# Property $(\tau)$ and geometry, IV

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# The Geometrisation Conjecture

Let $M$ be a closed orientable 3-manifold.

Geometrisation conjecture: '$M$ admits a canonical decomposition into geometric pieces'.

Canonical decomposition: This involves:

1. Expressing $M$ as a connected sum of prime 3-manifolds

2. Decomposing each prime piece along $\pi_1$-injective tori in a canonical way.

Geometric pieces: Each is of the form $X/G$, where

1. $X$ is a simply-connected, homogeneous, complete, Riemannian 3-manifold;

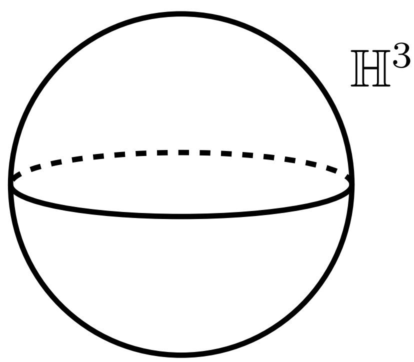
2. $G$ is a discrete group of isometries of $X$, acting freely.

8 possibilities for the geometry $X$:

$$\mathbb{E}^3, S^3, \mathbb{H}^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \mathrm{Nil}, \mathrm{Sol}, \widetilde{\mathrm{SL}(2,\mathbb{R})}$$

Of these, 7 are very well understood. The remaining one, hyperbolic geometry, is less so.

We will focus on hyperbolic 3-manifolds.




$$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$$

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|---|---|---|
| Finite volume hyperbolic 3-manifolds | $\longleftrightarrow$ | Torsion free lattices in $\mathrm{PSL}(2,\mathbb{C})$ |

Let $M$ = finite volume hyperbolic 3-manifold

Let $G = \pi_1(M)$.

Recall:

Conjectures:

$G$ has a finite index subgroup $G_i$ s.t.

1. $G_i$ is an amalgam/HNN
2. $b_1(G_i) > 0$
3. $b_1(G_i)$ is arbitrarily large
4. $G_i$ has free non-abelian quotient.

Very important lemma: Let $M$ be a compact orientable 3-manifold. Then

$$b_1(M) \geq b_1(\partial M)/2.$$

Proof: Use $\mathbb{Q}$ coefficients:

$$\begin{array}{ccccc}
H^1(M) & \to & H^1(\partial M) & \to & H^2(M, \partial M) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
H_2(M, \partial M) & \to & H_1(\partial M) & \to & H_1(M)
\end{array}$$

$$\begin{aligned}
&\mathrm{Ker}(H_1(\partial M) \to H_1(M)) \\
\cong &\mathrm{Im}(H^1(M) \to H^1(\partial M)) \\
\cong &\mathrm{Im}(H_1(\partial M) \to H_1(M))
\end{aligned}$$

So, each has dimension $b_1(\partial M)/2$. $\square$

This works for any field coefficients, and so

$$d_p(M) \geq d_p(\partial M)/2$$

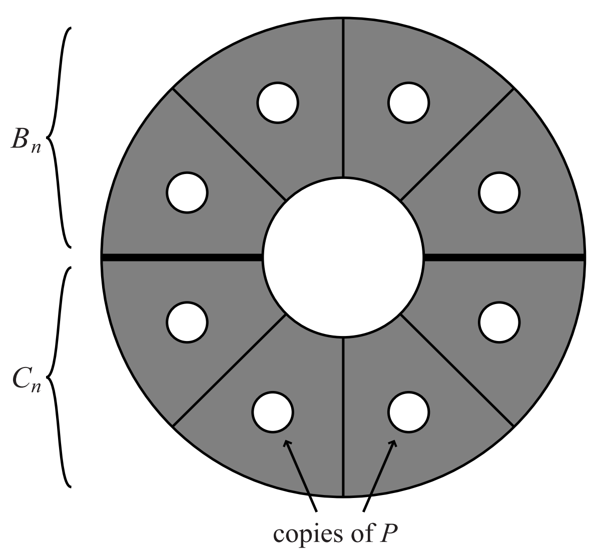
Theorem 3.1: [Cooper-Long-Reid]
Let $M$ be a finite volume hyperbolic 3-manifold with non-empty boundary. Then $\pi_1(M)$ is large.

