Property (τ) and Geometry, III

Marc Lackenby

University of Oxford

Recall:

Theorem 1.1: [L] Let G be a finitely presented group. Let $\{G_i\}$ be a nested sequence of finite index normal subgroups. Then one of the following must hold:

1. G_i is an amalgamated free product or HNN extension for all $i >> 0$;

2. G has Property (τ) with respect to $\{G_i\}$;

3. $\inf_i (d(G_i) - 1)/[G : G_i] = 0.$

$$\uparrow$$

Rank gradient

Natural questions:

How can we ensure that a group G has a nested sequence of finite index normal subgroups $\{G_i\}$

- without Property (τ)?
- with positive rank gradient?

We examined Q1 last time.

Let's consider Q2 now.

The mod p homology gradient is

$$\inf_i \frac{d_p(G_i) - 1}{[G : G_i]}.$$

Recall

$$d_p(G_i) = \dim(H_1(G_i; \mathbb{F}_p)).$$

The mod p homology gradient is clearly a lower bound for the rank gradient.

One can sometimes prove that it is positive eg:

Theorem 2.1: [L] Let G be a lattice in $\mathrm{PSL}(2, \mathbb{C})$. Suppose that either

1. G contains a torsion element, or
2. G is arithmetic.

Then G has a nested sequence of finite index subgroups G_i with positive mod p homology gradient, for some prime p.

Relationship with subgroup growth

Theorem 2.2: Let G be a finitely generated group, and let p be a prime. TFAE:

1. G has an infinite nested sequence of finite index subnormal subgroups, each with index a power of p, and with positive mod p homology gradient;

2. the pro-p completion of G has exponential subgroup growth.

Homology growth $\longleftrightarrow$ subgroup growth

(1) $\Rightarrow$ (2): Let G_i be a f.i. (subnormal) subgroup of G. The number of normal subgroups of G_i with index p is

$$\frac{p^{d_p(G_i)} - 1}{p - 1}.$$

(2) $\Rightarrow$ (1): Omitted.

Positive mod p homology gradient has stronger consequences than positive rank gradient:

Theorem 2.3: [L] Let G be a finitely presented group, and let

$$G \geq G_1 \triangleright G_2 \triangleright \ldots$$

be a sequence of finite index subgroups such that each G_{i+1} is normal in G_i and has index a power of p. Suppose that the mod p homology gradient of this sequence is positive. Then

1. G is large, or
2. G has Property (τ) w.r.t. $\{G_i\}$.

+ve rank gradient	$\rightsquigarrow$	amalgam/HNN/(τ)
+ve homology gradient	$\rightsquigarrow$	large or (τ)

Corollary 2.4: [L] Under the hypotheses of Theorem 2.3, G has Property (τ) w.r.t. some nested sequence of finite index subgroups.

Corollary 2.5: [L] If G is finitely presented and $\hat{G}_p$ has exponential subgroup growth, then G has Property (τ) w.r.t. some nested sequence of finite index subgroups.

A key step in the proof of Theorem 2.3:

Theorem 2.6: [L] Let K be a cell complex, and let B and C be subcomplexes s.t. $K = B \cup C$. Suppose that the maps

$$H^1(B; \mathbb{F}_p) \to H^1(B \cap C; \mathbb{F}_p)$$
$$H^1(C; \mathbb{F}_p) \to H^1(B \cap C; \mathbb{F}_p)$$

induced by inclusion are not injections. In the case $p = 2$, suppose also that the kernel of at least one of these maps has dimension more than 1. Then $\pi_1(K)$ is large.

Define the mod p Cheeger constant $h_p(M)$ of a closed manifold M to be

$$\inf \left\{ \frac{|H_1(S;\mathbb{F}_p)|}{\min\{|H_1(M_1;\mathbb{F}_p)|, |H_1(M_2;\mathbb{F}_p)|\}} \right\},$$

as S runs over all codimension 1 submanifolds S of M, dividing M into M_1 and M_2.

Corollary 2.7: [L] If

$$h_p(M) < \begin{cases} 1 & \text{if } p > 2; \\ 1/2 & \text{if } p = 2, \end{cases}$$

then $\pi_1(M)$ is large.

Let's assume Theorem 2.6 for the moment.

Outline of the proof of Theorem 2.3

Suppose that

1. G does not have Property (τ) w.r.t. $\{G_i\}$, and
2. $(G, \{G_i\})$ has positive mod p homology gradient.

We will show that G_i is large for all $i >> 0$, and hence so is G.

