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Finite covering spaces of 3-manifolds

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Abstract. Following Perelman's solution to the Geometrisation Conjecture, a 'generic' closed 3-manifold is known to admit a hyperbolic structure. However, our understanding of closed hyperbolic 3-manifolds is far from complete. In particular, the notorious Virtually Haken Conjecture remains unresolved. This proposes that every closed hyperbolic 3-manifold has a finite cover that contains a closed embedded orientable π_1-injective surface with positive genus.

I will give a survey on the progress towards this conjecture and its variants. Along the way, I will address other interesting questions, including: What are the main types of finite covering space of a hyperbolic 3-manifold? How many are there, as a function of the covering degree? What geometric, topological and algebraic properties do they have? I will show how an understanding of various geometric and topological invariants (such as the first eigenvalue of the Laplacian, the rank of mod p homology and the Heegaard genus) can be used to deduce the existence of π_1-injective surfaces, and more.

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1. Introduction

In recent years, there have been several huge leaps forward in 3-manifold theory. Most notably, Perelman [50, 51, 52] has proved Thurston's Geometrisation Conjecture [62], and, as a consequence, a 'generic' closed orientable 3-manifold is known to admit a hyperbolic structure. However, our understanding of closed hyperbolic 3-manifolds is far from complete. In particular, finite covers of hyperbolic 3-manifolds remain rather mysterious. Here, the primary goal is the search for closed embedded orientable π_1-injective surfaces (which are known as *incompressible*). The following conjectures remain notoriously unresolved. Does every closed hyperbolic 3-manifold have a finite cover that

1. is *Haken*, in other words, contains a closed embedded orientable incompressible surface (other than a 2-sphere)?

2. has positive first Betti number?

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3. fibres over the circle?
4. has arbitrarily large first Betti number?
5. has fundamental group with a non-abelian free quotient? (When a group has a finite-index subgroup with a non-abelian free quotient, it is known as *large*.)

The obvious relationships between these problems are shown in Figure 1. This figure also includes the Surface Subgroup Conjecture, which proposes that a closed hyperbolic 3-manifold contains a closed orientable π_1-injective surface (other than a 2-sphere), which need not be embedded. While this is not strictly a question about finite covers, one might hope to lift this surface to an embedded one in some finite cover of the 3-manifold.

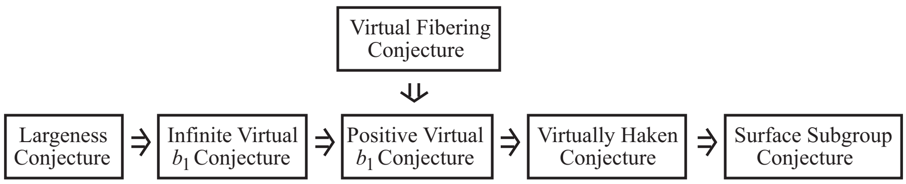

Figure 1.

There are many reasons why these questions are interesting. One source of motivation is that Haken manifolds are very well-understood, and so one might hope to use their highly-developed theory to probe general 3-manifolds. But the main reason for studying these problems is an aesthetic one. Embedded surfaces, particularly those that are π_1-injective, play a central role in low-dimensional topology and these conjectures assert that they are ubiquitous. In addition, these problems relate to many other interesting areas of mathematics, as we will see.

In order to tackle these conjectures, one is immediately led to the following questions, which are also interesting in their own right. How many finite covers does a hyperbolic 3-manifold have, as a function of the covering degree? How do natural geometric, topological and algebraic invariants behave in finite-sheeted covers, for example:

1. the spectrum of the Laplacian;
2. their Heegaard genus;
3. the rank of their fundamental group;
4. the order of their first homology, possibly with coefficients modulo a prime?

In this survey, we will outline some progress on these questions, and will particularly emphasise how an understanding of the geometric, topological and algebraic invariants of finite covers can be used to deduce the existence of incompressible surfaces, and more.

Many of the methods that we will discuss work only in dimension 3. However, many apply more generally to arbitrary finitely presented groups. We will also explain some of these group-theoretic applications.

An outline of this paper is as follows. In Section 2, we will give a summary of the progress to date (March 2010) towards the above conjectures. In Sections 3 and 4, we will explain the two main classes of covering space of a hyperbolic 3-manifold: congruence covers and abelian covers. In Section 5, we will give the best known lower bounds on the number of covers of a hyperbolic 3-manifold. In Sections 6 and 7, we will analyse the behaviour of various invariants in finite covers. In Section 8, we will explain how an understanding of this behaviour may lead to some approaches to the Virtually Haken Conjecture. In Section 9, we will examine arithmetic 3-manifolds and hyperbolic 3-orbifolds with non-empty singular locus, as these appear to be particularly tractable. In Section 10, we will consider some group-theoretic generalisations. And finally in Section 11, we will briefly give some other directions in theory of finite covers of 3-manifolds that have emerged recently.

2. The state of play

While the techniques developed to study finite covers of 3-manifolds are interesting and important, they have not yet solved the Virtually Haken Conjecture or its variants. In fact, our understanding of these conjectures is still quite limited. We focus, in this section, on the known unconditional results, and the known interconnections between the various conjectures.

The manifolds that are most well understood, but for which our knowledge is still far from complete, are the *arithmetic* hyperbolic 3-manifolds. We will not give their definition here, but instead refer the reader to Maclachlan and Reid's excellent book on the subject [48].

We start with the Surface Subgroup Conjecture. Here, we have the following result, due to the author [33].

Theorem 2.1. *Any arithmetic hyperbolic 3-manifold contains a closed orientable immersed π_1-injective surface with positive genus.*

In fact, a proof of the Surface Subgroup Conjecture for all closed hyperbolic 3-manifolds has recently been announced by Kahn and Markovic [25]. The details of this are still being checked. But the proof of Theorem 2.1 is still relevant because it falls into a general programme of the author for proving the Virtually Haken Conjecture.

The Virtually Haken Conjecture remains open at present, and is only known to hold in certain situations. Several authors have examined the case where the manifold is obtained by Dehn filling a one-cusped finite-volume hyperbolic 3-manifold. The expectation is that all but finitely many Dehn fillings of this manifold should be virtually Haken. This is not known at present. However, the following theorem, which is an amalgamation of results due to Cooper and Long [16] and Cooper and Walsh [19, 20], goes some way to establishing this.

Theorem 2.2. *Let X be a compact orientable 3-manifold with boundary a single torus, and with interior admitting a finite-volume hyperbolic structure. Then, infinitely many Dehn fillings of X are virtually Haken.*

One might hope to use the existence of a closed orientable immersed π_1-injective surface in a hyperbolic 3-manifold to find a finite cover of the 3-manifold which is Haken. However, the jump from the Surface Subgroup Conjecture to the Virtually Haken Conjecture is a big one. Currently, the only known method is the use of the group-theoretic condition called subgroup separability (see Section 11 for a definition). This is a powerful property, but not many 3-manifold groups are known to have it (see [4] for some notable examples). Indeed, the condition is so strong that when the fundamental group of a closed orientable 3-manifold contains the fundamental group of a closed orientable surface with positive genus that is separable, then this manifold virtually fibres over the circle or has large fundamental group. Nevertheless, subgroup separability is a useful and interesting property. For example, an early application of the condition is the following, due to Long [38].

Theorem 2.3. *Any finite-volume hyperbolic 3-manifold that contains a closed immersed totally geodesic surface has large fundamental group.*

The progress towards the Positive Virtual b_1 Conjecture is also quite limited. An experimental analysis [21] by Dunfield and Thurston of the 10986 manifolds in the Hodgson-Weeks census has found, for each manifold, a finite cover with positive b_1. This is encouraging, but the only known general results apply to certain classes of arithmetic 3-manifolds. The following is due to Clozel [15]. (We refer the reader to [48] for the definitions of the various terms in this theorem.)

Theorem 2.4. *Let M be an arithmetic hyperbolic 3-manifold, with invariant trace field k and quaternion algebra B. Assume that for every finite place ν where B ramifies, the completion k_ν contains no quadratic extension of $\mathbb{Q}_p$, where p is a rational prime and ν divides p. Then M has a finite cover with positive b_1.*

Again, the jump from positive virtual b_1 to infinite virtual b_1 is not known in general. However, it is known for arithmetic 3-manifolds, via the following result, which was first proved by Cooper, Long and Reid [18], but shortly afterwards, alternative proofs were given by Venkataramana [63] and Agol [2].

Theorem 2.5. *Suppose that an arithmetic hyperbolic 3-manifold M has $b_1 > 0$. Then M has finite covers with arbitrarily large b_1.*

The step from infinite virtual b_1 to largeness is known to hold in some circumstances. The following result is due to the author, Long and Reid [35].