Proof: $\partial M$ is a collection of tori. We may find a finite covering space $\tilde{M}$ with at least 3 boundary tori. Let $P$ be one of these.

$$b_1(\tilde{M}) \geq b_1(\partial \tilde{M})/2 \geq 3$$

So, $i^*\colon H^1(\tilde{M}) \to H^1(P)$ has non-trivial kernel, containing a non-zero element $\alpha$, say.

Let $M_n$ be the $n$-fold cyclic cover of $\tilde{M}$ arising from $\alpha$. Decompose it into 2 halves: $B_n$ and $C_n$.




$d_p(B_n)$ big (any prime $p$)
$d_p(C_n)$ big
$d_p(B_n \cap C_n)$ indep of $n$

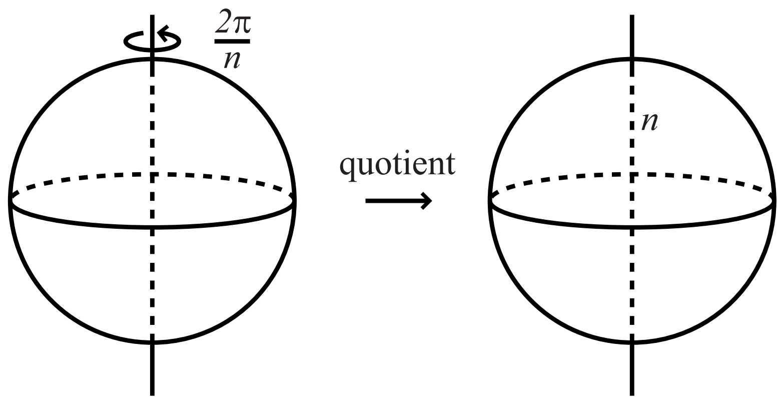
Corollary 2.7 $\Rightarrow$
$\pi_1(M_n)$ large

## Orbifolds

| | | |
|---|---|---|
| Lattices in $\mathrm{PSL}(2,\mathbb{C})$ maybe with torsion | $\longleftrightarrow$ | Finite volume hyperbolic 3-orbifolds |

A 3-orbifold is a space locally modelled on

$$B^3/(\text{finite subgroup of } O(3)).$$




## Basic principle

Easier $\longleftarrow \cdots\cdots \longrightarrow$ Harder

| 3-manifolds with boundary | 3-orbifolds | closed 3-manifolds |
|---|---|---|

Theorem 3.2: [L] Let $O = \mathbb{H}^3/G$ be a compact orientable 3-orbifold. Suppose that the singular locus is a link $L$, and that the singularity order of each component of $L$ is a prime $p$. Then

$$d_p(G) \geq |L|.$$

Sketch proof: Let $M$ be $O - \text{int}(N(L))$. This is a compact orientable 3-manifold. So,

$$d_p(M) \geq d_p(\partial M)/2 \geq |L|.$$

$G$ is obtained from $\pi_1(M)$ by quotienting $p^{\text{th}}$ powers of certain words (the meridians of the singular locus). This does not affect $d_p$. $\square$

So, as far as mod $p$ homology is concerned, these 3-orbifolds behave as though they have boundary.

Theorem 3.3: [L] Let $G$ be a lattice in $\mathrm{PSL}(2,\mathbb{C})$ with torsion. Then $G$ has a nested sequence of f.i. subgroups with positive mod $p$ homology gradient, for some prime $p$.