Proposition 2.8: Let G_i be a subnormal subgroup of G with index a power of p. Then

$$d_p(G_i) - 1 \leq [G : G_i](d_p(G) - 1).$$

compare: Reidermeister-Shreier

So, by replacing G by G_i for some big i, we may assume that $d_p(G_i)$ is approximately the mod p homology gradient of $(G, \{G_i\})$.

Let K = a finite 2-complex with $\pi_1(K) = G$, with a single 0-cell.

Let K_i = covering space corr. to G_i.

Let X_i = 1-skeleton of K_i.

Let A_i be a non-empty subset of $V(X_i)$ such that

$$|A_i| \leq |V(X_i)|/2 \text{ and } h(X_i) = |\partial A_i|/|A_i|.$$

Use A_i to decompose K_i into two sets B_i and C_i with small intersection.

Then

$$d_p(B_i) \gtrapprox d_p(K_i)/4$$
$$d_p(C_i) \gtrapprox d_p(K_i)/4$$
$$d_p(B_i \cap C_i) \text{ is small.}$$

(This is not completely obvious.)

Corollary 2.7$\Rightarrow$ $\pi_1(K_i) = G_i$ is large. $\square$

Towards the proof of Theorem 2.6

Use cohomology with mod p coefficients

$$H^1(K; \mathbb{F}_p) = \frac{\text{1-cocycles}}{\text{1-coboundaries}}$$

A 1-cochain is an assignment to each oriented 1-cell of an integer mod p (its weight)

It is a 1-cocycle if its evaluation around each 2-cell is zero.

It is a 1-coboundary if its evaluation around every closed loop is zero.

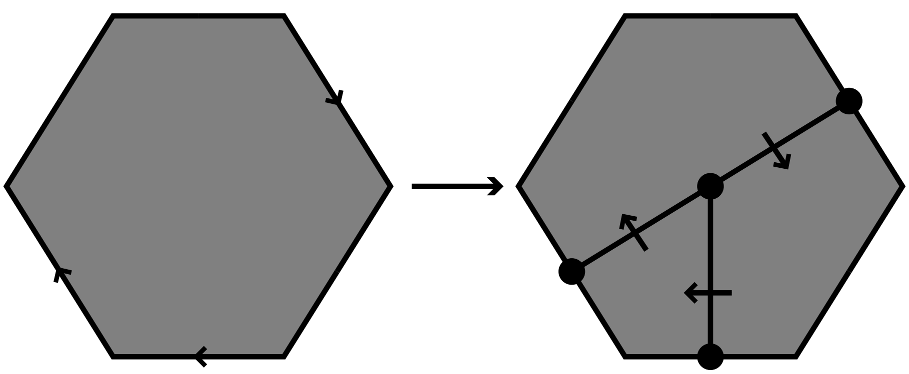
From a 1-cocycle, construct a subgraph of K called a regular cocycle Γ:

Place a vertex of Γ at the midpoint each 1-cell with non-zero weight.

Place a vertex of Γ into the centre of each 2-cell that has at least one 1-cell with non-zero weight in its boundary.

Join these up with edges.

Assign weights and transverse orientations in the obvious way.



A regular cocycle specifies a collapsing map

$$K \to L(p)$$

where $L(p)$ is the 2-complex with
a single 0-cell,
a single 1-cell,
a single 2-cell, winding p times around the 1-cell.

Proposition 2.9: Suppose that K contains two disjoint regular cocycles whose union is non-separating. Then