Theorem 2.6. *Let M be an arithmetic hyperbolic 3-manifold. Suppose that M has a finite cover with $b_1 \geq 4$. Then $\pi_1(M)$ is large.*

Thus, combining the above three theorems, many arithmetic hyperbolic 3-manifolds are known to have large fundamental groups. And by Theorem 2.1, they all contain closed orientable immersed π_1-injective surfaces with positive genus.

The remaining problem is the Virtual Fibering Conjecture. For a long time, this seemed to be rather less likely than the others, simply because there were very few manifolds that were known to be virtually fibred that were not already fibred. (Examples were discovered by Reid [54], Leininger [37] and Agol-Boyer-Zhang [3].) However, this situation changed recently, with work of Agol [1], which gives a useful sufficient condition for a 3-manifold to be virtually fibred. It has the following striking consequence.

Theorem 2.7. *Let M be an arithmetic hyperbolic 3-manifold that contains a closed immersed totally geodesic surface. Then M is virtually fibred.*

These manifolds were already known to have large fundamental group, by Theorem 2.3. However, virtual fibration was somewhat unexpected here.

Finally, we should mention that the Virtually Haken Conjecture and its variants are mostly resolved in the case when M is a compact orientable irreducible 3-manifold with non-empty boundary. Indeed it is a fundamental fact that $b_1(M) \geq b_1(\partial M)/2$. Hence, M trivially satisfies the Positive Virtual b_1 Conjecture. In fact, much more is true, by the following results of Cooper, Long and Reid [17].

Theorem 2.8. *Let M be a compact orientable irreducible 3-manifold with non-empty boundary, that is not an I-bundle over a disc, annulus, torus or Klein bottle. Then $\pi_1(M)$ is large.*

Theorem 2.9. *Let M be a compact orientable irreducible 3-manifold with non-empty boundary. Then $\pi_1(M)$ is trivial or free or contains the fundamental group of a closed orientable surface with positive genus.*

The main unsolved problem for 3-manifolds with non-empty boundary is therefore the Virtual Fibering Conjecture for finite-volume hyperbolic 3-manifolds.

3. Congruence covers

We start with some obvious questions. How can one construct finite covers of a hyperbolic 3-manifold? Do they come in different 'flavours'? Of course, any finite regular cover of a 3-manifold M is associated with a surjective homomorphism from $\pi_1(M)$ onto a finite group. But there is no systematic theory for such homomorphisms in general. There are currently just two classes of finite covering spaces of general hyperbolic 3-manifolds which are at all well-understood: congruence covers and abelian covers. We will examine these in more detail in this section and the one that follows it.

If Γ is the fundamental group of an orientable hyperbolic 3-manifold M, then the hyperbolic structure determines a faithful homomorphism $\Gamma \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$. When M has finite volume, one may in fact arrange that the image lies in $\mathrm{PSL}(2, R)$, where R is obtained from the ring of integers of a number field by inverting finitely many prime ideals. This permits the use of number theory.

Specifically, one can take any proper non-zero ideal I in R, and consider the composite homomorphsim

$$\Gamma \rightarrow \mathrm{PSL}(2,R) \rightarrow \mathrm{PSL}(2,R/I)$$

which is termed the *level I congruence homomorphism*. We denote it by ϕ_I. The kernel of such a homomorphism is called a *principal congruence subgroup*, and any subgroup that contains a principal congruence subgroup is *congruence*. We term the corresponding cover of M a *congruence cover*. Now, R/I is a finite ring, and in fact if I is prime, then R/I is a finite field. Hence, congruence covers always have finite degree.

There is an alternative approach to this theory, involving quaternion algebras, which is in many ways superior. It leads to the same definition of a congruence subgroup, but the congruence homomorphisms are a little different. However, we do not follow this approach here because it requires too much extra terminology.

Congruence subgroups are extremely important. They are used to prove the following foundational result.

Theorem 3.1. *The fundamental group Γ of a finite-volume orientable hyperbolic 3-manifold is residually finite. In fact, for all primes p with at most finitely many exceptions, Γ is virtually residually p-finite.*

The residual finiteness of Γ is established as follows. Let γ be any non-trivial element of Γ, and let $\hat{\gamma}$ be an inverse image of γ in $\mathrm{SL}(2,R)$. Then neither $\hat{\gamma} - 1$ nor $\hat{\gamma} + 1$ is the zero matrix, and so each has a non-zero matrix entry. Let x be the product of these entries. Since x is a non-zero element of R, it lies in only finitely many ideals I. Therefore $\hat{\gamma} - 1$ and $\hat{\gamma} + 1$ both have non-zero image in $\mathrm{SL}(2,R/I)$ for almost all ideals I. For each such I, the images of γ and the identity in $\mathrm{PSL}(2,R/I)$ are distinct, which proves residual finiteness.

To establish virtual residual p-finiteness, for some integral prime p, one works with the principal ideals (p^n) in R, where $n \in \mathbb{N}$. Provided p does not lie in any of the prime ideals that were inverted in the definition of R, (p^n) is a proper ideal of R. Let $\Gamma(p^n)$ denote the kernel of the level (p^n) congruence homomorphism. Then, by the above argument, for any non-trivial element γ of Γ, γ does not lie in $\Gamma(p^n)$ for all sufficiently large n. In particular, this is true of all non-trivial γ in $\Gamma(p)$. Now, the image of $\Gamma(p)$ under the level (p^n) congruence homomorphism lies in the subgroup of $\mathrm{PSL}(2,R/(p^n))$ consisting of elements that are congruent to the identity mod (p). This is a finite p-group. Hence, we have found, for each non-trivial element γ of $\Gamma(p)$, a homomorphism onto a finite p-group for which the image of γ is non-trivial, thereby proving Theorem 3.1.

The conclusions of Theorem 3.1 in fact hold more generally for any finitely generated group that is linear over a field of characteristic zero, with essentially the same proof. In fact, when studying congruence homomorphisms, one is led naturally to the extensive theory of linear groups. Here, the Strong Approximation Theorem of Nori and Weisfeiler [64] is particularly important. This deals with the images of the congruence homomorphisms $\phi_I \colon \Gamma \rightarrow \mathrm{PSL}(2,R/I)$, as I ranges over all the proper non-zero ideals of R, simultaneously. We will not give the precise statement here, because it also requires too much extra terminology. However, we note the following consequence, which has, in fact, a completely elementary proof.

Theorem 3.2. *There is a finite set S of prime ideals I in R with following properties.*

1. *For each prime non-zero ideal I of R that is not in S, $\mathrm{Im}(\phi_I)$ is isomorphic to $\mathrm{PSL}(2,q^n)$ or $\mathrm{PGL}(2,q^n)$, where q is the characteristic of R/I and $n > 0$.*
2. *For any finite set of prime non-zero ideals $I_1,\ldots,I_m$ in R, none of which lies in S, and for which the characteristics of the fields R/I_i are all distinct, the product homomorphism*
$$\prod_{i=1}^{m} \phi_{I_i} : \Gamma \to \prod_{i=1}^{m} \mathrm{PSL}(2, R/I_i)$$
has image equal to
$$\prod_{i=1}^{m} \mathrm{Im}(\phi_{I_i}).$$

This has the following important consequence for homology modulo a prime p, due to Lubotzky [41]. Given any prime p and group or space X, let $d_p(X)$ denote the dimension of $H_1(X;\mathbb{F}_p)$, as a vector space over the field $\mathbb{F}_p$.

Theorem 3.3. *Let Γ be the fundamental group of a finite-volume orientable hyperbolic 3-manifold. Let p be any prime integer, and let m be any natural number. Then Γ has a congruence subgroup $\tilde{\Gamma}$ such that $d_p(\tilde{\Gamma}) \geq m$.*

The proof runs as follows. For almost all integral primes q, there is a prime ideal I in R such that R/I is a field of characteristic q. Moreover, by Theorem 3.2, we may assume that $\mathrm{Im}(\phi_I)$ is isomorphic to $\mathrm{PSL}(2,q^n)$ or $\mathrm{PGL}(2,q^n)$, where $n \geq 1$. Inside $\mathrm{PSL}(2,q^n)$ or $\mathrm{PGL}(2,q^n)$, there is the subgroup consisting of diagonal matrices, which is abelian with order $(q^n-1)/2$ or (q^n-1). We now want to restrict to certain primes q, and to do this, we use Dirichlet's theorem, which asserts that there are infinitely many primes q such that $q \equiv 1 \pmod p$. When $p = 2$, we also require that $q \equiv 1 \pmod 4$. For these q, p divides the order of the subgroup of diagonal matrices, and so there is a subgroup of order p. We may find a set of m such primes $q_1,\ldots,q_m$ so that each q_i is the characteristic of R/I_i, where I_i is a prime ideal avoiding the finite set S described above. Then we have an inclusion of groups
$$(\mathbb{Z}/p)^m \leq \mathrm{Im}(\phi_{I_1}) \times \cdots \times \mathrm{Im}(\phi_{I_m}).$$
Now, Γ surjects onto the right-hand group. Let $\tilde{\Gamma}$ be the inverse image of the left-hand group. This is a congruence subgroup, and by construction, it surjects onto $(\mathbb{Z}/p)^m$. Hence, $d_p(\tilde{\Gamma}) \geq m$, thereby proving Theorem 3.3.