This $\Rightarrow$ at least exponential subgroup growth.

Idea of proof: Let $O$ be the 3-orbifold $\mathbb{H}^3/G$.

By passing to a finite index subgroup of $G$, we may assume that the singular locus of $O$ is a link $L$, and the singularity order of each component is a prime $p$ . Fix a component $K$ of $L$.

Find a sequence of finite covering spaces $O_i$ of $O$, such that the inverse image of $K$ is a disjoint union of copies of $K$. (This requires use of the Golod-Shafarevich inequality.)

Let $G_i$ be the finite index subgroup of $G$ corr. to $O_i$. Then, by Theorem 3.2,

$$d_p(G_i) \geq |\mathrm{sing}(O_i)| \geq [G : G_i].$$

Recall:

Theorem 2.3: [L] Let $G$ be a finitely presented group, and let

$$G \geq G_1 \triangleright G_2 \triangleright \dots$$

be a sequence of f.i. subgroups such that each $G_{i+1}$ is normal in $G_i$ and has index a power of $p$. Suppose that the mod $p$ homology gradient of this sequence is positive. Then

1. $G$ is large, or
2. $G$ has Property $(\tau)$ w.r.t. $\{G_i\}$.

Key question: Does $G$ have Property $(\tau)$ w.r.t. $\{G_i\}$ ?

If not, then $G$ is large.

This would follow from:

GS-$\tau$ Conjecture: [Lubotzky - Zelmanov]

Let $G = \langle X | R \rangle$. Suppose that

$$\frac{d_p(G)^2}{4} > |R| - |X| + d_p(G).$$

Then $G$ does not have Property $(\tau)$ (w.r.t some nested sequence of f.i. normal subgroups, each with index a power of $p$).

GS-$\tau$ Conjecture $\Rightarrow$ any lattice in $\mathrm{PSL}(2, \mathbb{C})$ with torsion is large.

## Arithmetic 3-manifolds

An important class of hyperbolic 3-manifolds.

Theorem 3.4: [Reid] Any arithmetic lattice in $\mathrm{PSL}(2, \mathbb{C})$ is commensurable with one containing torsion.

So the above results for 3-orbifolds also apply to arithmetic 3-manifolds:

1. Positive mod $p$ homology gradient
2. Exponential subgroup growth
3. GS-$\tau$ $\Rightarrow$ large. In fact:

   Theorem 3.5: [L-Long-Reid]
   The Geometrisation Conjecture and the Lubotzky-Sarnak Conjecture together imply that any arithmetic lattice in $\mathrm{PSL}(2, \mathbb{C})$ is large.

These techniques give many other results about arithmetic 3-manifolds, eg:

Theorem 3.6: [L-Long-Reid] Let $G$ be an arithmetic lattice in $\mathrm{PSL}(2,\mathbb{C})$ such that $b_1(G_i) \geq 4$ for some f.i. subgroup $G_i$ of $G$. Then $G$ is large.

Theorem 3.7: [L-Long-Reid] Let $G$ be an arithmetic lattice in $\mathrm{PSL}(2,\mathbb{C})$ containing one of the finite groups $A_4$, $S_4$ or $A_5$. Then $G$ is large.

Still quite mysterious.

Theorem 3.8: [L] Let $M$ be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ has a nested sequence of finite index subgroups $G_i$ such that

$$d_2(G_i) \geq \frac{[G:G_i]}{\sqrt{\log[G:G_i]}\log\log[G:G_i]}.$$

This is not quite linear.

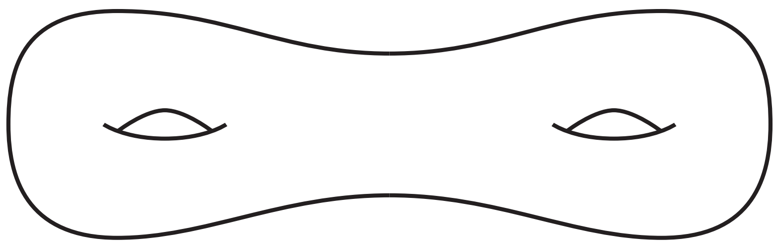
This $\Rightarrow$ fast subgroup growth.