$$\pi_1(K) \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z}).$$

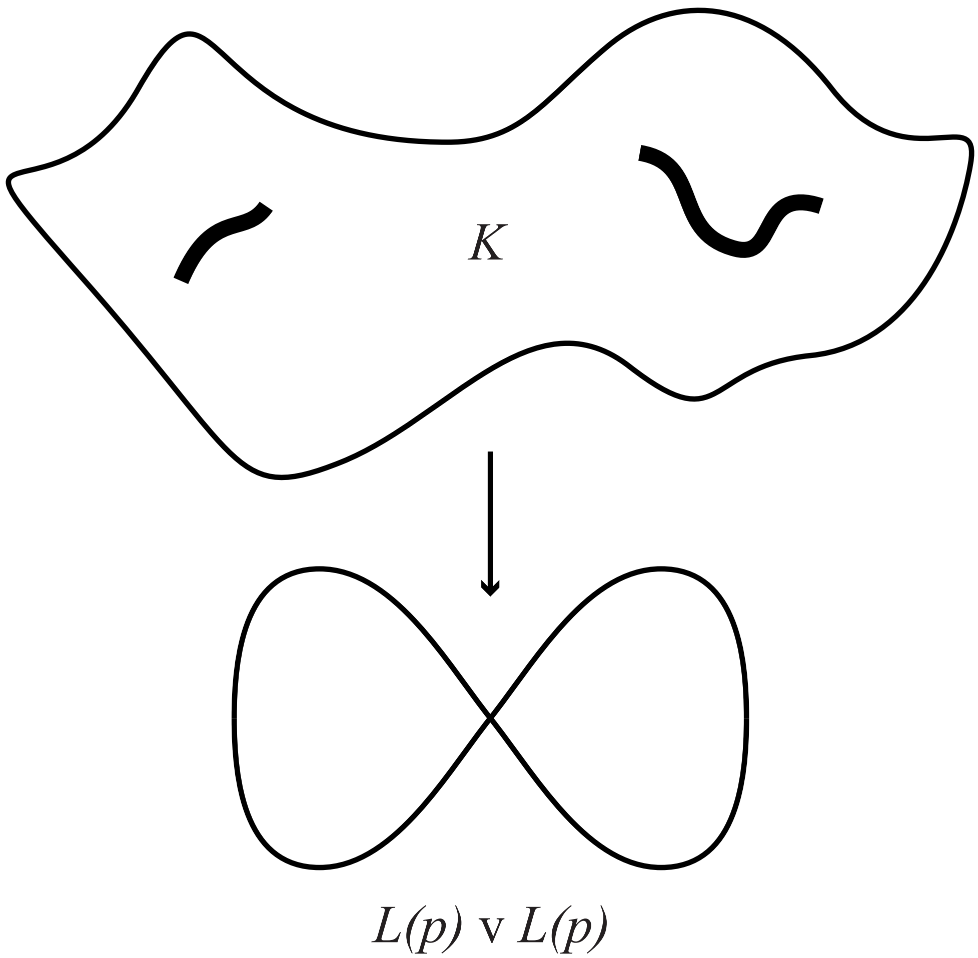
Hence, $\pi_1(K)$ is large if $p > 2$.

(There is a version when $p = 2$.)

Proof: We have a collapsing map

$$K \to L(p) \vee L(p).$$

Since the cocycles are non-separating, the induced homomorphism is surjective. $\square$



Proof of Theorem 2.6.

Find a regular cocycle in B representing a non-trivial element in the kernel of

$$H^1(B; \mathbb{F}_p) \to H^1(B \cap C; \mathbb{F}_p).$$

Can arrange that this is disjoint from $B \cap C$ and non-separating.

Find a similar regular cocycle in C.

Their union is non-separating in K.

Proposition 2.9 $\Rightarrow$ $\pi_1(K)$ is large. $\square$

Theorem 2.10: [L] Let G be a finitely presented group. Then TFAE:

1. G is large;

2. there is a sequence

$$G \geq G_1 \geq G_2 \ldots$$

of finite index subgroups, each normal in G_1, such that

(i) G_i/G_{i+1} is abelian for each i;

(ii)

$$\frac{\log |G_i/G_{i+1}|}{[G : G_i]} \to \infty$$

(iii)

$$\limsup_i \frac{d(G_i/G_{i+1})}{[G : G_i]} > 0.$$

Let's assume that each $[G_1 : G_i]$ is a power of p.

Idea of proof: (1) $\Rightarrow$ (2) is easy.

(2) $\Rightarrow$ (1):

(i) and (ii) $\Rightarrow$ G does not have (τ) w.r.t. $\{G_i\}$ [Lubotzky-Weiss]

(iii) $\Rightarrow$ the mod p homology gradient is positive.

Theorem 2.3 $\Rightarrow$ G is large.

Comments

1. For finitely presented groups, largeness is detectable by the profinite completion.

2. This gives a new proof of a theorem of Baumslag and Pride:

   If $G = \langle X | R \rangle$ where $|X| - |R| > 1$, then G is large.

3. Theorem 2.3 is false in general for finitely generated (infinitely presented) groups!

4. Condition (ii) can be dropped if each G_i/G_{i+1} is an elementary abelian p-group. This implies:

   Theorem 2.11. [L] If the derived p-series of a finitely presented group G has positive mod p homology gradient, then G is large.

   The proof, which is too complicated to give here, uses the theory of error-correcting codes!