Since $d_p(\tilde{\Gamma})$ is positive, the covering space corresponding to $\tilde{\Gamma}$ has a non-trivial regular cover with covering group that is an elementary abelian p-group (in other words is isomorphic to $(\mathbb{Z}/p)^m$ for some m). Thus, we are led to the following type of covering space.

4. Abelian covers

A covering map is *abelian* (respectively, *cyclic*) provided it is regular and the group of covering transformations is abelian (respectively, cyclic). A large amount of attention has been focused on the homology of abelian covers. Indeed, one of the earliest topological invariants, the Alexander polynomial, can be interpreted this way. However, the Alexander polynomial is only defined when the covering group is free abelian, and so we leave the realm of finite covering spaces. We therefore will not dwell too long on the Alexander polynomial, but to omit mention of it entirely would be remiss, especially as it has consequences also for certain finite cyclic covers, via the following result, due to Silver and Williams [60] (see also [55, 22]).

Theorem 4.1. *Let M be a compact orientable 3-manifold, let $\tilde{M}$ be an infinite cyclic cover and let $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$ be the resulting Alexander polynomial. Its Mahler measure is defined by*

$$M(\Delta) = |c| \prod_i \max\{1, |\alpha_i|\},$$

as α_i ranges over all roots of $\Delta(t)$, and c is the coefficient of the highest order term. Let M_n be the degree n cyclic cover of M that is covered by $\tilde{M}$. Then

$$\frac{\log |H_1(M_n)_{\rm tor}|}{n} \to \log M(\Delta),$$

as $n \to \infty$, where $H_1(M_n)_{\rm tor}$ denotes the torsion part of $H_1(M_n)$.

Thus, provided $\Delta(t)$ has at least one root off the unit circle, $|H_1(M_n)_{\rm tor}|$ has exponential growth as a function of n.

There are more sophisticated versions of this result, dealing for example with the case when $\tilde{M}$ is a regular cover with a free abelian group of covering transformations [60]. However, the theory only applies when $b_1(M)$ is positive, and so, in the absence of the solution to the Positive Virtual b_1 Conjecture, methods using the Alexander polynomial are not yet universally applicable in 3-manifold theory.

There is another important direction in the theory of abelian covers, which deals with homology modulo a prime p. Let Γ be the fundamental group of a compact orientable 3-manifold, and suppose that $\tilde{\Gamma}$ is a normal subgroup such that $\Gamma/\tilde{\Gamma}$ is an elementary abelian p-group. Then an important result of Shalen and Wagreich [59] gives a lower bound for the mod p homology of $\tilde{\Gamma}$. For simplicity, we will deal only with the case where $\Gamma/\tilde{\Gamma}$ is as big as possible, which is when $\tilde{\Gamma} = [\Gamma, \Gamma]\Gamma^p$.

Theorem 4.2. *Let Γ be the fundamental group of a compact orientable 3-manifold, and let $\tilde{\Gamma} = [\Gamma, \Gamma]\Gamma^p$. Then,*

$$d_p(\tilde{\Gamma}) \geq \binom{d_p(\Gamma)}{2}.$$

This has the consequence that when $d_p(\Gamma) \geq 3$, then also $d_p(\tilde{\Gamma}) \geq 3$. Hence, we may repeat the argument with $\tilde{\Gamma}$ in place of Γ. It is therefore natural to consider the *derived p-series* of Γ, which is defined by setting $\Gamma_0 = \Gamma$, and $\Gamma_{i+1} = [\Gamma_i, \Gamma_i](\Gamma_i)^p$ for $i \geq 0$. We deduce that when $d_p(\Gamma) \geq 3$, then the derived p-series is always strictly descending. Moreover, when $d_p(\Gamma) > 3$, then $d_p(\Gamma_i)$ tends to infinity. Note that $d_p(\Gamma_i)$ need not tend to infinity when $d_p(\Gamma) = 3$, as the example of the 3-torus demonstrates.

The original proof of Theorem 4.2 by Shalen and Wagreich used the following exact sequence of Stallings:

$$H_2(\Gamma; \mathbb{F}_p) \to H_2(\Gamma/\tilde{\Gamma}; \mathbb{F}_p) \to \frac{\tilde{\Gamma}}{[\tilde{\Gamma}, \Gamma](\tilde{\Gamma})^p} \to H_1(\Gamma; \mathbb{F}_p) \to H_1(\Gamma/\tilde{\Gamma}; \mathbb{F}_p) \to 0.$$

Let $d = d_p(\Gamma/\tilde{\Gamma}) = d_p(\Gamma)$. Then, $H_2(\Gamma/\tilde{\Gamma}; \mathbb{F}_p)$ is an elementary abelian p-group of rank $d(d+1)/2$ by the Künneth formula. However, by Poincaré duality, $H_2(\Gamma; \mathbb{F}_p)$ is an elementary abelian p-group of rank at most d. Thus, by exactness of the above sequence, $\tilde{\Gamma}/([\tilde{\Gamma}, \Gamma](\tilde{\Gamma})^p)$ has rank at least $d(d-1)/2$. But this is a quotient of $\tilde{\Gamma}/([\tilde{\Gamma}, \tilde{\Gamma}](\tilde{\Gamma})^p)$, which equals $H_1(\tilde{\Gamma}; \mathbb{F}_p)$. Hence, one obtains the required lower bound on $d_p(\tilde{\Gamma})$.

Although this argument is short, it is not an easy one for a geometric topologist to digest. In an attempt to try to understand it, the author found an alternative topological proof, which then led to a considerable strengthening of the theorem. The proof runs roughly as follows, focusing on the case $p = 2$ for simplicity.

Pick a generating set $\{x_1, \ldots, x_n\}$ for Γ such that the first d elements $x_1, \ldots, x_d$ form a basis for $H_1(\Gamma; \mathbb{F}_2)$, and so that each x_i is trivial in $H_1(\Gamma; \mathbb{F}_2)$ for $i > d$. Let K be a 2-complex with fundamental group Γ, with a single 0-cell and with 1-cells corresponding to the above generating set. Let $\tilde{K}$ be the covering space corresponding to $\tilde{\Gamma}$. We are trying to find a lower bound on $d_2(\tilde{\Gamma})$, which is the rank of the cohomology group $H^1(\tilde{K}; \mathbb{F}_2)$. Now, $H^1(\tilde{K}; \mathbb{F}_2)$ is equal to $\{$1-cocycles on $\tilde{K}\}/\{$1-coboundaries on $\tilde{K}\}$. Each 1-cocycle is, by definition, a 1-cochain that evaluates to zero on the boundary of each 2-cell of $\tilde{K}$. However, instead of examining all cochains, we only consider special ones, which are defined as follows. For each integer $1 \leq j \leq d$, each vertex of $\tilde{K}$ has a well-defined x_j-value, which is an integer mod 2. For $1 \leq i \leq j \leq d$, define the cochain $x_i \wedge x_j$ to have support equal to the x_i-labelled edges which start (and end) at vertices with x_j-value 1. The space spanned by these cochains clearly has dimension $d(d+1)/2$.

These cochains have the key property that if two closed loops in $\tilde{K}$ differ by a covering transformation, then their evaluations under one of these cochains are equal. Hence, when we consider the space spanned by these cochains, and determine whether any element of this space is a cocycle, we only need to consider one copy of each defining relation of Γ. It turns out that none of the cocycles in this space is a coboundary, except the zero cocycle, and so $d_2(\tilde{\Gamma}) \geq d(d+1)/2 - r$, where r is the number of 2-cells of K. In fact, by modifying these cocycles a little, the number of conditions that we must check can be reduced from r to $b_2(\Gamma; \mathbb{F}_2)$. So, we deduce that, if Γ is any finitely presented group and $\tilde{\Gamma} = [\Gamma, \Gamma]\Gamma^2$ and

$d = d_2(\Gamma)$, then

$$d_2(\tilde{\Gamma}) \geq d(d+1)/2 - b_2(\Gamma; \mathbb{F}_2).$$

And when Γ is the fundamental group of a compact orientable 3-manifold, Poincaré duality again gives that $b_2(\Gamma; \mathbb{F}_2) \leq d$, proving Theorem 4.2.