Maybe can't do better than this?? Probably too ambitious to try and prove positive mod $p$ homology gradient.

Let's return to rank gradient, or more precisely a variant known as Heegaard gradient.

## Heegaard splittings

A handlebody is a 'thickened up' graph. Its genus is the genus of its boundary surface.



Gluing two handlebodies with the same genus $g$, via a homeomorphism between their boundaries, gives a closed orientable 3-manifold $M$. This is said to be a Heegaard splitting for $M$. The boundary of each handlebody is a Heegaard surface. Every closed orientable 3-manifold has a Heegaard splitting. The Heegaard genus $g(M)$ for $M$ is the minimal genus of a Heegaard splitting.

The generators of $\pi_1$(one handlebody) gives a generating set for $\pi_1(M)$. So

$$d(\pi_1(M)) \leq g(M)$$

Let $\tilde{M} \to M$ be a finite-sheeted covering.

The inverse image of a Heegaard surface for $M$ is a Heegaard surface for $\tilde{M}$. So

$$g(\tilde{M}) - 1 \leq (g(M) - 1)\deg(\tilde{M} \to M)$$

compare: Reidermeister-Shreier

Define the Heegaard gradient of a sequence $M_i$ of finite-sheeted covers of $M$ to be

$$\inf_i \frac{g(M_i) - 1}{\deg(M_i \to M)}$$

Heegaard gradient $\geq$ rank gradient

Theorem 3.9: [L] Let $M$ be a closed orientable irreducible 3-manifold. Suppose that $M$ has a nested sequence of finite-sheeted regular covers $M_i$ such that

1. they have positive Heegaard gradient, and
2. $\pi_1(M)$ does not have Property $(\tau)$ w.r.t. $\pi_1(M_i)$.

Then $M_i$ is Haken for all $i >> 0$.

Heegaard gradient conjecture: [L] If $M_i$ does not have positive Heegaard gradient, then $M$ has a finite-sheeted cover that fibres over $S^1$.

| | | |
|---|---|---|
| Heegaard gradient conjecture + Lubotzky-Sarnak conjecture | $\Rightarrow$ | Virtually Haken conjecture |

Evidence for the Heegaard gradient conjecture:

Theorem 3.10: [L] Let $\{M_i \to M\}$ be a collection of finite regular covering spaces of a hyperbolic 3-manifold $M$. If $g(M_i) = o(\sqrt[4]{\deg(M_i \to M)})$, then $M_i$ fibres over $S^1$ for all $i >> 0$.

Theorem 3.11: [L] Let $h(M)$ be the Cheeger constant of $M$. Then

$$h(M) \leq 2\pi \frac{g(M) - 1}{\text{Volume}(M)}$$

So, $\quad (\tau) \;\Rightarrow$ Positive Heegaard gradient

So, for any $M$, at least one of the Lub-Sarnak conjecture and the Heegaard gradient conjecture is true!

The proof of both of these use minimal surface theory: 'most' minimal genus Heegaard surfaces can be realised as a minimal surface.

## Unsolved problems

1. Virtually Haken conjecture (and stronger forms)

2. GS-$\tau$ conjecture

3. Lubotzky-Sarnak conjecture

4. Heegaard gradient conjecture

5. If a finitely presented group does not have Property $(\tau)$, does it have a finite index subgroup with positive $b_1$?

6. If a finitely presented group $G$ has positive mod $p$ homology gradient for some nested sequence of finite index subgroups, must $G$ be large?

7. Are there finitely presented, residually finite groups of intermediate growth?

5 and 6 are unlikely, but answering them would be interesting.