text{no f.i. subgroup has } b_1 > 0 \\ & & \updownarrow \\ & & \text{"FAb"} \end{array}$$

Probably not

No, in general, for finitely generated groups (eg Grigorchuk group)

But one can characterise finitely presented groups that have a finite index subgroup G_i with $b_1(G_i) > 0$ in terms of the existence of a sequence of finite index subgroups $\{G_i\}$ s.t. $h(X(G/G_i))$ tends to 0 sufficiently quickly as a function of $[G : G_i]$.

Let's focus on closed hyperbolic 3-manifolds:

$$M = \mathbb{H}^3/G$$

where G = discrete group of isometries acting freely and cocompactly on $\mathbb{H}^3$.

Conjecture: [Lubotzky-Sarnak] G does not have Property (τ)

This is a consequence of the positive virtual b_1 conjecture, but it may be easier to prove. One approach:

GS-τ Conjecture: [Lubotzky-Zelmanov] Let $G = \langle X | R \rangle$, and let p be a prime. Suppose that

$$\frac{d_p(G)^2}{4} > |R| - |X| + d_p(G).$$

Then G does not have Property (τ) (w.r.t some nested sequence of f.i. normal subgroups, each with index a power of p).

Here

$$d_p(G) = \dim(H_1(G; \mathbb{F}_p)) = \dim\left(\frac{G}{[G,G]G^p}\right).$$

GS-τ $\Rightarrow$ Lubotzky-Sarnak

Theorem 1.3: [consequ. of Strong Approx. & Lubotzky alternative] Let G be a fin. gen. linear group that is not virtually soluble. Then, for any prime p, G has f.i. subgroups K where $d_p(K)$ is arbitrarily large.

Theorem 1.4: Let G be the fundamental group of a closed 3-manifold. Then G has a presentation $\langle X|R\rangle$ where $|X| = |R|$.

Proof: Pick a cell structure for M.
For $i = 0, \ldots, 3$, let C_i be the number of i-cells.

Can ensure $C_0 = C_3 = 1$.

$$\begin{aligned} &C_0 - C_1 + C_2 - C_3 \\ &= b_0(M) - b_1(M) + b_2(M) + b_3(M) = 0. \end{aligned}$$

So, $C_1 = C_2$.
$C_1 \longrightarrow$ generators X, $\qquad C_2 \longrightarrow$ relations R. $\square$

GS-$\tau \Rightarrow$ Lubotzky-Sarnak (continued)

Let $G = \pi_1(M)$.

Find a f.i. subgroup G_1 of G with $d_p(G_1) \geq 5$.

$G_1 = \pi_1$(closed 3-manifold) and so $G_1 = \langle X | R \rangle$, where $|X| = |R|$. So,

$$d_p(G_1)^2/4 > |R| - |X| + d_p(G_1).$$

GS-$\tau \Rightarrow G_1$ does not have Property (τ).