This is not the end of the story, because it turns out that these cochains are just the first in a whole series of cochains, each with an associated integer, which is its 'level' ℓ. The ones above are those with level $\ell = 1$. An example of a level 2 cochain has support equal to the x_1-labelled edges which start at the vertices for which the x_2-value and x_3-value are both 1. By considering cochains at different levels, we can considerably strengthen Theorem 4.2, as follows. Again, there are more general versions which deal with the general case where $\Gamma/\tilde{\Gamma}$ is an elementary abelian p-group, but we focus on the case where $\tilde{\Gamma} = [\Gamma, \Gamma]\Gamma^2$.

Theorem 4.3. *Let Γ be the fundamental group of a compact orientable 3-manifold, and let $\tilde{\Gamma} = [\Gamma, \Gamma]\Gamma^2$. Then, for each integer ℓ between 1 and $d_2(\Gamma)$,*

$$d_2(\tilde{\Gamma}) \geq d_2(\Gamma) \binom{d_2(\Gamma)}{\ell} - \sum_{j=1}^{\ell+1} \binom{d_2(\Gamma)}{j}.$$

Setting $\ell = \lfloor d_2(\Gamma)/2 \rfloor$ and using Stirling's formula to estimate factorials, we deduce the following [31].

Theorem 4.4. *Let Γ be the fundamental group of a compact orientable 3-manifold such that $d_2(\Gamma) > 3$. Let $\{\Gamma_i\}$ be the derived 2-series of Γ. Then, for each $\lambda < \sqrt{2/\pi}$,*

$$d_2(\Gamma_{i+1}) \geq \lambda 2^{d_2(\Gamma_i)} \sqrt{d_2(\Gamma_i)},$$

for all sufficiently large i.

This is not far off the fastest possible growth of homology of a finitely generated group. By comparison, when $\{\Gamma_i\}$ is the derived 2-series of a non-abelian free group, then

$$d_2(\Gamma_{i+1}) = 2^{d_2(\Gamma_i)}(d_2(\Gamma_i) - 1) + 1.$$

Theorem 4.4 can be used to produce strong lower bounds on the number of covering spaces of a hyperbolic 3-manifold, as we will see in the following section.

5. Counting finite covers

How many finite covers does a 3-manifold have? This question lies in the field of subgroup growth [45], which deals with the behaviour of the following function. For a finitely generated group Γ and positive integer n, let $s_n(\Gamma)$ be the number of subgroups of Γ with index at most n.

The fastest possible growth rate of $s_n(\Gamma)$, as a function of n, is clearly achieved when Γ is a non-abelian free group. In this case, $s_n(\Gamma)$ grows slightly faster than exponentially: it grows like $2^{n \log n}$. More generally, any large finitely generated group has this rate of subgroup growth.

By comparison, the subgroup growth of the fundamental group of a hyperbolic 3-manifold group has a lower bound that grows slightly slower than exponentially, as the following result [31] of the author demonstrates.

Theorem 5.1. *Let Γ be the fundamental group of a finite-volume hyperbolic 3-manifold. Then,*

$$s_n(\Gamma) > 2^{n/(\sqrt{\log(n)} \log \log n)}$$

for infinitely many n.

The proof is a rapid consequence of Theorems 4.4 and 3.3. Theorem 3.3 gives a finite index subgroup $\tilde{\Gamma}$ of Γ with $d_2(\tilde{\Gamma}) > 3$. Then Theorem 4.4 implies that the mod 2 homology of the derived 2-series of $\tilde{\Gamma}$ grows rapidly. And if Γ_i is a subgroup of Γ with index n, then clearly

$$s_{2n}(\Gamma) \geq 2^{d_2(\Gamma_i)}.$$

Thus, in the landscape of finite covers of a hyperbolic 3-manifold, abelian covers appear to play a major role. Certainly, there are far more of them than there are congruence covers, by the following result of Lubotzky [43], which estimates $c_n(\Gamma)$, which is the number of congruence subgroups of Γ with index at most n.

Theorem 5.2. *Let Γ be the fundamental group of an orientable finite-volume hyperbolic 3-manifold. Then, there are positive constants a and b such that*

$$n^{a \log n / \log \log n} \leq c_n(\Gamma) \leq n^{b \log n / \log \log n},$$

for all n.

The lower bound provided by Theorem 5.1 is not sharp in general, because there are many examples where Γ is large. Indeed, the Largeness Conjecture asserts that this should always be the case.

There is another important situation when we know that the lower bound of Theorem 5.1 can be improved upon, due to the following result of the author [30].

Theorem 5.3. *Let Γ be the fundamental group of either an arithmetic hyperbolic 3-manifold or a finite-volume hyperbolic 3-orbifold with non-empty singular locus. Then, there is a real number $c > 1$ such that $s_n(\Gamma) \geq c^n$ for infinitely many n.*

Like Theorem 5.1, this is proved by finding lower bounds on the rank of the mod p homology of certain finite covers. We will give more details in Section 9, where the covering spaces of 3-orbifolds and arithmetic 3-manifolds will be examined more systematically.

6. The behaviour of algebraic invariants in finite covers

As we have seen, it is important to understand how the homology groups can grow in a tower of finite covers. Thus, we are led to the following related invariants of a group Γ:

1. the first Betti number $b_1(\Gamma)$,
2. the torsion part $H_1(\Gamma)_{\mathrm{tor}}$ of first homology,
3. the rank $d_p(\Gamma)$ of mod p homology,
4. the rank of Γ, denoted $d(\Gamma)$, which is the minimal number of generators.

For each of these invariants, it is natural to consider its growth rate in a nested sequence of finite index subgroups Γ_i. For example, one can define the *rank gradient* which is

$$\liminf_i \frac{d(\Gamma_i)}{[\Gamma : \Gamma_i]}.$$

The *mod p homology gradient* and *first Betti number gradient* are defined similarly. The latter is in fact, by a theorem of Lück [47], related to the first L^2 Betti number of Γ (denoted $b_1^{(2)}(\Gamma)$). More precisely, when Γ_i is a nested sequence of finite-index normal subgroups of a finitely presented group Γ, and their intersection is the identity, then their first Betti number gradient is equal to $b_1^{(2)}(\Gamma)$. When Γ is the fundamental group of a finite-volume hyperbolic 3-manifold, $b_1^{(2)}(\Gamma)$ is known to be zero, and hence $b_1(\Gamma_i)$ always grows sub-linearly as a function of the covering degree $[\Gamma : \Gamma_i]$. Interestingly, there is no corresponding theory for mod p homology gradient, and the following question is at present unanswered.

Question. *Let Γ be a finitely presented group, let Γ_i be a nested sequence of finite-index normal subgroups that intersect in the identity. Then does their mod p homology gradient depend only on Γ and possibly p, but not the sequence Γ_i?*

This is unknown, but it seems very likely that the mod p homology gradient is always zero when Γ is the fundamental group of a finite-volume hyperbolic 3-manifold and the subgroups intersect in the identity. However, if we drop the condition that the subgroups intersect in the identity, then there is an interesting situation where positive mod p homology gradient is known to hold, by the following result of the author [30].

Theorem 6.1. *Let Γ be the fundamental group of either an arithmetic hyperbolic 3-manifold or a finite-volume hyperbolic 3-orbifold with non-empty singular locus. Then, for some prime p, Γ has a nested strictly descending sequence of finite-index subgroups with positive mod p homology gradient.*

We will explore this in more detail in Section 9. But we observe here that it rapidly implies Theorem 5.3. The existence of such a sequence of finite-index subgroups seems to be a very strong conclusion. In fact, the following is unknown.

Question. *Suppose that a finitely presented group Γ has a strictly descending sequence of finite-index subgroups with positive mod p homology gradient, for some prime p. Does this imply that Γ is large?*

We will give some affirmative evidence for this in Section 10. Somewhat surprisingly, this question is related to the theory of error-correcting codes (see [32]).

We have mostly focused on the existence of very fast homology growth for certain covers of hyperbolic 3-manifolds. But one can also consider the other end of the spectrum, and ask how slowly the homology groups of a tower of covers can grow. In this context, the following theorem of Boston and Ellenberg [7] is striking (see also [12]).

Theorem 6.2. *There is an example of a closed hyperbolic 3-manifold, with fundamental group that has a sequence of nested finite-index normal subgroups Γ_i which intersect in the identity, such that $b_1(\Gamma_i) = 0$ and $d_3(\Gamma_i) = 3$ for all i.*

This is proved using the theory of pro-p groups. This is a particularly promising set of techniques, which will doubtless have other applications to 3-manifold theory.

7. The behaviour of geometric and topological invariants in finite covers

In addition to the above algebraic invariants, it seems to be important also to understand the behaviour of various geometric and topological invariants in a tower of covers, including the following:

1. the first eigenvalue of the Laplacian,
2. the Cheeger constant,
3. the Heegaard genus.

We will recall the definitions of these terms below.

It is well known that the Laplacian on a closed Riemannian manifold M has a discrete set of eigenvalues, and hence there is a smallest positive eigenvalue, denoted $\lambda_1(M)$. This exerts considerable control over the geometry of the manifold. In particular, it is related to the *Cheeger constant* $h(M)$, which is defined to be

$$\inf_S \frac{\text{Area}(S)}{\min\{\text{Volume}(M_1), \text{Volume}(M_2)\}},$$

as S ranges over all codimension-one submanifolds that divide M into submanifolds M_1 and M_2. It is a famous theorem of Cheeger [14] and Buser [11] that if M is closed Riemannian n-manifold with Ricci curvature at least $-(n-1)a^2$ (for some $a \geq 0$), then

$$h(M)^2/4 \leq \lambda_1(M) \leq 2a(n-1)h(M) + 10(h(M))^2.$$

A consequence is that if M_i is a sequence of finite covers of M, then $\lambda_1(M_i)$ is bounded away from zero if and only if $h(M_i)$ is. In this case, $\pi_1(M)$ is said to have *Property* (τ) with respect to the subgroups $\{\pi_1(M_i)\}$. As the definition implies, this depends only on the fundamental group $\pi_1(M)$ and the subgroups $\{\pi_1(M_i)\}$, and not on the choice of particular Riemannian metric on M. Also, $\pi_1(M)$ is said to have *Property* (τ) if it has Property (τ) with respect to the collection of all its finite-index subgroups. Since any finitely presented group is the fundamental group of some closed Riemannian manifold, this is therefore a property that is or is not enjoyed by any finitely presented group. In fact, one can extend the definition to finitely generated groups that need not be finitely presented. (See [42, 46] for excellent surveys of this concept.)

The reason for the 'τ' terminology is that Property (τ) is a weak form of Kazhdan's Property (T). In particular, any finitely generated group with Property (T) also has Property (τ) [49]. A harder result is due to Selberg [58], which implies that $\mathrm{SL}(2,\mathbb{Z})$ has Property (τ) with respect to its congruence subgroups.

The simplest example of a group without Property (τ) is $\mathbb{Z}$. Also, if there is a surjective group homomorphism $\Gamma \to \overline{\Gamma}$ and $\overline{\Gamma}$ does not have Property (τ), then nor does Γ. Hence, if a group has a finite-index subgroup with positive first Betti number, then it does not have Property (τ). Strikingly, it remains an open question whether the converse holds in the finitely presented case.

Question. *If a finitely presented group does not have Property (τ), then must it have a finite-index subgroup with positive first Betti number?*

The assumption that the group is finitely presented here is critical. For example, Grigorchuk's group [23] is residually finite and amenable, and hence does not have Property (τ), and yet it is a torsion group, and so no finite-index subgroup has positive first Betti number. Although a positive answer to this question is unlikely, it might have some striking applications. For example, every residually finite group with sub-exponential growth does not have Property (τ). So, a positive answer to the above question might be a step in establishing that such groups are virtually nilpotent, provided they are finitely presented.

One small piece of evidence for an affirmative answer to the question is given by the following theorem of the author [29], which relates the behaviour of λ_1 and h to the existence of a finite-index subgroup with positive first Betti number.

Theorem 7.1. *Let M be a closed Riemannian manifold. Then the following are equivalent:*

- *there exists a tower of finite covers $\{M_i\}$ of M with degree d_i, where each $M_i \to M_1$ is regular, and such that $\lambda_1(M_i)d_i \to 0$;*
- *there exists a tower of finite covers $\{M_i\}$ of M with degree d_i, where each $M_i \to M_1$ is regular, and such that $h(M_i)\sqrt{d_i} \to 0$;*
- *there exists a finite-index subgroup of $\pi_1(M)$ with positive first Betti number.*

However, it remains unlikely that the above question has a positive answer. Hence, the following conjecture [44] about 3-manifolds is *a priori* much weaker than the Positive Virtual b_1 Conjecture.

Conjecture (Lubotzky-Sarnak). *The fundamental group of any closed hyperbolic 3-manifold does not have Property* (τ).

This is a natural question for many reasons. It is known that if Γ is a lattice in a semi-simple Lie group G, then whether or not Γ has Kazhdan's Property (T) depends only on G. It remains an open question whether a similar phenomenon holds for Property (τ), but if it did, then this would of course imply the Lubotzky-Sarnak Conjecture.

An 'infinite' version of the Lubotzky-Sarnak Conjecture is known to hold, according to the following result of the author, Long and Reid [36].

Theorem 7.2. *Any closed hyperbolic 3-manifold has a sequence of infinite-sheeted covers M_i where $\lambda_1(M_i)$ and $h(M_i)$ both tend to zero.*

Of course, $\lambda_1(M_i)$ and $h(M_i)$ need to be defined appropriately, since each M_i has infinite volume. In the case of $\lambda_1(M_i)$, this is just the bottom of the spectrum of the Laplacian on L^2 functions on M_i. To define $h(M_i)$, one considers all compact codimension-zero submanifolds of M_i, one evaluates the ratio of the area of their boundary to their volume, and then one takes the infimum. Just as in the finite-volume case, there is a result of Cheeger [14] which asserts that $\lambda_1(M_i) \geq h(M_i)^2/4$. Also, in the case of hyperbolic 3-manifolds M_i, these quantities are related to another important invariant $\delta(M_i)$, which is the critical exponent. A theorem of Sullivan [61] asserts that

$$\lambda_1(M_i) = \begin{cases} \delta(M_i)(2-\delta(M_i)) & \text{if } \delta(M_i) \geq 1 \\ 1 & \text{if } \delta(M_i) \leq 1. \end{cases}$$

The proof of Theorem 7.2 relies crucially on a recent result of Bowen [9], which asserts that, given any closed hyperbolic 3-manifold M and any finitely generated discrete free convex-cocompact subgroup F of $\mathrm{PSL}(2,\mathbb{C})$, there is an arbitrarily small 'perturbation' of F which places a finite-index subgroup of F as a subgroup of $\pi_1(M)$. Starting with a group F where $\delta(F)$ is very close to 2, the critical exponent of this perturbation remains close to 2. Thus, this produces subgroups of $\pi_1(M)$ with critical exponent arbitrarily close to 2. By Sullivan's theorem, the corresponding covers M_i of M have $\lambda_1(M_i)$ arbitrarily close to zero. By Cheeger's theorem, $h(M_i)$ also tends to zero, proving Theorem 7.2.

Although the infinite version of the Lubotzky-Sarnak Conjecture does not seem to have any immediate consequence for finite covering spaces of closed hyperbolic 3-manifolds, it can be used to produce surface subgroups. Indeed, it is a key step in the proof of Theorem 2.1. We will give more details in Section 9.

In addition to understanding $\lambda_1(M_i)$ and $h(M_i)$ for finite covering spaces M_i, it also seems to be important to understand the growth rate of their Heegaard genus. Recall that any closed orientable 3-manifold M can be obtained by gluing two handlebodies via a homeomorphism between their boundaries. This is a *Heegaard splitting* for M, and the image of the boundary of each handlebody is a *Heegaard surface*. The minimal genus of a Heegaard surface in M is known as the *Heegaard genus* $g(M)$. A related quantity is the *Heegaard Euler characteristic* $\chi_-^h(M)$, which is $2g(M)-2$. These are widely-studied invariants of 3-manifolds, and there is now a well-developed theory of Heegaard splittings [56]. It is therefore natural to consider the *Heegaard gradient* of a sequence of finite covers $\{M_i\}$, which is

$$\liminf_i \frac{\chi_-^h(M_i)}{\mathrm{degree}(M_i \to M)}.$$

Somewhat surprisingly, the Cheeger constant and the Heegaard genus of a closed hyperbolic 3-manifold are related by the following inequality of the author [29].

Theorem 7.3. *Let M be a closed orientable hyperbolic 3-manifold. Then*

$$h(M) \leq \frac{8\pi(g(M)-1)}{\mathrm{Volume}(M)}.$$

A consequence is that if the Heegaard gradient of a sequence of finite covers of M is zero, then the corresponding subgroups of $\pi_1(M)$ do not have Property (τ). Equivalently, if a sequence of finite covers has Property (τ), then these covers have positive Heegaard gradient.

We now give a sketch of the proof of Theorem 7.3. Any Heegaard splitting for a 3-manifold M determines a 'sweepout' of the manifold by surfaces, as follows. The Heegaard surface divides the manifold into two handlebodies, each of which is a regular neighbourhood of a core graph. Thus, there is a 1-parameter family of copies of the surface, starting with the boundary of a thin regular neighbourhood of one core graph and ending with the boundary of a thin regular neighbourhood of the other graph. Consider sweepouts where the maximum area of the surfaces is as small as possible. Then, using work of Pitts and Rubinstein [53], one can arrange that the surfaces of maximal area tend (in a certain sense) to a minimal surface S, which is obtained from the Heegaard surface possibly by performing some compressions. Since S is a minimal surface in a hyperbolic 3-manifold, Gauss-Bonnet implies that its area is at most $-2\pi\chi(S) \leq 4\pi(g(M)-1)$. Hence, we obtain a sweepout of M by surfaces, each of which has area at most this bound (plus an arbitrarily small $\epsilon > 0$). One of these surfaces divides M into two parts of equal volume. This decomposition gives the required upper bound on the Cheeger constant $h(M)$.

There is an important special case when the Heegaard gradient of a sequence of finite covers is zero. Suppose that M fibres over the circle with fibre F. Then it is easy to construct a Heegaard splitting for M with genus at most $2g(F)+1$, where $g(F)$ is the genus of F. Hence, the finite cyclic covers of M dual to F have uniformly bounded Heegaard genus. In particular, their Heegaard gradient is zero. This is the only known method of constructing sequences of finite covers of a hyperbolic 3-manifold with zero Heegaard gradient. And so we are led to the following conjecture of the author, called the Heegaard Gradient Conjecture [29].

Conjecture. *A closed orientable hyperbolic 3-manifold has zero Heegaard gradient if and only if it virtually fibres over the circle.*

This remains a difficult open problem. However, a qualitative version of it is known to be true. More specifically, if a closed hyperbolic 3-manifold has a sequence of finite covers with Heegaard genus that grows 'sufficiently slowly', then these covers are eventually fibred, by the following result of the author [27].

Theorem 7.4. *Let M be a closed orientable hyperbolic 3-manifold, and let M_i be a sequence of finite regular covers, with degree d_i. Suppose that $g(M_i)/\sqrt[4]{d_i} \to 0$. Then, for all sufficiently large i, M_i fibres over the circle.*

The proof of this uses several of the results mentioned above. Using Theorem 7.3, the hypothesis that $g(M_i)/\sqrt[4]{d_i} \to 0$ implies that $h(M_i)d_i^{3/4} \to 0$. Hence, by Theorem 7.1, we deduce that some finite-sheeted cover of M has positive first Betti number. In fact, if we go back to the proof of Theorem 7.1, we see that this is true of each M_i sufficiently far down the sequence, and with further work, one can actually prove that these manifolds fibre over the circle.

We will see that the two conjectures introduced in this section, the Lubotzky-Sarnak Conjecture and the Heegaard Gradient Conjecture, may be a route to proving the Virtually Haken Conjecture.

8. Two approaches to the Virtually Haken Conjecture

The two conjectures introduced in the previous section can be combined to form an approach to the Virtually Haken Conjecture, via the following theorem of the author [29].

Theorem 8.1. *Let M be a closed orientable irreducible 3-manifold, and let M_i be a tower of finite regular covers of M such that*

1. *their Heegaard gradient is positive, and*
2. *they do not have Property (τ).*

Then, for all sufficiently large i, M_i is Haken.

Hence, the Lubotzky-Sarnak Conjecture and the Heegaard Gradient Conjecture together imply the Virtually Haken Conjecture. For, assuming the Lubotzky-Sarnak Conjecture, a closed orientable hyperbolic 3-manifold M has a tower of finite regular covers without Property (τ). If these have positive Heegaard gradient, then by Theorem 8.1, they are eventually Haken. On the other hand, if they have zero Heegaard gradient, then by the Heegaard Gradient Conjecture, M is virtually fibred.

The proof requires some ideas from the theory of Heegaard splittings. A central concept in this theory is the notion of a *strongly irreducible* Heegaard surface S, which means that any compression disc on one side of S must intersect any compression disc on the other side. A key theorem of Casson and Gordon [13] implies that if a closed orientable irreducible 3-manifold has a minimal genus Heegaard splitting that is *not* strongly irreducible, then the manifold is Haken. A quantified version of this is as follows. Suppose that S is a Heegaard surface for the closed 3-manifold M, and that there are d disjoint non-parallel compression discs on one side of S that are all disjoint from d disjoint non-parallel compression discs on the other side of S. Then, either $g(M) \leq g(S) - (d/6)$ or M is Haken.

Suppose now that M_i is a sequence of covers of M as in Theorem 8.1. Let S be a minimal genus Heegaard surface for M. Its inverse image in each M_i is a Heegaard surface S_i. We are assuming that the Heegaard gradient of these covers is positive. Hence (by replacing M by some M_i if necessary), we may assume that $g(M_i)$ is roughly $g(S_i)$. Now, we are also assuming that these covers do not have Property (τ). Hence, there is a way of decomposing M_i into two pieces A_i and B_i with large volume, and with small intersection. By using compression discs on one side of S_i that lie in A_i and compression discs on the other side of S_i that lie in B_i, we obtain d_i discs on each side of S_i which are all disjoint and non-parallel, and where d_i grows linearly as a function of the covering degree of $M_i \to M$. Hence, by the quantified version of Casson-Gordon, if M_i is not Haken, then $g(M_i)$ is substantially less than $g(S_i)$, which is a contradiction, thereby proving Theorem 8.1.

Of course, it remains unclear whether the hypotheses of Theorem 8.1 always hold, and hence the Virtually Haken Conjecture remains open. However, there is another intriguing approach. By using results of Bourgain and Gamburd [8] which give lower bounds on the first eigenvalue of the Laplacian on certain Cayley graphs of $\mathrm{SL}(2, p)$, Long, Lubotzky and Reid [39] were able to establish the following theorem.

Theorem 8.2. *Let M be a closed orientable hyperbolic 3-manifold. Then M has a sequence of finite covers M_i with Property (τ) and such that the subgroups $\pi_1(M_i)$ of $\pi_1(M)$ intersect in the identity.*

Combining this with Theorem 7.3, we deduce that these covers have positive Heegaard gradient. Now, Theorem 8.2 does not provide a *tower* of finite regular covers, but it is not unreasonable to suppose that this can be achieved. Hence, by replacing M by some M_i if necessary, we may assume that the Heegaard gradient of these covers is very close to $\chi_-^h(M)$. Let S be any minimal genus Heegaard surface for M. Its inverse image S_i in M_i is a Heegaard splitting of M_i, and it therefore is nearly of minimal genus. It is reasonable to conjecture that there is a minimal genus splitting $\overline{S}_i$ for M_i with geometry 'approximating' that of S_i. Now, S_i inevitably fails to be strongly irreducible when the degree of $M_i \to M$ is large, via a simple argument that counts compression discs and their points of intersection. One might conjecture that this is also true of $\overline{S}_i$, which would therefore imply that M_i is Haken for all sufficiently large i. Of course, this is somewhat speculative, and the conjectural relationship between S_i and $\overline{S}_i$ may not hold. But it highlights the useful interaction between Heegaard splittings, Property (τ) and the Virtually Haken Conjecture.

9. Covering spaces of hyperbolic 3-orbifolds and arithmetic 3-manifolds

The material in the previous section is, without doubt, rather speculative. However, the ideas behind it have been profitably applied in some important special cases. It seems to be easiest to make progress when analysing finite covers of either of the following spaces:

1. hyperbolic 3-orbifolds with non-empty singular locus;
2. arithmetic hyperbolic 3-manifolds.

There is a well-developed theory of orbifolds, their fundamental groups and their covering spaces. We will only give a very brief introduction here, and refer the reader to [57] for more details.

Recall that an orientable hyperbolic 3-orbifold O is the quotient of hyperbolic 3-space $\mathbb{H}^3$ by a discrete group Γ of orientation-preserving isometries. This group may have non-trivial torsion, in which case it does not act freely. The images in O of points in $\mathbb{H}^3$ with non-trivial stabiliser form the *singular locus* $\mathrm{sing}(O)$. This is a collection of 1-manifolds and trivalent graphs. Each 1-manifold and each edge of each graph has an associated positive integer, its *order*, which is the order of the finite stabiliser of corresponding points in $\mathbb{H}^3$. For any positive integer n, $\mathrm{sing}_n(O)$ denotes the closure of the union of singular edges and 1-manifolds that have order a multiple of n. The underlying topological space of a 3-orbifold O is always a 3-manifold, denoted $|O|$.

One can define the fundamental group $\pi_1(O)$ of any orbifold O, which is, in general, different from the usual fundamental group of $|O|$. When O is hyperbolic, and hence of the form $\mathbb{H}^3/\Gamma$, its fundamental group is Γ. One can also define the notion of a covering map between orbifolds. In the hyperbolic case, these maps are of the form $\mathbb{H}^3/\Gamma' \to \mathbb{H}^3/\Gamma$, for some subgroup Γ' of Γ. Note that this need not be a cover in the usual topological sense.

The following result of the author, Long and Reid [35] allows one to apply orbifold technology in the arithmetic case.

Theorem 9.1. *Any arithmetic hyperbolic 3-manifold is commensurable with a 3-orbifold O with non-empty singular locus. Indeed, one may arrange that every curve and arc of the singular locus has order 2 and that there is at least one singular vertex.*

The main reason why 3-orbifolds are often more tractable than 3-manifolds is the following lower bound on the rank of their homology [30]. For an orbifold O and prime p, we let $d_p(O)$ denote $d_p(\pi_1(O))$.

Theorem 9.2. *Let O be a compact orientable 3-orbifold. Then for any prime p, $d_p(O) \geq b_1(\mathrm{sing}_p(O))$.*

The reason is that $\pi_1(O)$ can be computed by starting with the usual fundamental group of the manifold $O - \mathrm{sing}(O)$ and then quotienting out powers of the meridians of the singular locus, where the power is the relevant edge's singularity order. If this order is a multiple of p, then quotienting out this power of this meridian has no effect on d_p. On the other hand, if the order of a singular edge or curve is coprime to p, then we may replace these points by manifold points without changing d_p. Hence, $d_p(O) = d_p(|O| - \mathrm{int}(N(\mathrm{sing}_p(O))))$. Now, the latter space is a compact orientable 3-manifold M with boundary, and it is a well-known consequence of Poincaré duality that $d_p(M)$ is at least $d_p(\partial M)/2$. From this, the required inequality rapidly follows.

So, as far as mod p homology is concerned, orbifolds O where $\mathrm{sing}_p(O)$ is non-empty behave as though they have non-empty boundary. And 3-manifolds with non-empty boundary are often much more tractable than closed ones.

Theorem 9.2 is the basis behind Theorem 6.1. Here, we are given a finite-volume hyperbolic 3-orbifold O with non-empty singular locus. The main case is when O is closed. Let p be a prime that divides the order of some edge or curve in the singular locus. We first show that one can find a finite cover $\tilde{O}$ where $\mathrm{sing}(\tilde{O})$ is a non-empty collection of simple closed curves with singularity order p, and where $d_p(\tilde{O}) \geq 11$, using techniques that are generalisations of those in Section 3. Let λ and μ be a longitude and meridian of some component L of $\mathrm{sing}(\tilde{O})$, viewed as elements of $\pi_1(\tilde{O})$. Then, using the Golod-Shafarevich inequality [40], we can show that $\pi_1(\tilde{O})/\langle\langle \lambda, \mu \rangle\rangle$ is infinite, and in fact has an infinite sequence of finite-index subgroups. These pull back to finite-index subgroups of $\pi_1(\tilde{O})$, which determine a sequence of covering spaces O_i. Because these subgroups contain the normal subgroup $\langle\langle \lambda, \mu \rangle\rangle$, the inverse image of L in each O_i is a disjoint union of copies of L. Hence, there is a linear lower bound on the number of components of $\mathrm{sing}_p(O_i)$ as a function of the covering degree. Therefore, by Theorem 9.2, $d_p(O_i)$ grows linearly, as required.

For any closed orientable 3-manifold M, there are obvious inequalities $g(M) \geq d(\pi_1(M)) \geq d_p(M)$, and the same is true for closed orientable 3-orbifolds (with an appropriate definition of Heegaard genus). Hence, Theorem 6.1 provides a sequence of finite covers of the orbifold O with positive Heegaard gradient. If we also knew that these covers did not have Property (τ), then by (an orbifold version of) Theorem 8.1, we would deduce that they are eventually Haken. In fact, we would get much more than this. We would be able to deduce that $\pi_1(O)$ is large, via the following theorem of the author [32].

Theorem 9.3. *Let Γ be a finitely presented group, let p be a prime and suppose that $\Gamma \geq \Gamma_1 \triangleright \Gamma_2 \triangleright \dots$ is a sequence of finite-index subgroups, where each Γ_{i+1} is normal in Γ_i and has index a power of p. Suppose that*

1. *the subgroups Γ_i have positive mod p homology gradient, and*
2. *the subgroups Γ_i do not have Property (τ).*

Then Γ is large.

We will explain the proof of this and related results in the next section. Similar reasoning also gives the the following theorem [35].

Theorem 9.4. *The Lubotzky-Sarnak Conjecture implies that any closed hyperbolic 3-orbifold that has at least one singular vertex has large fundamental group. In particular, the Lubotzky-Sarnak Conjecture implies that every arithmetic hyperbolic 3-manifold has large fundamental group.*

It is quite striking that the Lubotzky-Sarnak Conjecture, which is a question solely about the spectrum of the Laplacian, should have such far-reaching consequences for arithmetic hyperbolic 3-manifolds.

The way that this is proved is as follows. One starts with the closed hyperbolic 3-orbifold O with at least one singular vertex. Its fundamental group therefore contains a finite non-cyclic subgroup. For simplicity, suppose that this is $\mathbb{Z}/2 \times \mathbb{Z}/2$ (which is the case considered in [35]). One can then pass to a finite cover $\tilde{O}$ where every arc and circle of the singular locus has order 2 and which has at least one singular vertex. Any finite cover of the underlying manifold $|\tilde{O}|$ induces a finite cover O_i of $\tilde{O}$ where $\mathrm{sing}(O_i)$ is the inverse image of $\mathrm{sing}(\tilde{O})$. Since $\mathrm{sing}(\tilde{O})$ contains a trivalent vertex, $b_1(\mathrm{sing}_2(O_i))$ grows linearly as a function of the covering degree. Hence, $\{\pi_1(O_i)\}$ has positive mod 2 homology gradient. With some further work, and using the solution to the Geometrisation Conjecture, we may arrange that $|\tilde{O}|$ has a hyperbolic structure or has a finite cover with positive b_1. Hence, assuming the Lubotzky-Sarnak Conjecture, one can find finite covers $|O_i|$ with Cheeger constants tending to zero. Thus, $\pi_1(O)$ is large, by Theorem 9.3.

The above arguments are closely related to those behind Theorem 2.1. In fact, we can prove the following stronger version [33].

Theorem 9.5. *Let Γ be the fundamental group of a finite-volume hyperbolic 3-orbifold or 3-manifold. Suppose that Γ has a finite non-cyclic subgroup or is arithmetic. Then Γ contains the fundamental group of a closed orientable surface with positive genus.*

The proof runs as follows. One uses the same finite cover $\tilde{O}$ as above. We do not know that the Lubotzky-Sarnak Conjecture holds, but we have Theorem 7.2, which provides a sequence of infinite-sheeted covers $|O_i|$ of $|\tilde{O}|$ with Cheeger constants tending to zero. These induce covers O_i of $\tilde{O}$. One can use the singular locus of O_i to find a finite cover with more than one end. It then follows quickly that $\pi_1(O_i)$ contains a surface subgroup.

To make further progress with finite covers, it seems to be necessary to establish the Lubotzky-Sarnak Conjecture. But there is an important special case where this holds trivially: when the manifold or orbifold has a finite cover with positive first Betti number. For example, suppose that O is a compact orientable 3-orbifold with singular locus that contains a simple closed curve C. Suppose also that there is a surjective homomorphism $\pi_1(O) \to \mathbb{Z}$ that sends $[C]$ to zero. Then the resulting finite cyclic covers have linear growth of mod p homology (where p divides the order of C) and also their Cheeger constants tend to zero. So, by Theorem 9.3, $\pi_1(O)$ is large. Using this observation, the author, Long and Reid were able to prove the following [35].

Theorem 9.6. *Let Γ be the fundamental group of a finite-volume hyperbolic 3-manifold or 3-orbifold. Suppose that Γ is arithmetic or contains $\mathbb{Z}/2\times\mathbb{Z}/2$. Suppose also that Γ has a finite-index subgroup $\tilde{\Gamma}$ with $b_1(\tilde{\Gamma}) \geq 4$. Then Γ is large.*

This is significant because such a finite-index subgroup $\tilde{\Gamma}$ is known to exist in many cases. Indeed, arithmetic techniques, due to Clozel [15], Labesse-Schwermer [26], Lubotzky [44] and others, often provide a congruence subgroup with positive first Betti number. Then, using a theorem of Borel [6], one can find congruence subgroups with arbitrarily large first Betti number. The consequence of Theorem 9.6 is that one can in fact strengthen the conclusion to deduce that these groups are large.

10. Group-theoretic generalisations

We have discussed several topological results, such as Theorem 8.1, which are helpful in tackling the Virtually Haken Conjecture. It is natural to ask whether there are more general group-theoretic versions of these theorems. In many cases, there are. For example, the following is a version of Theorem 8.1, due to the author [28].

Theorem 10.1. *Let Γ be a finitely presented group, and let $\{\Gamma_i\}$ be a nested sequence of finite-index normal subgroups. Suppose that*

1. *their rank gradient is positive, and*
2. *they do not have Property (τ).*

Then, for all sufficiently large i, Γ_i is an amalgamated free product or HNN extension.

We now give an indication of the proof. As is typical with arguments in this area, one starts with a finite cell complex K with fundamental group Γ. Let K_i be the finite covering space corresponding to Γ_i. The hypothesis that Γ does not have Property (τ) with respect to $\{\Gamma_i\}$ implies that one can form a decomposition of K_i into two sets B_i and C_i with large volume but small intersection. Via the Seifert - van Kampen theorem, this then determines a decomposition of Γ_i into a graph of groups. We must show that this is a non-trivial decomposition. In other words, we must ensure that neither $\pi_1(B_i)$ nor $\pi_1(C_i)$ surjects onto Γ_i. This is where the hypothesis that $\{\Gamma_i\}$ has positive rank gradient is used. The number of 1-cells of B_i (or C_i) gives an upper bound to the rank of $\pi_1(B_i)$, and this is a definite fraction of the total number of 1-cells of K_i. Hence if $\pi_1(B_i)$ or $\pi_1(C_i)$ were to surject onto Γ_i, one could use this to deduce that the rank of Γ_i was too small.

We have also seen Theorem 9.3, which starts with the stronger hypothesis of positive mod p homology gradient, and which ends with the strong conclusion of largeness. The proof follows similar lines, but now the goal is to show that neither $H_1(B_i;\mathbb{F}_p)$ nor $H_1(C_i;\mathbb{F}_p)$ surjects onto $H_1(\Gamma_i;\mathbb{F}_p)$. Instead of using the Seifert -

van Kampen theorem, the Meyer-Vietoris theorem is used. One deduces that if $H_1(B_i;\mathbb{F}_p)$ or $H_1(C_i;\mathbb{F}_p)$ were to surject onto $H_1(\Gamma_i;\mathbb{F}_p)$, then $H_1(\Gamma_i;\mathbb{F}_p)$ would be too small, contradicting the assumption that the subgroups Γ_i have positive mod p homology gradient. Hence, again we get a graph of groups decomposition for Γ_i. This induces a graph of groups decomposition for $\tilde{\Gamma}_i = [\Gamma_i,\Gamma_i](\Gamma_i)^p$. Its underlying graph has valence at least p at each vertex. And $\tilde{\Gamma}_i$ surjects onto the fundamental group of this graph, which is a non-abelian free group (when $p \neq 2$), as required.

One might wonder whether Theorem 9.3 remains true even if we do not assume that the subgroups $\{\Gamma_i\}$ do not have Property (τ). The above proof breaks down. But is the hypothesis that the subgroups Γ_i have positive mod p homology gradient enough to deduce largeness? As mentioned in Section 6, this question relates to error-correcting codes. More details can be found in [32]. However, there is one interesting and natural situation where the hypothesis of positive mod p homology gradient is enough to deduce largeness, according to the following theorem of the author [34].

Theorem 10.2. *Let Γ be a finitely presented group. Suppose that its derived p-series has positive mod p homology gradient. Then Γ is large.*

The main part of the proof is showing that if the derived p-series of Γ has positive mod p homology gradient, then it does not have Property (τ). Hence, by Theorem 9.3, Γ is large. Once again, let K be a finite 2-complex with fundamental group Γ, and let K_i be the covering space corresponding to the subgroup Γ_i in the derived p-series. One needs to show that the Cheeger constant of K_i is arbitrarily small. This is achieved by finding non-trivial 1-cocycles on K_{i-1} with small support size, compared with the total number of edges of K_{i-1}. If $\tilde{K}_{i-1}$ denotes the cyclic covering space dual to such a cocycle, then the inverse image of the cocycle determines a decomposition of $\tilde{K}_{i-1}$ into two parts with large volume and small intersection. Since K_i finitely covers $\tilde{K}_{i-1}$, it too has small Cheeger constant. In fact, one keeps track of not just one cocycle on K_{i-1}, but several of them, and one uses these to create cocycles on K_i with slightly smaller relative support size, and so on. This is achieved using the technology explained in Section 4 for constructing cocycles on abelian covers, together with an elementary theorem from coding theory, known as the Plotkin bound.

11. Subgroup separability, special cube complexes and virtual fibering

There are many other interesting directions in the theory of finite covers of 3-manifolds, which we can only briefly discuss here.

The first of these is the notion of subgroup separability. A subgroup H of a group Γ is *separable* if for every element $\gamma \in \Gamma$ that does not lie in H, there is a homomorphism ϕ from Γ onto a finite group such that $\phi(\gamma) \notin \phi(H)$. A group Γ is said to be *LERF* if every finitely generated subgroup is separable. The relevance of this concept to 3-manifolds arises from the following theorem [38].

Theorem 11.1. *Let M be a compact orientable irreducible 3-manifold, and suppose that $\pi_1(M)$ has a separable subgroup that is isomorphic to the fundamental group of a closed orientable surface with positive genus. Then either M is virtually fibred or $\pi_1(M)$ is large. In particular, M is virtually Haken.*

This raises the question of which 3-manifolds have LERF fundamental group. There are examples of certain graph 3-manifolds M for which $\pi_1(M)$ is not LERF [10]. But it is conjectured that the fundamental group of every closed hyperbolic 3-manifold is LERF. A piece of evidence for this conjecture is given by the following important theorem, which is an amalgamation of work by Agol, Long and Reid [4] and Bergeron, Haglund and Wise [5].

Theorem 11.2. *Let M be an arithmetic hyperbolic 3-manifold which contains a closed immersed totally geodesic surface. Then every geometrically finite subgroup of $\pi_1(M)$ is separable.*

There is an important new concept, introduced by Haglund and Wise [24], that relates to subgroup separability. They considered a certain type of cell complex, known as a *special cube complex*. A group is said to be *virtually special* if it has a finite index subgroup which is the fundamental group of a compact special cube complex. One major motivation for introducing this concept is the following theorem of Haglund and Wise [24].

Theorem 11.3. *Let Γ be a word-hyperbolic group that is virtually special. Then every quasi-convex subgroup of Γ is separable.*

In the 3-dimensional case, it is known that this condition is equivalent to having 'enough' surface subgroups that are separable. Indeed Theorem 11.2 is proved in the case when M is closed by using the surface subgroups arising from totally geodesic surfaces to deduce that $\pi_1(M)$ is virtually special.

This is related to work of Agol [1]. He introduced a condition on a group, called RFRS. We will not give the definition of this here, but we note that if a group is virtually special then it is virtually RFRS. Agol was able to show that this condition can be used to prove that a 3-manifold virtually fibres over the circle.

Theorem 11.4. *Let M be a compact orientable irreducible 3-manifold with boundary a (possibly empty) collection of tori. Suppose that $\pi_1(M)$ is virtually RFRS. Then M has a finite cover that fibres over the circle.*

The hypotheses that $\pi_1(M)$ is virtually RFRS or virtually special are strong ones. However, Wise has recently raised the possibility of showing that if a compact orientable hyperbolic 3-manifold M has a properly embedded orientable incompressible surface that is not a sphere or a virtual fibre, then $\pi_1(M)$ is virtually special, by using induction along a hierarchy for M. While this would not say anything about the Virtually Haken Conjecture itself, it would be a very major development, as it would nearly reduce all the other conjectures to it. For example, combined with Theorem 11.4, it would show that every finite-volume orientable hyperbolic Haken 3-manifold is virtually fibred.

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